A GENERALIZATION OF A THEOREM OF KAPLANSKY AND RINGS WITH INVOLUTION

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I. Kaplansky has shown that if R is a semisimple ring each of whose elements is *power-central* (that is, if for each element x of R, there exists a positive integer n(x) such that $x^{n(x)}$ is in the center Z of R), then R is a commutative ring.

In this paper, we generalize Kaplansky's theorem to a ring with involution each of whose *symmetric* elements is power-central. We show that if the ring R has no nil right ideals, then all norms xx^* and all traces $x + x^*$ in the ring are central elements. If we weaken the assumption of no nil right ideals and assume merely that R has no nil ideals, the conclusion still holds, provided the least positive exponent n(s), for which $s^{n(s)} \in Z$, remains bounded as s ranges over the subset of symmetric elements in R (see Theorem 4).

Since semisimple rings R have no nil right ideal other than 0 (for brevity, we refer to them as rings *with no nil right ideal*), the first part of Theorem 4 generalizes Kaplansky's theorem.

I. N. Herstein has established the following extension of Kaplansky's theorem: A ring R with no nil ideal all of whose elements are power-central is a commutative ring [4]. The second part of Theorem 4 generalizes Herstein's theorem in the case of *bounded* exponents. The conclusion is the best one could expect; for if R consists of the 2-by-2 matrices over a field of characteristic 2, its symmetric elements under the involution $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ are the matrices $\begin{pmatrix} a & b \\ c & a \end{pmatrix}$, which are all square-central.

Whether the assumption "with no nil right ideal" in Theorem 4 can be replaced by its two-sided version "with no nil ideal" is an open question equivalent to a question of K. McCrimmon (see Section 4).

We break the proof of Theorem 4 into three steps. In Section 1 we prove the result for the case of division rings (see Theorems 1 and 2). In Section 2, we extend the results of Section 1 to certain *-prime rings (see Theorem 3), and in Section 3 we establish the general result by reduction to the preceding case. The author was inspired by Herstein's proof of [5, Theorem 3.2.2], and similar techniques are used. Finally, in Section 4, we give an example of a division ring all of whose symmetric elements are square-central, but not all of whose elements are central, and we conclude the paper with some open questions.

1. DIVISION RINGS

Some notation and conventions: Throughout this paper, R denotes a ring with involution in which

Received July 21, 1971, and November 10, 1972.

This research was supported by Grant A78676 of the NRC of Canada and by a summer fellowship of the Canadian Mathematical Congress (Branch at Université Laval). The author would like to thank the referee for many valuable suggestions.

Michigan Math. J. 20 (1973).

(1) each symmetric element is power-central.

By

$$S = S(R) = \{s \mid s = s^*\}$$

we denote the subset of symmetric elements, by $Z = Z(R)$ the center of R, by $Z^+ = Z \cap S$ the subring of central symmetric elements, and by $Z^0 = Z^+ - \{0\}$ the subset of nonzero central symmetric elements.

For division rings, we can replace condition (1) with the following:

(2) For each $s \in S$, there exist an $r \geq 1$ and a polynomial $p = p(t)$ in t with rational integer coefficients such that $s^r - s^{r+1}p(s) \in Z$.

As a digression, we prove the following result.

PROPOSITION 1. *If* R *is a division ring with involution, then* (1) *and* (2) *are equivalent conditions.*

Proof. Obviously, (1) implies (2). Conversely, let $s \in Z$, and let $F = Z^+(s)$ be the subdivision ring generated by Z^+ and s. This is a subfield of symmetric elements satisfying (2). By a theorem of Herstein [3, Theorem 1], F has one of the following forms:

(i) $F = Z^+$;

(ii) F is a purely inseparable extension of Z^+;

(iii) F is an algebraic extension of a finite field.

In each case, one sees at once that $s^{n(s)} \in Z^+ \subseteq Z$ for some $n(s) \geq 1$.

In the course of the proof of Proposition 1, we saw that if $s \notin Z$, then F must be of type (ii) or (iii). Now Herstein and S. Montgomery have recently proved that a division ring with involution in which the symmetric elements are algebraic over a finite field must be a field [7, Theorem 1]. Therefore, if (iii) were to occur, Z^+ would be algebraic over a finite field, and consequently S would be algebraic over a finite field (by (1)). It would follow that $R = Z$, which is in conflict with the assumption $x \notin Z$. We are therefore left with (ii). Recall that this means that R has characteristic $p \neq 0$ and that there exists a $k = k(s)$ such that $s^{p^k} \in Z$. We have thus established the following result.

PROPOSITION 2. *A symmetric element* s *of* R *that is not purely inseparable (over the center) is central.*

The next proposition, which improves slightly Remark 6 in [1], is not true in the case of characteristic 2 (see Example 1).

PROPOSITION 3. *Let* R *be a* 2*-torsion-free division ring. If* S *is algebraic but not contained in* Z, *then* S *contains a separable element.*

Proof. We may assume that R has characteristic $p > 2$. If S were inseparable, a partial duplication of [5, Lemma 3.2.1] would yield some $s \in S - Z$ of the form $s = cs - sc$, for some $c \in R$. Writing $\sigma = (c - c^*)/2$, we get the relations

$$s = \sigma s - s\sigma, \qquad \sigma - 1 = s\sigma s^{-1}, \qquad \sigma^2 + 1 - 2\sigma = s\sigma^2 s^{-1}.$$

Because $\sigma^2 \in S$, either the subfield $Z^+(\sigma^2)$ generated by Z^+ and σ^2 contains a proper subfield separable over Z^+, or $(\sigma^2)^{p^m} \in Z$ for some $m \geq 1$. Since

$Z^+(\sigma^2) \subseteq S$, we must conclude that $\sigma^{2p^m} \in Z$. From this it follows that

$$(\sigma^2 + 1 - 2\sigma)^{p^m} = \sigma^{2p^m} + 1 - 2^{p^m}\sigma^{p^m} = (s\sigma^2 s^{-1})^{p^m} = s\sigma^{2p^m} s^{-1} = \sigma^{2p^m},$$

$$\sigma^{p^m} = \frac{1}{2^{p^m}} \in Z, \qquad (\sigma - 1)^{p^m} = \sigma^{p^m} - 1 = (s\sigma s^{-1})^{p^m} = \sigma^{p^m},$$

which is impossible. Combining Propositions 1 to 3, we get the following theorem.

THEOREM 1. *For a* 2*-torsion-free division ring* R *with involution, the following conditions are equivalent:*

(i) *each symmetric element* s *satisfies a condition* $s^r - s^{r+1}p(s) \in Z$, *where* $r = r(s) \geq 1$, *and where* $p = p(t)$ *denotes a polynomial with rational integral coefficients;*

(ii) *each symmetric element is power-central.*

The conditions imply that all symmetric elements are central.

Theorem 1 generalizes [3, Theorem 1], and in the case of 2-torsion-free rings, it extends [7, Theorem 2] and [2, Theorem 1.5]. Now to the case of characteristic 2.

PROPOSITION 4. *If* $S \not\subseteq Z$, *there exists a pair of symmetric elements* a *and* b *in* R *with the properties*

(i) $ab + ba = 1$,

(ii) $ab \notin Z$,

(iii) $\{1, a, b, ab\}$ *is linearly independent over* Z,

(iv) *the subdivision ring* Z(a, b) *generated by* Z, a, *and* b *has a center* $Z(a^2, b^2)$ *with the basis* $\{1, a, b, ab\}$ *over its center.*

Proof. Assume, by way of contradiction, that ab = ba for all a and b in S. This implies that R satisfies a polynomial identity of degree at most 4. Consequently, R is at most 4-dimensional. Let M be a maximal subfield of R. In $R \otimes_Z M$, define an involution (on generators) by

$$(a \otimes m)^* = a^* \otimes m \qquad (a \in R,\ m \in M).$$

Again, any two symmetric elements in $R \otimes_Z M$ commute. Now consider the natural M-isomorphism from $R \otimes_Z M$ onto M_2. We can equip M_2 with an involution that leaves fixed every element in M such that every two symmetric matrices commute. It follows that all symmetric elements of M_2 must be central. Thus all symmetric elements of R are central, a contradiction. We must conclude that there are two noncommuting symmetric elements u and v in R. In view of Proposition 3, there exist two positive integers n and m such that $u^{2^n}, v^{2^m} \in Z$. Let n and m denote the least integers for which u^{2^n} commutes with v and v^{2^m} commutes with $u^{2^{n-1}}$. Set $a = u^{2^{n-1}}$ and $b' = v^{2^{m-1}}$. Clearly, $a^2(b'^2)$ commutes with b'(a), but a does not commute with b'. If $b = (ab' + b'a)^{-1}b'$, condition (i) is satisfies. A routine inspection shows that (ii), (iii), and (iv) also hold.

PROPOSITION 5. *If* $S \not\subseteq Z$, *the involution is of the first kind.*

Proof. Let a and b be defined as in Proposition 4. It suffices to show that the involution leaves each element of $Z(a^2, b^2)$ fixed. Thus one may assume that $R = Z(a, b)$ and that consequently $Z = Z(a^2, b^2)$. Let $\xi \in Z$, and let $x = a + \xi b$. Then

$$x^* = a + \xi^* b,$$

$$s = xx^* = a^2 + \xi^* ab + \xi ba + \xi\xi^* b^2 = z + (\xi + \xi^*) ab \quad \text{for some } z \in Z,$$

$$\cdots\cdots\cdots\cdot,$$

$$s^{2^k} = z_k + (\xi + \xi^*)^{2^k} ab \quad \text{for some } z_k \in Z \ (k = 1, 2, \cdots).$$

Assume, by way of a contradiction, that $\xi + \xi^* \neq 0$. If $s \in Z$, then $(\xi + \xi^*) ab \in Z$; consequently $ab \in Z$, which is impossible. One must conclude that $s \notin Z$. By Proposition 2, there exists a positive integer k such that $s^{2^k} \in Z$, so that $(\xi + \xi^*)^{2^k} ab \in Z$. Since $\xi + \xi^* \neq 0$, we see that $(\xi + \xi^*)^{2^k} \neq 0$. Consequently, $ab \in Z$, which is again impossible. This shows that $\xi + \xi^* = 0$, that is, $\xi = \xi^*$.

PROPOSITION 6. *If* $S \not\subseteq Z$, *there is a separable* **-closed subfield* $M \supsetneqq Z$.

Proof. Let a and b be defined as in Proposition 4, and let $x = ab$. We see that $x^* = ba$ and $xx^* = x^*x$. By Proposition 2, a and b are purely inseparable over Z, and therefore there exists a positive integer k such that $a^{2^k} \in Z$ and $b^{2^k} \in Z$. It follows that

$$(xx^*)^{2^k} \in Z \quad \text{and} \quad x^{2^k} + (x^*)^{2^k} = 1 \in Z.$$

We may therefore assume that R contains an element x with $x + x^* = 1$ and $xx^* \in Z$. From the equation $x^2 + (x + x^*) + xx^* = 0$, which has two distinct roots x and x^*, it follows that x is separable over Z but $x \notin Z$.

PROPOSITION 7. *If* $S \not\subseteq Z$, *then the centralizer* Γ *of any subfield* M *of* R, *which is maximal with respect to the properties of being separable over* Z *and* *-*closed, is a commutative subfield.*

Proof. Clearly, Γ is a *-closed subdivision ring containing M in its center; therefore we can regard it as a division ring with involution. If $S \cap \Gamma$ were not contained in Z, then the involution on Γ would be of the first kind (Proposition 5). Hence each element of M would be fixed by the involution and $M \subseteq S$. Since M is separable, $M \subseteq Z$ (Proposition 2), which is in conflict with Proposition 6. One must conclude that $S \cap \Gamma \subseteq Z$. From this it follows that $yy^* = y^*y$ for each $y \in \Gamma$. Let $y \in \Gamma$ with $y \neq y^*$ and $y \notin M$. By maximality, $M(y) = M$. One must conclude that the difference set $\Gamma - M$ consists entirely of symmetric elements. By the hypothesis, $x^{n(x)} \in Z \subseteq M$ for each $x \in \Gamma - M$. By Kaplansky's result, Γ is commutative.

PROPOSITION 8. *If* $S \not\subseteq Z$, R *contains maximal subfields that are* 2-*dimensional over the center.*

Proof. By Proposition 6, R contains a subfield M that is maximal with respect to the properties of being separable and *-closed. By Proposition 2, $M \cap S \subseteq Z$. Consequently, M is a quadratic extension of Z. Since M is separable, it must be of dimension 2 over Z. By [5, Theorem 4.3.2], the centralizer Γ of M has the center M. By Proposition 7, $\Gamma = M$; that is, M is a maximal subfield.

THEOREM 2. *If* R *is a division ring with involution, the following conditions are equivalent:*

(i) *each symmetric element* s *satisfies the condition* $s^r - s^{r+1}p(s) \in Z$;

(ii) *each symmetric element is power-central;*

(iii) *for each* $x \in R$, $x + x^*$ *and* xx^* *are central elements.*

The conditions imply that R *must be commutative or* 4*-dimensional over its center.*

Proof. (iii) ⇒ (ii). In fact, $s^2 = ss^* \in Z$, for each $s \in S$.

(ii) ⇔ (i). By Proposition 3, (ii) ⇒ (i). In view of Theorem 1, we may assume that the characteristic of R is equal to 2 and that $S \not\subseteq Z$. By Proposition 8, R contains a maximal subfield M that is 2-dimensional over Z. Since R is a right-Artinian algebra over Z, and since M is a finite-dimensional algebra over Z, it follows that $R \otimes_Z M$ is right Artinian [see 8, p. 116, footnote 2]. Because the latter ring is a primitive ring of linear transformations of R regarded as a left vector space over M, it follows that R is finite-dimensional over M, and consequently R is finite-dimensional. Therefore R is 4-dimensional. By Proposition 4, R contains a subdivision ring $A = Z(a, b)$ that is 4-dimensional over its center $C = Z(a^2, b^2)$. It follows that $R = A$ and $Z = C$. Writing

$$x = \xi_0 1 + \xi_1 a + \xi_2 b + \xi_3 ab$$

and using the property that the involution is of the first kind (Proposition 5), we verify that $x + x^* \in Z$ and that

$$xx^* = \xi_0\xi_3 + \xi_0^2 + \xi_1^2 a^2 + \xi_1\xi_2 + \xi_2^2 b^2 + \xi_3^2 a^2 b^2 \in Z \qquad \text{for all } x \in R.$$

From the relations $xx^* \in Z$ and $x + x^* \in Z$, it follows at once that R is either commutative or 4-dimensional.

2. *-PRIME RINGS

In this section, we extend Theorems 1 and 2 to rings R with involution and with the following properties:

(3) R is a *-prime ring; that is, nonzero two-sided ideals of R that are closed under the involution (*symmetric* ideals) have nonzero product;

(4) there exists a nonnilpotent symmetric element s_0 in R such that $s_0^{k(I)} \in I$ for each nonzero symmetric ideal I of R.

Hereafter, we assume that R satisfies (3) and (4).

PROPOSITION 9. Z^0 *is a nonempty subsemigroup of regular elements.*

Proof. By (4), there exists an element $s_0 \in S$ that is not nilpotent. Because s_0 is power-central, there exists an $n \geq 1$ such that $s_0^n \in Z - \{0\}$ and consequently $s_0^n \in Z^0 = Z^+ - \{0\}$. This shows that Z^0 is not empty. Next, let $s \in Z^0$. Let $x \in R$ with $sx = 0$. Because $s \in Z^+$, we see that $(s)(x, x^*) = 0$, where (s) is the ideal generated by s, and where (x, x^*) is the ideal generated by x and x^*. Since these ideals are symmetric, condition (3) above implies $x = 0$. This shows that s is regular. Hence Z^0 is closed under multiplication.

Evidently, the subset Z^0 has the Ore property; therefore one can form the partial ring of fractions

$$Q = R(Z^0)^{-1} = \left\{ \frac{x}{z} \,\middle|\, x \in R,\ z \in Z^0 \right\},$$

upon which one defines the involution $\left(\frac{x}{z}\right)^* = \frac{x^*}{z}$.

PROPOSITION 10. (i) Q *is the total ring of fractions of* R.

(ii) *Each symmetric element of* Q *is nilpotent or invertible.*

(iii) Q *is semisimple.*

Proof. (i) Let x be a regular element, and let $s = xx^*$. This is again a regular element. Consequently, $s^n \in Z^0$, and therefore s^n is invertible in Q. Since $s^n = (xx^*)^n = x(x^*x)^{n-1}x^*$, it follows that x is left-invertible in Q. Similarly, x is right-invertible.

(ii) Let $u \in S(Q)$. There exist elements $a \in R$ and $z \in Z^0$ such that $u = a/z = a^*/z$. From this it follows that $a = a^* \in S(R)$. If a is nilpotent, then a/z is nilpotent for $z^{-1} \in Z(Q)$. If a is not nilpotent, then $a^{n(a)} \in Z^0$, and consequently a is invertible in Q. Then $u = a/z$ is invertible.

(iii) Let J be the Jacobson radical of Q. Let J_0 be the restriction to R. Since J is symmetric, J_0 is symmetric. If $J \neq 0$, then $J_0 \neq 0$. By (4), $s_0^k \in J_0$, with $s_0^n \in Z^0$. Therefore $s_0^k \in J$, where s_0 is an invertible element of Q. It follows that $Q = J$. But Q has a unit and cannot be radical. We conclude that Q is semisimple.

J. M. Osborn [12, Theorem 2] has shown that every 2-torsion-free ring Q with properties (ii) and (iii) in Proposition 10 is of one of the following types:

(5) a division ring;

(6) a direct product of a division ring by itself with the interchanging involution;

(7) a 2-by-2 matrix over a 2-torsion-free field with involution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

More recently, K. McCrimmon [11, Theorem 2] showed that rings with characteristic 2 and properties (ii) and (iii) are subject to the same classification (with the obvious rectification on the characteristic of the ground field in case (7)). These results lead immediately to the following theorem.

THEOREM 3. *Let* R *be a* *-*prime ring whose symmetric elements are power-central. Suppose that* R *has at least one nonnilpotent symmetric element* s_0 *such that* $s_0^{k(I)} \in I$ *for each symmetric ideal* $I \neq 0$ *of* R. *Then, for each* $x \in R$, *the elements* xx^* *and* $x + x^*$ *are central.*

Proof. Let Q be the partial ring of fractions of R with respect to Z^0 (Proposition 9). It suffices to prove the theorem for Q. By Osborn's and McCrimmon's results, Q is of one of the types (5), (6), and (7). In the last case, the result is obvious. In case (6), write $Q = D \oplus D^*$, where D is a division ring. For each $x \in D$, set $s = s(x) = x + x^* \in S(Q)$. Now, each symmetric element s of Q is of the form $s = a/z$, where $a = a^*$ and $z^{-1} \in Z(Q)$. Since $a^n \in Z \subseteq Z(Q)$ for some $n \geq 1$,

$$s^n = \frac{a^n}{z^n} \in Z(Q) = Z(D) \oplus Z(D^*).$$

From this it follows that $x^n \in Z(D)$. By Kaplansky's result, $D = Z(D)$. This implies that $Q = Z(Q)$. In case (5), the theorem follows at once from Theorem 2.

3. RINGS WITH NO NIL RIGHT IDEALS

If R is a subdirect product of rings R_ι with induced involutions such that for each $s_\iota \in R_\iota$, the elements $x_\iota x_\iota^*$ and $x_\iota + x_\iota^*$ belong to $Z(R_\iota)$ for all indices ι, then obviously xx^* and $x + x^*$ belong to $Z(R)$ for all $x \in R$. In order to prove Theorem 4, it is therefore sufficient to find at least one subdirect representation of the ring R all of whose factors R_ι have properties (3) and (4) in Theorem 3.

PROPOSITION 11. *If* R *is a ring with involution, then there exists a symmetric ideal* N *such that*

(i) $N \cap S$ *is nil,*

(ii) *the factor ring* R/N, *equipped with the induced involution* $x + N \rightarrow x^* + N$, *is a subdirect product of rings with properties* (3) *and* (4).

Proof. For each nonnilpotent symmetric element u of R, define the subset

$$A(u) = \{u^k \mid k = 1, 2, \cdots\}.$$

Clearly, A(u) is a multiplicative system excluding 0. By Zorn's Lemma, there exists a symmetric ideal $P = P(u)$, maximal with respect to the exclusion of A(u). A routine inspection shows that the factor ring $R(u) = R/P(u)$ is a *-prime ring having properties (3) and (4) with respect to the symmetric element $s_0 = u + P(u)$. Let

$$N = \bigcap \{P(u) \mid u \text{ is a nonnilpotent symmetric element}\}.$$

This set is a symmetric ideal such that $N \cap S$ is nil (by construction).

Does the hypothesis in Theorem 4 imply that $N = 0$? This, in conjunction with Proposition 11, would at once imply Theorem 4. In fact, we shall prove the following result.

PROPOSITION 12. (i) *If* R *has no nil right ideals* (*except* 0), *then* $N = 0$.

(ii) *If* R *has no nil ideals, and if the integers* n(s) *in the relations* $s^{n(s)} \in Z$ *have a finite maximum, then* $N = 0$.

Proof. (i) Let $s \in N \cap S$ and $s^2 = 0$. Let $d = sx + x^*s \in N \cap S$, for an arbitrary $x \in R$. There is an n such that

$$d^n = 0 = (sx)^n + (x^*s)^n + s\,y_n\,s$$

for some $y_n \in R$. From this it follows that $0 = d^n s = (sx)^n s$, and consequently $(sx)^{n+1} = 0$. This implies that the right ideal generated by s is nil. By hypothesis, $s = 0$. It follows that $N \cap S = 0$. Thus $x + x^* = xx^* = 0$ for each $x \in N$. Therefore $x^2 = 0$ for all $x \in N$. By a result of J. Levitzki (see [7, Lemma 1]), either $N = 0$ or R contains a nil ideal different from 0. We conclude that $N = 0$.

(ii) By the argument above, if $N \cap S \neq 0$, there exists an $s \in N \cap S$, with $s^2 = 0$ and $s \neq 0$, such that the right ideal generated by s is nil of finite index. Again, by Levitzki's result, R would contain a nil ideal, and therefore $N \cap S = 0$. As above, this implies that $N = 0$. We have now proved our main result.

THEOREM 4. *Let* R *be a ring with involution and with center* Z, *and suppose that* (1) *each symmetric element* s *of* R *is power-central.*

(i) *If the only nil right ideal of* R *is* 0, *then for each* $x \in R$, xx^* *and* $x + x^*$ *are central elements.*

(ii) *If* R *has no nil ideal, and if the integers* n(s) *in the relations* $s^{n(s)} \in Z$ *have a finite maximum as* s *ranges over the symmetric elements, then* xx^* *and* $x + x^*$ *are central, for each* $x \in R$.

COROLLARY 1. *If* R *is a* 2-*torsion-free ring having no nil right ideal in which each symmetric element is power-central, then each symmetric element of* R *is central.*

Proof. By Theorem 4, $2s \in Z$, for each $s \in S$. This implies that

$$2sx - 2xs = (2s)x - x(2s) = 0,$$

and consequently $2(sx - xs) = 0$ for all $x \in R$. Since R is 2-torsion-free, $sx - xs = 0$. Therefore $s \in Z$ for all $s \in S$.

COROLLARY 2. *A ring* R *satisfying the hypotheses in Theorem* 4 *is a quadratic integral extension of its center with a standard identity of degree at most* 4.

COROLLARY 3. *A semisimple ring* R *satisfying the hypotheses in Theorem* 4 *is a subdirect product of rings of one of the following types:*

(i) *fields;*

(ii) 4-*dimensional division rings;*

(iii) *the product of a field by itself;*

(iv) 2-*by*-2 *matrices over a field.*

4. AN EXAMPLE AND OPEN QUESTIONS

Example (a 4-dimensional division ring R as described in Theorem 2). Let $F = Z_2(s, t)$ be the field of rational functions in two commuting variables s and t, with coefficients in the field Z_2 of integers modulo 2. Let $K = F(y)$ be the simple extension of F obtained by adjoining an indeterminate y subject to the condition $y^2 - y = t$. Let $R = K \oplus K\times$ ($\times$ is a multiplicative symbol), and for $u, v, c, d \in K$, define

$$(u + v\times)(c + d\times) = (uc + s(1 + d)b) + (du + b(1 + c)\times).$$

Let $a = \times$, $b = s^{-1}(y\times)$. Then $ab + ba = 1$. Thus R is a division ring with the basis $\{1, a, b, ab\}$ and center $Z = F$. The map

$$v = \xi_0 1 + \xi_1 a + \xi_2 b = \xi_3 ab \to v^* = v + \xi_3 \qquad (\xi_\iota \in Z(R))$$

turns Q into a quaternion algebra with an involution of the first kind. Here $S \not\subseteq Z$, but each norm and each trace is central.

Concerning part (i) of Theorem 4, K. McCrimmon has recently asked the following question [10].

Question 1 (McCrimmon). Let R be a ring with involution with symmetric elements S. If I is a symmetric ideal such that $I \cap S$ is nil, is I always nil? This question is equivalent with the following.

Question 2. Let R be a ring without nil ideals. If each symmetric element is power-central, is each norm and each trace central?

To see that Questions 1 and 2 are equivalent, one can proceed as follows. Assuming an affirmative answer to Question 1, one can choose the ideal N in Proposition 11 to be a nil ideal. As we saw earlier, Theorem 3 would immediately give an affirmative answer to Question 2. Conversely, let R be a ring in which S is nil. Let N be the nil radical of R, and let R_1 be the factor ring R/N. By construction, R_1 has no nil ideal. If s_1 is a norm or a trace of R_1, there is, respectively, a norm or a trace s in R that is mapped back on s_1. Because s is nilpotent, s_1 must also be nilpotent, and consequently s_1 is power-central. By the assumption, s_1 is central. Since s_1 is nilpotent and R_1 has no nil ideal, $s_1 = 0$. From this it follows that $xx^* = x + x^* = 0$ for all $x \in R_1$. Since $x = -x^*$, x^2 is symmetric, and $x^4 = 0$ for all $x \in R_1$. If now $N \neq R$, then R_1 contains a nil ideal different from $\{0\}$ (Levitzki's result). One must conclude that $R = N$, and therefore R is nil. Thus a more accurate generalization of Herstein's theorem (no boundedness assumption when there are no nil ideals) is equivalent with an affirmative answer to McCrimmon's question. As McCrimmon has pointed out, an affirmative answer to his question would follow from an affirmative answer to a question of A. Kurosch [10].

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