A SUFFICIENT CONDITION THAT AN OPERATOR BE NORMAL

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The following theorem answers a question raised by H. Radjavi and P. Rosenthal [1, p. 197].

THEOREM. *Let* T *be a (bounded) operator defined on a Hilbert space* H. *Suppose*

(a) T *is quasi-similar to a normal operator* N, *that is, there exist one-to-one operators* X *and* Y *with dense ranges such that* TX = XN *and* NY = YT, *and*

(b) *each hyperinvariant subspace of* T *is reducing.*

Then T *is normal.*

Proof. Let E be the spectral measure of N. Then $E(\sigma)H$ is hyperinvariant for N for every Borel set σ.

Let $\mathscr{L}$ be a hyperinvariant subspace of N. Put

$$q(\mathscr{L}) = \bigvee_{A \in \{T\}'} \overline{AX\mathscr{L}}.$$

Then (see B. Sz.-Nagy and C. Foiaş [2, pp. 76-78])

(i) $q(\mathscr{L})$ is hyperinvariant for T,

(ii) $\overline{Yq(\mathscr{L})} = \mathscr{L}$,

(iii) $\mathscr{L} \subset \mathscr{L}'$ implies $q(\mathscr{L}) \subset q(\mathscr{L}')$,

(iv) $\bigvee_\alpha q(\mathscr{L}_\alpha) = q\left(\bigvee_\alpha \mathscr{L}_\alpha\right)$,

(v) $q(\{0\}) = \{0\}$ and $q(H) = H$.

By (i) and the hypothesis (b), $q(\mathscr{L})$ is reducing for T. Let $P = P_{q(\mathscr{L})}$ and $Q = P_{q(\mathscr{L}^\perp)}$ (for each closed subspace $\mathscr{M}$, we denote by $P_\mathscr{M}$ the orthogonal projection from H onto $\mathscr{M}$). Then PT = TP. Since $q(\mathscr{L}^\perp)$ is hyperinvariant for T, we see that QPQ = PQ. Hence PQ = QP. On the other hand, by (ii), we have the relation

$$\overline{Yq(\mathscr{L})} \cap \overline{Yq(\mathscr{L}^\perp)} = \mathscr{L} \cap \mathscr{L}^\perp = 0.$$

Hence $q(\mathscr{L}) \cap q(\mathscr{L}^\perp) = 0$. Also,

$$\overline{q(\mathscr{L}) + q(\mathscr{L}^\perp)} \supseteq \overline{X\mathscr{L} + X\mathscr{L}^\perp} = \overline{X(\mathscr{L} + \mathscr{L}^\perp)} = \overline{XH} = H.$$

Therefore $P = I - Q$, or $q(\mathscr{L}^\perp) = q(\mathscr{L})^\perp$.

Now, for disjoint Borel sets σ and τ, we have the relation $E(\sigma)H \perp E(\tau)H$ or $E(\sigma)H \subset (E(\tau)H)^\perp$. Hence

$$q(E(\sigma)H) \subseteq q((E(\tau)H)^\perp) = q(E(\tau)H)^\perp.$$

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Hence $q(E(\sigma)H) \perp q(E(\tau)H)$. For each Borel set σ, let $F(\sigma) = P_{q(E(\sigma)H)}$. Then

$$F(\sigma)F(\tau) = F(\tau)F(\sigma) = \{0\},$$

if $\sigma \cap \tau = \emptyset$. Together with properties (i) to (v), this shows that F is a spectral measure. Let $M = \int z\, dF_z$. Then M is a normal operator.

We claim that $E(\sigma)Y = YF(\sigma)$. By (ii), $\overline{YF(\sigma)H} = E(\sigma)H$. If $F(\sigma)y = y$, then $y \in F(\sigma)H$ and hence $Yy \in E(\sigma)H$. Therefore $E(\sigma)Yy = Yy = YF(\sigma)y$ in this case. If $F(\sigma)y = 0$, then $F(\sigma^c)y = y$ (the symbol σ^c stands for the complement of σ); because $\overline{YF(\sigma^c)H} = E(\sigma^c)H$, we can assert that $E(\sigma^c)Yy = Yy$, or $E(\sigma)Yy = 0$. Hence $E(\sigma)Yy = YF(\sigma)y = 0$ in this case. Therefore $E(\sigma)Y = YF(\sigma)$.

In particular, $E(\sigma) = 0$ if and only if $F(\sigma) = 0$. Hence N and M have the same spectrum. If f is a step function such that $|f(z) - z|$ is small on this spectrum, then $\|f(N) - N\|$ and $\|f(M) - M\|$ are small, and $f(N)Y = Yf(M)$. Hence $NY = YM$. Thus $YT = YM$. Since Y is one-to-one, $M = T$. Therefore T is normal.

REFERENCES

1. H. Radjavi and P. Rosenthal, *Invariant subspaces*. Ergebnisse, Band 77. Springer-Verlag, New York, 1973.

2. B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space*. Akadémiai Kiadó, Budapest; North-Holland Publ. Co., Amsterdam, 1970.

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