INNER FUNCTIONS IN THE POLYDISC AND MEASURES ON THE TORUS

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The purpose of this note is to prove a proposition about the structure of R. P. measures, and to use this result in the consideration of some problems about inner functions in the polydisc. In the first section, we give the result about R. P. measures. In the second section, we apply this result to get some information about factoring inner functions, and in the last section we consider some examples.

1. In what follows, C denotes the space of complex numbers, U the open unit disc in C, and T the boundary of U. We consider Borel measures μ carried on $T^N \subseteq C^N$, where N is a positive integer. We fix positive integers k and ℓ such that $k + \ell = N$, and we write

$$C^N = C^k \times C^\ell, \qquad T^N = T^k \times T^\ell, \qquad U^N = U^k \times U^\ell.$$

If $z \in C^N$, we write $z = (\zeta, w)$, where $\zeta \in C^k$ and $w \in C^\ell$. If n is an N-tuple of integers, we write $n = (\alpha, \beta)$, where α is a k-tuple and β is an ℓ-tuple. We use the usual multi-index notation $z^n = z_1^{n_1} \cdots z_N^{n_N}$. If μ is a Borel measure on T, we let

$$\hat{\mu}(n) = \int_{T^N} \bar{z}^n \, d\mu(z),$$

and if $E \subseteq T^N$ is a Borel set, we let μ_E denote the restriction of μ to E. If μ is a Borel measure on T^N, we denote by $\pi\mu$ the measure on T^k such that $(\pi\mu)(E) = \mu(E \times T^\ell)$ for every Borel set $E \subseteq T^k$. We note that if f is a continuous function on T^k, then

$$\int_{T^k} f(\zeta)\, d(\pi\mu)(\zeta) = \int_{T^N} f(\zeta)\, d\mu(\zeta, w).$$

A real Borel measure μ on T^N is said to be an R. P. *measure* if $\hat{\mu}(n) = 0$ whenever not all the n_i have the same sign. The R. P. measures are the measures whose Poisson integrals are the real parts of holomorphic functions (see [3, p. 33]). Finally, m_k denotes Haar measure on T^k, and m_ℓ denotes Haar measure on T^ℓ.

PROPOSITION 1. *Let* μ *be an* R. P. *measure on* T^N, *and let* $E \subseteq T^k$ *be a Borel set such that* $m_k(E) = 0$; *then* $\mu_{E \times T^\ell} = (\pi\mu)_E \times m_\ell$.

Proof. Fix an ℓ-tuple $\beta \neq 0$, say $\beta_i > 0$ for some i. Then, since μ is an R. P. measure, we see that $\hat{\mu}(\alpha, \beta) = 0$ unless $\alpha_j \geq 0$ for $j = 1, \cdots, k$. Now

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$$\hat{\mu}(\alpha, \beta) = \int_{T^N} \bar{\zeta}^\alpha \bar{w}^\beta d\mu(\zeta, w) = \int_{T^N} \bar{\zeta}^\alpha d(\bar{w}^\beta \mu)(\zeta, w) = \int_{T^k} \bar{\zeta}^\alpha d(\pi \bar{w}^\beta \mu)(\zeta).$$

In other words, $(\pi \bar{w}^\beta \mu)\hat{}\,(\alpha) = 0$ unless $\alpha_j \geq 0$ for $j = 1, \cdots, k$. It follows from a theorem of S. Bochner [1] that the measure $\pi \bar{w}^\beta \mu$ is absolutely continuous with respect to Haar measure m_k. If $\beta_i < 0$ for some i, then the same argument shows that $(\pi \bar{w}^\beta \mu)\hat{}\,(\alpha) = 0$, unless $\alpha_j \leq 0$ for $j = 1, \cdots, k$, and we may still conclude from Bochner's theorem that $\pi \bar{w}^\beta \mu$ is absolutely continuous with respect to m_k. It follows that if $\beta \neq 0$, then $\pi \bar{w}^\beta \mu$ is absolutely continuous with respect to m_k. Hence $(\pi \bar{w}^\beta \mu)_E \equiv 0$, since $m_k(E) = 0$. Hence, for *each* α, we have the relation $(\pi \bar{w}^\beta \mu)\hat{}_E\,(\alpha) = 0$, in other words,

$$\int_E \bar{\zeta}^\alpha d(\pi \bar{w}^\beta \mu)(\zeta) = 0$$

for all α. But

$$\int_E \bar{\zeta}^\alpha d(\pi \bar{w}^\beta \mu)(\zeta) = \int_{E \times T^\ell} \bar{\zeta}^\alpha \bar{w}^\beta d\mu(\zeta, w);$$

that is, $(\mu_{E \times T^\ell})\hat{}\,(\alpha, \beta) = 0$ if $\beta \neq 0$. Note that when $\beta = 0$, then

$$(\mu_{E \times T^\ell})\hat{}\,(\alpha, 0) = (\widehat{\pi\mu})_E(\alpha);$$

hence we see that the measures $\mu_{E \times T^\ell}$ and $(\pi\mu)_E \times m_\ell$ have the same Fourier coefficients, so that they are equal. The proof is complete.

2. If ϕ is a bounded holomorphic function in U^N, then $\lim_{r \to 1^-} \phi(rz) = \phi^*(z)$ exists for almost all $z \in T^N$. If $|\phi^*(z)| = 1$ almost everywhere, we say that ϕ is an *inner function*. If μ is a positive singular (with respect to Haar measure) R. P. measure on T^N and f is the function whose real part is the Poisson integral of μ, then $\phi = \exp(-f)$ is an inner function. Moreover, every nonvanishing inner function arises in this way. These matters are discussed in [3]. If ϕ and ψ are inner functions in U^N, we say that ϕ divides ψ if ψ/ϕ is an inner function. If ϕ and ψ are nonvanishing inner functions with corresponding R. P. measures μ and ν, then ϕ divides ψ if and only if $0 \leq \mu \leq \nu$. Therefore the problem of finding all divisors of ψ is equivalent to finding all R. P. *measures* μ such that $0 \leq \mu \leq \nu$. Finally, we point out that if ϕ is a nonvanishing inner function whose R. P. measure is μ, then ϕ is independent of the last ℓ variables if and only if $\mu = \sigma \times m_\ell$, where σ is an R. P. measure on T^k.

PROPOSITION 2. *Let ϕ be a nonvanishing inner function in* U^N *with* R. P. *measure μ on* T^N. *Let ϕ_1 be a nonvanishing inner function in* U^k *with* R. P. *measure σ on* T^k. *Then ϕ_1 divides ϕ in* U^N *if and only if $\sigma \leq \pi\mu$.*

Proof. If ϕ_1 divides ϕ in U^N, then the preceding remarks imply that $\sigma \times m_\ell \leq \mu$, and hence $\sigma = \pi(\sigma \times m_\ell) \leq \pi\mu$. On the other hand, suppose $\sigma \leq \pi\mu$. Since ϕ_1 is an inner function, σ is singular. Let E be a Borel set in T^k such that $m_k(E) = 0$ and σ is carried on E. Then, by Proposition 1,

$$\mu_{E \times T^\ell} = (\pi\mu)_E \times m_\ell \geq \sigma \times m_\ell,$$

and hence $\sigma \times m_\ell \leq \mu_{E \times T^\ell} \leq \mu$.

PROPOSITION 3. *Let ϕ be a nonvanishing inner function in U^N, and let ϕ_1 be an inner function in U^k; then ϕ_1 divides ϕ in U^N if and only if the function $F(\zeta) = \phi(\zeta, 0)/\phi_1(\zeta)$ is bounded in U^k.*

Proof. If ϕ_1 divides ϕ in U^N, then $\phi(\zeta, 0)/\phi_1(\zeta)$ is certainly bounded in U^k. To see the other implication, let μ be the R. P. measure for ϕ; in other words, let $\phi = e^{-f}$, where the real part of f is the Poisson integral of μ. Because $\phi(\zeta, 0) = \exp\{-f(\zeta, 0)\}$, we see that

$$\Re f(\zeta, 0) = \int_{T^N} P_\zeta(\theta)\, d\mu(\theta, \phi) = \int_{T^k} P_\zeta(\theta)\, d(\pi\mu)(\theta),$$

where $P_\zeta(\theta)$ is the Poisson kernel [3]. If σ is the R. P. measure for ϕ_1, then $\phi_1 = \exp(-g)$, where $\Re g(\zeta) = \int_{T^k} P_\zeta(\theta)\, d\sigma(\theta)$. Now it follows that

$$\phi(\zeta, 0)/\phi_1(\zeta) = \exp\{-(f - g)\} \quad \text{and} \quad \Re(f(\zeta) - g(\zeta)) = \int_{T^k} P_\zeta(\theta)\, d(\pi\mu - \sigma)(\theta).$$

Since the quotient $F(\zeta) = \phi(\zeta, 0)/\phi_1(\zeta)$ is bounded and ϕ_1 is an inner function, F must be bounded by 1. Hence the Poisson integral of $\pi\mu - \sigma$ is always nonnegative, so that $\pi\mu - \sigma \geq 0$. The result now follows from Proposition 2.

We shall see from the examples how this can fail if ϕ is allowed to take the value 0.

PROPOSITION 4. *Let ϕ be a nonvanishing inner function in U^N. Then ϕ has a u n i q u e factorization in the form*

$$\phi(z_1, \cdots, z_N) = \phi_1(z_1)\,\psi(z_1, \cdots, z_N),$$

where ϕ_1 and ψ are inner functions and ψ has no factor depending on z_1 alone.

Proof. We take $k = 1$ and $\ell = N - 1$ in the previous propositions, and we consider the Lebesgue decomposition of the measure $\pi\mu$, where μ is the R. P. measure for ϕ. We see that $\pi\mu = hm_1 + \sigma$. Let ϕ_1 be the inner function corresponding to the measure σ. (Since $k = 1$, every measure is R. P.) Since $\sigma \leq \pi\mu$, ϕ_1 divides ϕ in U^N, by Proposition 2. If $\psi = \phi/\phi_1$, then the R. P. measure for ψ is $\mu - \sigma \times m_\ell = \nu$. Because $\pi\nu = \pi\mu - \sigma = hm_1$, there are no positive singular measures $\tau \leq \pi\mu$, and therefore ψ has no factor depending only on z_1 (again by Proposition 2). Finally, the uniqueness of the decomposition follows from the uniqueness of the Lebesgue decomposition of $\pi\mu$.

It is easy to see that if ϕ is any inner function in U^N and $k + \ell = N$, then we can write $\phi(\zeta, w) = \phi_1(\zeta)\,\psi(\zeta, w)$, where ϕ_1 and ψ are inner functions and ψ has no factor depending on ζ alone. However, we shall see that when $k > 1$, we no longer get uniqueness, even if ϕ has no zeros.

3. First we show that Proposition 3 can fail if ϕ is allowed to vanish. It is easy to find an inner function $\phi(\zeta, w)$ in U^2 such that $\phi(0, 0) = 0$ (and hence $\phi(\zeta, 0)/\zeta$ is

bounded) but ζ does not divide $\phi(\zeta, w)$ in U^2. We go a bit further and show that there exist an inner function ϕ in U^2 and a *nonvanishing* inner function ϕ_1 in U such that $\phi(\zeta, 0)/\phi_1(\zeta)$ is bounded but ϕ_1 does not divide ϕ in U^2. To get such functions ϕ and ϕ_1, let ϕ_1 be a singular inner function in U, and pick γ so that $|\gamma| < 1$ and $B = (\phi_1 + \gamma)/(1 + \bar{\gamma}\phi_1)$ is a Blaschke product. This is possible, by a well-known theorem of Frostman [2, pp. 109-113]. Let ψ be another singular inner function, define $\alpha = \gamma\psi(0)$, and let

$$\phi(\zeta, w) = \frac{B(\zeta)\psi(w) - \alpha}{1 - \bar{\alpha} B(\zeta)\psi(w)}.$$

Clearly, ϕ is an inner function in U^2, and since

$$\phi(\zeta, 0) = \psi(0)\frac{B(\zeta) - \gamma}{1 - \bar{\alpha}\psi(0) B(\zeta)},$$

it follows that $\phi(\zeta, 0)/\phi_1(\zeta)$ is bounded. However, if ϕ_1 were to divide ϕ in U^2, there would exist an inner function Φ in U^2 such that

$$\psi(w)\frac{B(\zeta) - \alpha/\psi(w)}{1 - \bar{\alpha} B(\zeta)\psi(w)} = \phi_1(\zeta)\Phi(\zeta, w).$$

This would imply that ϕ_1 divides the inner part of $B - \alpha/\psi(w)$ for all $|w| < 1$. But the inner part of $B - \delta$ is a Blaschke product, for all δ outside a set of capacity zero, again by Frostman's theorem. This gives a contradiction, because $\alpha/\psi(w)$ covers a neighborhood of γ, as w varies over U. Hence $\phi(\zeta, 0)/\phi_1(\zeta)$ is bounded, but ϕ_1 does not divide ϕ in U^2.

Next we show that the uniqueness part of Proposition 4 can fail if $k > 1$. We give an example of a nonvanishing inner function ϕ in U^3 with two essentially different factorizations $\phi = \phi_1\psi$, where ϕ_1 depends only on the first two variables and ψ has no factors that depend only on the first two variables. If ϕ is a nonvanishing inner function in U^3 with measure μ, and $\pi\mu$ is its projection on T^2, then finding a factorization of the type indicated is equivalent to finding a singular R. P. measure σ on T^2 such that $0 \leq \sigma \leq \pi\mu$ and σ is maximal with respect to this property, in other words, such that if τ is a singular R. P. measure satisfying the relations $\sigma \leq \tau \leq \pi\mu$, then $\tau = \sigma$. Therefore, to produce our example we need a singular positive R. P. measure μ on T^3 such that there exist two different singular R. P. measures σ_1 and σ_2 on T^2, both maximal with respect to the property of being less than $\pi\mu$. We sketch the construction and omit the verification of the details. We begin with three measures on T:

$$d\lambda(\theta) = [17/8 + \cos\theta + \cos 2\theta]\frac{d\theta}{2\pi},$$

$$d\sigma_1(\theta) = \frac{d\theta}{2\pi},$$

$$d\sigma_2(\theta) = (9/8 + \cos\theta)\frac{d\theta}{2\pi}.$$

All are positive, and $\sigma_1 \leq \lambda$, $\sigma_2 \leq \lambda$. We map T into T^2 by the map $z \mapsto (z, \bar{z})$. This induces measures $\bar{\lambda}$, $\bar{\sigma}_1$, $\bar{\sigma}_2$ on T^2 from the measures λ, σ_1, and σ_2. They are singular with respect to Haar measure m_2 on T^2, and $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are R. P. measures. Define the measure τ on T^2 by the formula

$$d\tau(\theta, \phi) = [1 - \cos(\theta - \phi)] \frac{d\theta\, d\phi}{(2\pi)^2},$$

and let $\nu = \tau + \bar{\lambda}$. Then $\nu \geq 0$, ν is an R. P. measure, and $\bar{\lambda}$ is the singular part of ν. Moreover, $\bar{\sigma}_1 \leq \nu$ and $\bar{\sigma}_2 \leq \nu$, and among singular R. P. measures they are maximal with respect to this property. Therefore we need only show that there is a positive singular R. P. measure μ on T^3 such that $\pi\mu = \nu$. To this end, we map T^2 into T^3 by the correspondence $(\zeta, w) \mapsto (\zeta, w, \bar{\zeta}^2 \bar{w}^2)$. This induces a measure μ_1 from the measure τ, and we can easily verify that $\mu = \mu_1 + \bar{\lambda} \times m_1$ is a positive, singular R. P. measure and that $\pi\mu = \nu$.

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