BLASCHKE PRODUCTS WITH DERIVATIVE IN H^p AND B^p

David Protas

1. INTRODUCTION

Let U denote the open unit disc in the complex plane. The Hardy class H^p ($0 < p < \infty$) consists of all functions analytic in U for which

$$\|f\|_{H^p}^p = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt$$

is finite. The class B^p ($0 < p < 1$) consists of all functions analytic in U for which

$$\|f\|_{B^p} = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} |f(re^{it})| (1-r)^{1/p-2} dt dr$$

is finite. See [4] for a general discussion of H^p , and see [5] for basic properties of B^p . We note here the well-known result that $H^p \subset B^p$ for all p in the interval $(0, 1)$.

A function analytic and bounded in U is said to be an inner function if its boundary values have modulus 1 almost everywhere. Every inner function ϕ has a factorization $\phi = mbs$, where m is a monomial, b is a Blaschke product with zeros $\{a_n\}$ ($a_n \neq 0$), and s is a singular inner function; that is,

$$b(z) = \prod_n \frac{\bar{a}_n}{|a_n|} \frac{a_n - z}{1 - \bar{a}_n z},$$

where $0 < |a_n| < 1$ and $\sum (1 - |a_n|) < \infty$, and

$$s(z) = \exp \left[- \int_0^{2\pi} [(e^{it} + z)/(e^{it} - z)] d\mu(t) \right],$$

where μ is a finite positive singular measure.

In [2], J. G. Caughran and A. L. Shields raised the question whether there exists a singular inner function s whose derivative s' is in $H^{1/2}$. In [3], M. R. Cullen showed among other things that the derivative of every singular inner function lies in B^p for all p ($0 < p < 1/2$), and he conjectured that the derivative of no singular inner function lies in $B^{1/2}$. H. A. Allen and C. L. Belna disproved this conjecture by giving examples in [1] of singular inner functions with derivatives in B^p ($0 < p < 2/3$). In this paper, we shall investigate the corresponding properties for Blaschke products. Since the derivative of each finite Blaschke product is obviously in every H^p -space, we need only consider infinite Blaschke products.

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2. DERIVATIVE IN B^p

It is easily seen from Theorem 5 of P. L. Duren, B. W. Romberg, and A. L. Shields [5] that the derivative of every Blaschke product is in B^p for all p ($0 < p < 1/2$). Also, W. Rudin showed in [6] that if the zeros $\{a_n\}$ of a Blaschke product b satisfy the condition

$$\sum_n (1 - |a_n|) \log \frac{1}{1 - |a_n|} < \infty,$$

then $b' \in B^{1/2}$. The following theorem implies that for each $p < 1$ there exist infinite Blaschke products with derivative in B^p .

THEOREM 1. *Let b be a Blaschke product with zeros $\{a_n\}$ such that*

$$\sum_n (1 - |a_n|)^\alpha < \infty$$

for some α ($0 < \alpha < 1$). Then $b' \in B^{1/(1+\alpha)}$.

Proof. It is easily seen that

$$b'(z) = \sum_n \beta_n(z) (1 - |a_n|^2) / (1 - \bar{a}_n z)^2,$$

where $\beta_n(z) = b(z) (1 - \bar{a}_n z) / (z - a_n)$, and this implies that

$$|b'(z)| \leq \sum_n (1 - |a_n|^2) / |1 - \bar{a}_n z|^2 \leq 2 \sum_n (1 - |a_n|) / |1 - \bar{a}_n z|^2$$

for all $|z| < 1$. Therefore, for $0 < r < 1$,

$$\int_0^{2\pi} |b'(re^{it})| dt \leq 2 \sum_n (1 - |a_n|) \int_0^{2\pi} \frac{dt}{|1 - \bar{a}_n re^{it}|^2} = 4\pi \sum_n \frac{1 - |a_n|}{1 - r^2 |a_n|^2}.$$

The inequalities

$$2(1 - r^2 |a_n|^2) \geq 2(1 - r|a_n|) \geq 2 - r^2 - |a_n|^2 \geq 1 - r + 1 - |a_n|$$

now imply that

$$\int_0^{2\pi} |b'(re^{it})| dt \leq 8\pi \sum_n \frac{1 - |a_n|}{1 - r + 1 - |a_n|}.$$

If we write $p = 1/(1 + \alpha)$, then $1/p - 2 = \alpha - 1$; setting $1 - |a_n| = c_n$, we now obtain the estimate

$$\begin{aligned} \int_0^1 \frac{c_n(1-r)^{\alpha-1}}{1-r+c_n} dr &= \int_0^1 \frac{c_n s^{\alpha-1}}{s+c_n} ds \leq \int_0^{c_n} s^{\alpha-1} ds + \int_{c_n}^1 c_n s^{\alpha-2} ds \\ &= \frac{c_n^\alpha}{\alpha} + \frac{c_n^\alpha - c_n}{1-\alpha} \leq \frac{c_n^\alpha}{\alpha(1-\alpha)}. \end{aligned}$$

It follows immediately that

$$\|b'\|_{B^p} \leq \frac{4}{\alpha(1-\alpha)} \sum_n (1-|a_n|)^\alpha.$$

3. DERIVATIVE IN H^p

We now prove that for each $p < 1$, there exists an infinite Blaschke product with derivative in H^p .

THEOREM 2. *Let b be a Blaschke product with zeros $\{a_n\}$ such that*

$$\sum_n (1-|a_n|)^\alpha < \infty$$

for some α ($0 < \alpha < 1/2$). Then $b' \in H^{1-\alpha}$.

Proof. Let $1/2 < p < 1$. As in the proof of the first theorem, we have the inequality

$$|b'(z)| \leq 2 \sum_n (1-|a_n|)/|1-\bar{a}_n z|^2$$

for all $|z| < 1$. Since $0 < p < 1$, this implies that

$$|b'(z)|^p \leq 2^p \sum_n (1-|a_n|)^p / |1-\bar{a}_n z|^{2p}.$$

Since $p > 1/2$, there exists a constant C (see [4, p. 65]) such that for $n = 1, 2, \dots$ and all r ($0 < r < 1$),

$$\begin{aligned} \int_0^{2\pi} \frac{dt}{|1-\bar{a}_n r e^{it}|^{2p}} &= \int_0^{2\pi} \frac{dt}{(1+r^2|a_n|^2-2r|a_n|\cos t)^p} \\ &\leq C(1-r|a_n|)^{1-2p} \leq C(1-|a_n|)^{1-2p}. \end{aligned}$$

Therefore we conclude that

$$\|b'\|_{H^p}^p \leq 2^p C \sum_n (1-|a_n|)^{1-p}.$$

For $\alpha = 1/2$, we get the following result, which is similar to Rudin's result for $B^{1/2}$.

THEOREM 3. Let b be a Blaschke product with zeros $\{a_n\}$ such that

$$\sum_n (1 - |a_n|)^{1/2} \log \frac{1}{1 - |a_n|} < \infty.$$

Then $b' \in H^{1/2}$.

This theorem can be proved exactly in the same manner as Theorem 2, once we have established the following lemma to replace the estimate from [4] that we used above.

LEMMA.
$$\int_0^{2\pi} \frac{dt}{(1 + \rho^2 - 2\rho \cos t)^{1/2}} = O\left(\log \frac{1}{1 - \rho}\right) \quad \text{as } \rho \rightarrow 1.$$

Proof. Elementary computations give the estimate

$$\begin{aligned} \int_0^{2\pi} \frac{dt}{(1 + \rho^2 - 2\rho \cos t)^{1/2}} &= 2 \int_0^\pi \frac{dt}{((1 - \rho)^2 + 4\rho \sin^2(t/2))^{1/2}} \\ &\leq 2 \int_0^{1-\rho} \frac{dt}{1 - \rho} + 2 \int_{1-\rho}^\pi \frac{dt}{2\sqrt{\rho} t/\pi} = O\left(\log \frac{1}{1 - \rho}\right). \end{aligned}$$

It would be interesting to know whether the convergence of $\sum (1 - |a_n|)^\alpha$ for some α ($1/2 < \alpha < 1$) implies that $b' \in H^p$ for some p depending only on α .

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California State University
Northridge, California 91324