THE DISTRIBUTION OF VALUES OF MULTIPLICATIVE FUNCTIONS

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1. INTRODUCTION. E. Landau [1] has shown that, as $x \to \infty$, the number of positive integers not exceeding x which are representable as the sum of two squares is asymptotic to

$$Bx(\log x)^{-1/2}, \tag{1}$$

where

$$B = \left(2^{-1} \prod_{p\equiv 3 (\mathrm{mod}\ 4)} \frac{1}{1-p^{-2}}\right)^{1/2}.$$

P. Lévy [2] gave a simple heuristic derivation of (1) without determining B; his argument also led him to the conjecture that, if $r_2(m)$ is the number of representations of m as the sum of the squares of two positive integers, and $R_k(x)$ is the number of $m \leq x$ for which $r_2(m) = k$, then

$$R_k(x) \sim \frac{Bx}{\log^{1/2} x} \cdot \frac{e^{-\theta}\theta^k}{k!}, \quad \text{where } \theta = c \log^{1/2} x.$$

In probabilistic terms, this means roughly that, out of the integers m for which $r_2(m) > 0$, those for which $r_2(m)$ has a specified value have a Poisson distribution, with parameter θ.

It will be shown here that this is not the case, and that in fact the asymptotic behavior of $R_k(x)$ depends rather strongly on the arithmetic structure of k, as well as on its size. This is not very surprising, since r_2, being a multiplicative function, must be considered in probability language as a product of random variables, while the usual theory applies to sums of random variables. Thus A. Wintner [3] has shown that if f is an additive function [so that $f(mn) = f(m) + f(n)$ whenever $(m, n) = 1$] with the property that $f(p) = 1$ and $f(p^\alpha) > 0$ for all primes p and all $\alpha > 1$, then the number of solutions of $f(m) = k$ which do not exceed x is asymptotic to

$$\frac{x(\log\log x)^{k-1}}{(k-1)!\log x};$$

this is a "Poisson distribution" with parameter $\log\log x$. In particular, if $\omega(m)$ is the total number of prime divisors of m, $\tau(m)$ is the number of divisors of m, and $f(m) = \omega(\tau(m))$, then f satisfies Wintner's hypotheses, so that the integers m for which $\tau(m)$ has a specified number of prime factors are Poisson distributed; that this is not true of the m for which $\tau(m)$ has a specified value is shown in §2.

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It is well known that, with the definition of $r_2(m)$ above, the equation $r_2(m) = \tau(m')$ holds if $r_2(m) > 0$, where m' is the factor of m consisting of all the primes of the form $4t + 1$ which divide m. Because of this close relation, and because of the independent interest in the τ function, §2 is devoted to an investigation of the asymptotic behavior of the number $T_k(x)$ of integers m not exceeding x for which $\tau(m) = k$. In §3, the estimation of $R_k(x)$ is effected.

2. THE ESTIMATION OF $T_k(x)$. It is clear that, for each k, the multiplicative structure of the numbers m for which $\tau(m) = k$ is determined by that of k. Thus, if $k = 2$, then $m = p$; if $k = 3$, $m = p^2$; if $k = 4$, $m = pq$ or $m = p^3$, and so forth. In general, if $\tau(m) = k$, then m is an integer of the form $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, where $p_1, \cdots, p_r$ are distinct primes and $(\alpha_1 + 1) \cdots (\alpha_r + 1)$ is a factorization of k. Thus

(2) $$T_k(x) = \sum_k{}' \, T(x|\alpha_1, \cdots, \alpha_r),$$

where $T(x|\alpha_1, \cdots, \alpha_r)$ is the number of integers $m \leq x$ of the form $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, the primes $p_1, \cdots, p_r$ being distinct; here (and hereafter)

$$\sum_k{}' \, F(\alpha_\mu, \cdots, \alpha_\nu)$$

denotes a summation over all sets $\alpha_\mu, \cdots, \alpha_\nu$ of positive integers such that

$$(\alpha_\mu + 1) \cdots (\alpha_\nu + 1) = k, \quad \alpha_\mu \leq \cdots \leq \alpha_\nu.$$

THEOREM 1. *If* $\alpha_1 = \cdots = \alpha_\nu < \alpha_{\nu+1} \leq \cdots \leq \alpha_r$, *then*

(3) $$T(x|\alpha_1, \cdots, \alpha_r) \sim \frac{A_1}{(\nu - 1)!} \, \frac{x^{1/\alpha_1} \log_2^{\nu-1} x}{\log x}$$

as $x \to \infty$, *where* $\log_2 x = \log\log x$ *and*

$$A_1 = A_1(\alpha_1, \cdots, \alpha_r) = \alpha_1 \sum{}'' \left(p_{\nu+1}^{-\alpha_{\nu+1}} \cdots p_r^{-\alpha_r}\right)^{1/\alpha_1},$$

the summation Σ'' *being over all sets of distinct primes* $p_{\nu+1}, \cdots, p_r$ *which yield distinct terms of the form written. Hence, if*

$$k = P_1^{\nu_1} \cdots P_s^{\nu_s} = P_1^{\nu_1} k_1,$$

where the P_i *are primes with* $P_1 < P_2 < \cdots < P_s$, *then*

(4) $$T_k(x) \sim \frac{A_2}{(\nu_1 - 1)!} \cdot \frac{x^{(P_1 - 1)^{-1}} \log_2^{\nu_1 - 1} x}{\log x},$$

where

$$A_2 = \sum_{k_1}{}' A_1(\alpha_{\nu_1+1}, \cdots, \alpha_r), \quad \alpha_1 = P_1 - 1.$$

It is clear from (3) that the dominant terms in the sum occurring in (2) are those in which, first, α_1 is minimal, and second, ν is maximal; that is, those corresponding to the factorization

$$k = \underbrace{P_1 \cdots P_1}_{\nu_1} (\alpha_{\nu_1+1} + 1) \cdots (\alpha_r + 1)$$

of k in which $\alpha_1 = \cdots = \alpha_{\nu_1} = P_1 - 1$. This gives (4).

For a set of α's as described in the hypothesis, put

$$a_n = \begin{cases} 1 & \text{if } n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \text{ for some distinct primes } p_1, \cdots, p_r, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$T(x|\alpha_1, \cdots, \alpha_r) = \sum_{n \leq x} a_n.$$

Put

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} = \sum{}'' p_1^{-\alpha_1 s} \cdots p_r^{-\alpha_r s}$$

and

$$P(s) = \sum_p p^{-s};$$

then $f(s)$ can be expressed as a polynomial in

$$P(s), P(2s), \cdots, P((\alpha_1 + \cdots + \alpha_r)s).$$

For example, if $r = 1$, then $f(s) = P(\alpha_1 s)$, while if $r = 2$, $\alpha_1 = 1$, and $\alpha_2 = 2$, then $f(s) = P(s)P(2s) - P(3s)$. In general, this polynomial has rational coefficients, and it does not involve any $P(\alpha s)$ with $\alpha < \alpha_1$. We write

$$(5) \qquad f(s) = h_0 (P_1(\alpha_1 s))^{\nu} + h_1(P_1(\alpha_1 s))^{\nu-1} + \cdots + h_{\nu},$$

where now the coefficients h_μ are polynomials in various functions $P(\kappa_i s)$ with each $\kappa_i > \alpha_1$. In particular, it is easily seen that

$$(6) \qquad h_0 = \frac{1}{\nu!} \sum{}'' p_{\nu+1}^{-\alpha_{\nu+1} s} \cdots p_r^{-\alpha_r s},$$

and this sum can be written in the desired form.

Let $G(\alpha)$ be the region of complex numbers $s = \sigma + it$ such that

$$\sigma \geq \alpha^{-1}(1 - a/\log^b \alpha t) \qquad \text{for } t \geq c/\alpha,$$

$$\sigma \geq \alpha^{-1}(1 - a/\log^b c) = \theta_\alpha \quad \text{for } |t| \leq c/\alpha,$$

$$\sigma \geq \alpha^{-1}(1 - a/\log^b (-\alpha t)) \quad \text{for } t \leq - c/\alpha.$$

It is well known (see, for example, [4], p. 179) that, for suitable positive numbers a, b, and c, the zeta-function is regular in $G(1)$ except for a pole of order 1 at $s = 1$, and that $\zeta(s) \neq 0$ for all $s \in G(1)$. Hence if $G'(\alpha)$ results from $G(\alpha)$ by making a linear cut from θ_α to $1/\alpha$, then $\log \zeta(\alpha s)$ is regular in $G'(\alpha)$, except at $s = \alpha^{-1}$, and $\log \zeta(\alpha s) - \log(\alpha s - 1)^{-1}$ is regular throughout $G(\alpha)$. It is known also ([4], p. 239) that, for

$$|t| \geq c/\alpha \quad \text{and} \quad \alpha\sigma \geq 1 - a(\log^b |\alpha t|)^{-1},$$

the inequality

$$|\log \zeta(\alpha s)| < d' \log^b |\alpha t|$$

holds, when the constant d' is sufficiently large.

We have

$$\log \zeta(\alpha s) - P(\alpha s) = \sum_{\substack{m,p \\ m \geq 2}} 1/mp^{\alpha m s},$$

and this series is absolutely convergent for $\sigma > (2\alpha)^{-1}$. Hence for $a < 1/2$ the function $P(\alpha s) - \log(\alpha s - 1)^{-1}$ is regular in $G(\alpha)$, $P(\alpha s)$ is regular in $G'(\alpha)$ excluding $s = \alpha^{-1}$, and

$$|P(\alpha s)| < d'' \log^b |\alpha t|$$

for $\alpha |t| \geq c$ and $\alpha\sigma \geq 1 - a(\log^b |\alpha t|)^{-1}$.

For a sufficiently small and $\alpha > \alpha_1$, the pole of $P(\alpha s)$ lies outside $G(\alpha_1)$; we choose a small enough to meet all requirements so far stated. Then the coefficients h_μ in (5) are regular throughout $G(\alpha_1) = G$. Hence $f(s)$ is regular in G' excluding $s = \alpha_1^{-1}$; for

$$|\alpha_1 t| \geq c \quad \text{and} \quad \alpha_1\sigma \geq 1 - a/\log^b |\alpha_1 t|,$$

the inequality

$$|f(s)| < d \log^b |t|$$

holds; and there exist functions $\phi_1, \cdots, \phi_\nu$, regular in G, such that the function

$$f(s) - \phi_1(s)\log(\alpha_1 s - 1)^{-1} - \cdots - \phi_\nu(s) \log^\nu (\alpha_1 s - 1)^{-1} \tag{7}$$

is regular in G. [In particular, $\phi_\nu(s) = h_0$, as given by (6).] By a well-known theorem ([4], pp. 183-185),

$$2\pi i \sum_{n\le x} a_n \log x/n = \int_{2-i\infty}^{2+i\infty} x^s s^{-2} f(s)\,ds,$$

the path of integration being the line $\sigma = 2$. By a standard argument (*loc. cit.* pp. 186, 240), this can be replaced by

$$2\pi i\, S(x) = -\int_{\theta}^{\beta} x^s s^{-2} f(s)\,ds - \int_{\beta}^{\theta} x^s s^{-2} f(s)\,ds + O(x^{\beta}/\log^m x),$$

where $\theta = \theta_{\alpha_1}$ and $\beta = \alpha_1^{-1}$, the path of the first integral is the upper edge of the cut, that of the second is the lower, m is arbitrary, and

$$S(x) = \sum_{n\le x} a_n \log x/n.$$

Now let $\delta = \delta(x)$ be a monotone decreasing function which approaches zero as $x \to \infty$; later it will be chosen precisely. We have

$$\int_{x}^{x+\delta x} T(u\,|\alpha_1, \cdots, \alpha_r)u^{-1}\,du = \int_{x}^{x+\delta x} \sum_{n\le u} a_n u^{-1}\,du$$

$$= \sum_{n\le x+\delta x} a_n \int_{n}^{x+\delta x} u^{-1}\,du - \sum_{n\le x} a_n \int_{n}^{x} u^{-1}\,du$$

$$= \sum_{n\le x} a_n \int_{x}^{x+\delta x} u^{-1}\,du + \sum_{x}^{x+\delta x} a_n \int_{n}^{x+\delta x} u^{-1}\,du$$

$$= \sum_{n\le x} a_n \log(1+\delta) + \sum_{x}^{x+\delta x} a_n \log\frac{x+\delta x}{n}$$

$$= \log(1+\delta)\sum_{n\le x} a_n + \sum_{n\le x+\delta x} a_n \log\frac{x+\delta x}{n} - \sum_{n\le x} a_n \log\left(\frac{x+\delta x}{n}\cdot\frac{x}{x}\right)$$

$$= \log(1+\delta)\sum_{n\le x} a_n + S(x+\delta x) - S(x) - \log(1+\delta)\sum_{n\le x} a_n$$

$$= S(x+\delta x) - S(x).$$

But

$$\int_x^{x+\delta x} T(u|\alpha_1, \cdots, \alpha_r)u^{-1}\,du \leq T(x+\delta x|\alpha_1, \cdots, \alpha_r)\log(1+\delta)$$

$$= T(x+\delta x|\alpha_1, \cdots, \alpha_r)(\delta + o(\delta)),$$

and

$$\int_x^{x+\delta x} T(u|\alpha_1, \cdots, \alpha_r)u^{-1}\,du \geq T(x|\alpha_1, \cdots, \alpha_r)(\delta + o(\delta)).$$

Hence

$$\limsup_{x\to\infty} \frac{T(x|\alpha_1, \cdots, \alpha_r)}{\{S(x+\delta x) - S(x)\}/\delta} \leq 1 \tag{8}$$

and

$$\liminf_{x\to\infty} \frac{T(x+\delta x|\alpha_1, \cdots, \alpha_r)}{\{S(x+\delta x) - S(x)\}/\delta} \geq 1, \tag{9}$$

and we need an asymptotic expression for $S(x+\delta x) - S(x)$. We have

$$2\pi i\{S(x+\delta x) - S(x)\}$$

$$= -\left(\int_\theta^\beta + \int_\beta^\theta\right)(x+\delta x)^s s^{-2} f(s)\,ds - \left(\int_\theta^\beta + \int_\beta^\theta\right) x^s s^{-2} f(s)\,ds + O(x^\beta/\log^4 x)$$

$$= -\int_\theta^\beta \rho(x, \delta, s)s^{-2}f(s)\,ds - \int_\beta^\theta \rho(x, \delta, s)s^{-2}f(s)\,ds + O(x^\beta/\log^4 x),$$

where $\rho(x, \delta, s) = (x+\delta x)^s - x^s$. Using (7), we have

$$2\pi i\{S(x+\delta x) - S(x)\}$$

$$= -\left(\int_\theta^\beta + \int_\beta^\theta\right)\rho(x, \delta, s)s^{-2}\{\phi_\nu(s)\log^\nu(\alpha_1 s - 1)^{-1} + \phi_{\nu-1}(s)\log^{\nu-1}(\alpha_1 s - 1)^{-1}$$

$$+ O(\log^{\nu-2}(\alpha_1 s - 1)^{-1})\}\,ds + O(x^\beta/\log^4 x).$$

In moving across the cut, $\log(\alpha_1 s - 1)^{-1}$ must be replaced by $\log(\alpha_1 s - 1)^{-1} + 2\pi i$, so that

$$2\pi i\{S(x+\delta x) - S(x)\}$$

$$= \int_\theta^\beta \rho(x, \delta, s)s^{-2}\phi_\nu(s)\{(\log(\alpha_1 s - 1)^{-1} + 2\pi i)^\nu - \log^\nu(\alpha_1 s - 1)^{-1}\}\,ds$$

$$+\int_\theta^\beta \rho(x,\ \delta,\ s)s^{-2}\phi_{\nu-1}(s)\{(\log(\alpha_1 s-1)^{-1}+2\pi i)^{\nu-1}-\log^{\nu-1}(\alpha_1 s-1)^{-1}\}ds$$

$$+O\left(\int_\theta^\beta \rho(x,\ \delta,\ s)s^{-2}\log^{\nu-2}(\alpha_1 s-1)^{-1}ds\right)+O(x^\beta/\log^4 x)$$

$$=\nu\int_\theta^\beta \rho(x,\ \delta,\ s)s^{-2}\phi_\nu(s)\log^{\nu-1}(\alpha_1 s-1)^{-1}ds$$

$$+O\left(\int_\theta^\beta \rho(x,\delta,s)s^{-2}\log^{\nu-2}(\alpha_1 s-1)^{-1}ds\right)+O(x^\beta/\log^4 x).$$

Now

$$\int_\theta^\beta \rho(x,\ \delta,\ s)s^{-2}\phi_\nu(s)\log^{\nu-1}(\alpha_1 s-1)^{-1}ds$$

$$=\int_\theta^\beta \rho(x,\ \delta,\ s)s^{-2}\phi_\nu(s)\{\log(1-\alpha_1 s)^{-1}-\pi i\}^{\nu-1}ds$$

$$=\int_\theta^\beta \rho(x,\ \delta,\ s)s^{-2}\phi_\nu(s)\log^{\nu-1}(1-\alpha_1 s)^{-1}ds$$

$$+O\left(\int_\theta^\beta \rho(x,\ \delta,\ s)s^{-2}\log^{\nu-2}(1-\alpha_1 s)^{-1}ds\right),$$

and

$$\int_\theta^\beta \rho(x,\ \delta,\ s)s^{-2}\phi_\nu(s)\log^{\nu-1}(1-\alpha_1 s)^{-1}ds$$

$$=\int_\theta^\beta \rho(x,\ \delta,\ s)\{\beta^{-2}\phi_\nu(\beta)+O(\beta-s)\}\{\log^{\nu-1}(\beta-s)^{-1}+O(\log^{\nu-2}(\beta-s)^{-1})\}ds$$

$$=\alpha_1^2\phi_\nu(\beta)\int_\theta^\beta \rho(x,\ \delta,\ s)\log^{\nu-1}(\beta-s)^{-1}ds+O\left(\int_\theta^\beta \rho(x,\ \delta,\ s)\log^{\nu-2}(\beta-s)^{-1}ds\right).$$

Moreover,

$$\int_\theta^\beta \rho(x,\ \delta,\ s)\log^{\nu-1}(\beta-s)^{-1}ds=\int_0^{\beta-\theta}\rho(x,\ \delta,\ \beta-y)(-\log y)^{\nu-1}dy$$

$$= \int_0^{\beta-\theta} \{(x+\delta x)^{\beta-y} - x^{\beta-y}\} (-\log y)^{\nu-1}\, dy$$

$$= x^\beta \int_0^{\beta-\theta} x^{-y} \{(1+\delta)^{\beta-y} - 1\} (-\log y)^{\nu-1}\, dy$$

$$= x^\beta \int_0^{\beta-\theta} x^{-y} \{(\beta - y)\delta + O(\delta^2)\} (-\log y)^{\nu-1}\, dy$$

$$= \frac{\delta x^\beta}{\alpha_1} \int_0^{\beta-\theta} x^{-y} (-\log y)^{\nu-1} dy - \delta x^\beta \int_0^{\beta-\theta} x^{-y}\, y\, (-\log y)^{\nu-1}\, dy$$

$$+ O\left(\delta^2 x^\beta \int_0^{\beta-\theta} x^{-y} (-\log y)^{\nu-1} dy\right)$$

$$= \frac{\delta x^\beta [1+O(\delta)]}{\alpha_1 \log x} \int_0^{(\beta-\theta)\log x} e^{-z} (\log_2 x - \log z)^{\nu-1} dz$$

$$- \frac{\delta x^\beta}{\log^2 x} \int_0^{(\beta-\theta)\log x} e^{-z}\, z (\log_2 x - \log z)^{\nu-1} dz$$

$$= \frac{\delta x^\beta [1+O(\delta)]}{\alpha_1 \log x} \log_2^{\nu-1} x \int_0^{(\beta-\theta)\log x} e^{-z} dz$$

$$+ O\left(\frac{\delta x^\beta [1+O(\delta)]}{\alpha_1 \log x} \log_2^{\nu-2} x \int_0^{(\beta-\theta)\log x} e^{-z} \log z\, dz\right) + O\left(\frac{\delta x^\beta \log_2^{\nu-1} x}{\log^2 x}\right)$$

$$= \frac{\delta x^\beta \log_2^{\nu-1} x}{\log x} + O\left(\frac{\delta^2 x^\beta \log_2^{\nu-1} x}{\log x}\right) + O\left(\frac{\delta x^\beta \log_2^{\nu-2} x}{\log x}\right).$$

It follows that

$$S(x+\delta x) - S(x) = \alpha_1 \nu \phi_\nu(\beta) \frac{\delta x^\beta \log_2^{\nu-1} x}{\log x}$$
$$+ O\left(\frac{\delta^2 x^\beta \log_2^{\nu-1} x}{\log x}\right) + O\left(\frac{\delta x^\beta \log_2^{\nu-2} x}{\log x}\right) + O\left(\frac{x^\beta}{\log^4 x}\right),$$

and for $\delta = \delta(x) = (\log x)^{-2}$ we obtain

$$\frac{S(x+\delta x) - S(x)}{\delta} \sim \alpha_1 \nu \phi_\nu(\beta) \frac{x^\beta \log_2^{\nu-1} x}{\log x}.$$

Here the function on the right is of sufficiently slow growth that it has asymptotically equal values for asymptotically equal arguments; together with the relations (8) and (9), this clearly implies that

$$T(x|\alpha_1, \cdots, \alpha_r) \sim \alpha_1 \nu \phi_\nu(\beta) \frac{x^\beta \log_2^{\nu-1} x}{\log x}$$

$$= \frac{\alpha_1}{(\nu-1)!} \sum{}'' \left(p_{\nu+1}^{\alpha_{\nu+1}} \cdots p_r^{\alpha_r}\right)^{-1/\alpha_1} \frac{x^\beta \log_2^{\nu-1} x}{\log x}.$$

3. THE ESTIMATION OF $R_k(x)$. Throughout this section the letters p and q will be used exclusively to designate primes congruent to 1 and 3 (mod 4), respectively. To emphasize this, the symbols "(1)" and "(3)" will be adjoined to summation and product symbols, when appropriate. We define

$$\zeta_1(s) = \prod_{(1)} (1 - p^{-s})^{-1} \quad \text{for } \Re s > 1,$$

$$\zeta_3(s) = \prod_{(3)} (1 - q^{-s})^{-1} \quad \text{for } \Re s > 1,$$

$$L(s) = \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-s} = \prod_{(1)} (1 - p^{-s})^{-1} \prod_{(3)} (1 + q^{-s}) \quad \text{for } \Re s > 1,$$

$$P_1(s) = \sum_{(1)} p^{-s} \qquad \text{for } \Re s > 1.$$

The series expansion for $L(s)$ converges for $\Re s > 0$, so that it provides an analytic continuation of $L(s)$ over this latter region; $L(s)$ is therefore regular for $\Re s > 0$. Moreover, it is known ([4], pp. 462-466) that $L(s)$ has all the properties asserted for $\zeta(s)$ in the paragraph following (6), in the region $G(1)$, except that it has no pole at $s = 1$.

As noted earlier, if $n = 2^\alpha n'n''$, where $n' = \prod p_i^{\alpha_i}$, $n'' = \prod q_i^{\beta_i}$, then

$$r_2(n) = \begin{cases} \tau(n') & \text{if } n'' \text{ is a square.} \\ 0 & \text{otherwise.} \end{cases}$$

Thus if we again let $(\alpha_1 + 1) \cdots (\alpha_r + 1)$ be a factorization of k in which

$$\alpha_1 = \cdots = \alpha_\nu < \alpha_{\nu+1} \leq \cdots \leq \alpha_r,$$

and put

$$b_n = \begin{cases} 1 & \text{if } n' = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \text{ with } p_i \neq p_j, \text{ and } n'' \text{ is a square,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g(s) = \sum_{n=1}^{\infty} b_n n^{-s},$$

then $g(s)$ is the generating function of those solutions of the equation $r_2(n) = k$ which correspond to the given factorization of k. We have

$$\begin{aligned} g(s) &= \sum_{(1)}{}'' p_1^{-\alpha_1 s} \cdots p_r^{-\alpha_r s} (1 + 2^{-s} + 2^{-2s} + \cdots) \cdot \prod_{(3)} (1 + q^{-2s} + q^{-4s} + \cdots) \\ &= \sum_{(1)}{}'' p_1^{-\alpha_1 s} \cdots p_r^{-\alpha_r s} (1 - 2^{-s})^{-1} \cdot \prod_{(3)} (1 - q^{-2s})^{-1} \\ &= \sum_{(1)}{}'' p_1^{-\alpha_1 s} \cdots p_r^{-\alpha_r s} (1 - 2^{-s})^{-1} \cdot \zeta_3(2s). \end{aligned}$$

Now

(11) $$\zeta(s) = (1 - 2^{-s})^{-1} \zeta_1(s) \zeta_3(s),$$

so that

$$\zeta(s) L(s) = (1 - 2^{-s})^{-1} \zeta_1^2(s) \zeta_3(2s),$$

and

(12) $$\zeta_1^2(s) = (1 - 2^{-s}) \zeta(s) L(s) / \zeta_3(2s).$$

Thus $\zeta_1(s)$ is regular for $\Re s > 1/2$, except for a branch point at $s = 1$, and by (11), the same is true of $\zeta_3(s)$. Moreover, by the argument used before and by (12), the functions $P_1(s) - \log \zeta_1(s)$ and $P_1(s) - 2^{-1} \log (s - 1)^{-1}$ are regular for $\Re s > 1/2$. Consequently, writing

$$g(s) = \left\{ \frac{1}{\nu!} \sum_{(1)}{}'' p_{\nu+1}^{-\alpha_{\nu+1} s} \cdots p_r^{-\alpha_r s} P_1^{\nu}(\alpha_1 s) + O(P_1^{\nu-1}(\alpha_1 s)) \right\} (1 - 2^{-s})^{-1} \zeta_3(2s),$$

we see that the behavior of $g(s)$ depends essentially on the relative sizes of α_1 and 2. Accordingly, we consider separately the cases $\alpha_1 = 1$, $\alpha_1 = 2$, $\alpha_1 \geq 3$. In summary, the result is this:

THEOREM 2. a) *If* k *is even, say* $k = 2^{\nu_1} k_1$ *and* $2 \nmid k_1$, *then*

$$R_k(x) \sim \frac{\zeta_3(2)}{2^{\nu_1 - 1} (\nu_1 - 1)!} \sum_{k_1}{}' \sum_{(1)}{}'' p_{\nu_1+1}^{-\alpha_{\nu_1+1}} \cdots p_r^{-\alpha_r} \cdot \frac{x \log_2^{\nu_1 - 1} x}{\log x}.$$

b) *If* $2 \nmid k$, *and* $3 \nmid k$, *then*

$$R_k(x) \sim \frac{2+\sqrt{2}}{\pi} \zeta_3^{1/2}(2) \sum_k{}' \sum_{(1)}{}'' \left(p_1^{-\alpha_1} \cdots p_r^{-\alpha_r}\right)^{1/2} \cdot \frac{x^{1/2}}{\log^{1/2} x}.$$

c) *If* $2 \nmid k$ *and* $3 \mid k$, *so that* $k = 3^{\nu_1} k_1$, *where* $2 \nmid k_1$ *and* $3 \nmid k_1$, *then*

$$R_k(x) \sim \frac{2(\sqrt{2}+1)}{\pi^{1/2} \nu_1 !} \sum_{k_1}{}' \sum_{(1)}{}'' \left(p_{\nu_1+1}^{-\alpha_{\nu_1+1}} \cdots p_r^{-\alpha_r}\right)^{1/2} \cdot \frac{x^{1/2} \log_2^{\nu_1} x}{\log^{1/2} x}.$$

Case a) k *even*. Here there exist factorizations of k for which $\alpha_1 = 1$, and as was seen in §2, the dominant terms in the estimate for $R_2(x)$ arise from those factorizations in which α_1 is minimal and ν is maximal: $\nu = \nu_1$.

The function $P_1(s) - 2^{-1} \log (s-1)^{-1}$ is regular in G(1) and, for

$$|t| \geq c \quad \text{and} \quad \sigma \geq 1 - a \log^{-b} |t|,$$

we have

$$|P_1(s)| < d \log^b |t|.$$

Thus if we write

$$g(s) = \psi_\nu(s) 2^{-\nu} \log^\nu (s-1)^{-1} + \cdots,$$

where the remaining terms are of lower order at $s = 1$ than the term written, and where

$$\psi_\nu(s) = \frac{\zeta_3(2s)}{\nu!(1-2^{-s})} \sum_{(1)}{}'' p_{\nu+1}^{-\alpha_{\nu+1} s} \cdots p_r^{-\alpha_r s},$$

then the method used in §2 leads directly to a) in the statement of Theorem 2.

Case b) $2 \nmid k$ *and* $3 \nmid k$. Since in this case $\alpha_1 \geq 3$, the last factor in the representation (10) of g(s) now predominates (that is, has the pole with greatest real part). We write $g(s|\alpha_1, \cdots; \alpha_r)$ in place of g(s), and put

$$g_1(s) = \sum_k{}' g(s|\alpha_1, \cdots, \alpha_r) = (1-2^{-s})^{-1} \sum_k{}' \sum_{(1)}{}'' p_1^{-\alpha_1 s} \cdots p_r^{-\alpha_r s} \zeta_3(2s) = \phi(s) \zeta_3(2s).$$

Then $\phi(s)$ is regular for $\sigma > 1/3$, and

$$g_1(s) = \sum_{\substack{n \\ r_2(n)=k}} n^{-s} = \sum_{n=1}^{\infty} c_n n^{-s},$$

where c_n is 1 or 0 according as $r_2(n)$ is or is not equal to k. We again have

$$2\pi i \sum_{n\leq x} c_n \log x/n = \int_{2-i\infty}^{2+i\infty} x^s s^{-2} g_1(s)\,ds.$$

From (11) and (12) it is easily seen that

$$\zeta_3^2(s) = (1 - 2^{-s})\zeta(s)\zeta_3(2s)/L(s),$$

so that ζ_3 has a branch point at $s = 1$:

$$\zeta_3(s) = \left(\frac{1}{2}\frac{\zeta_3(2)}{\pi/4}\right)^{1/2}(s - 1)^{-1/2} + \cdots$$

and

$$\zeta_3(2s) = (\zeta_3(2)/\pi)^{1/2}(s - 1/2)^{-1/2} + \cdots.$$

Hence the function

$$g_1(s) = (\zeta_3(2)/\pi)^{1/2}\phi(1/2)(s - 1/2)^{-1/2} + \cdots$$

has a branch point at $s = 1/2$, but is otherwise regular for $\sigma > 1/3$. The situation is now almost identical with that encountered by Landau in his proof of (1); paralleling the development there, one easily obtains the statement b) in Theorem 2.

Case c) $2 \nmid k$, $3 \mid k$. Here $\alpha_1 = 2$ for some factorizations of k; considering such factorizations, we write

$$g(s) = \sum_{(1)}{}'' p_1^{-2s} \cdots p_{\nu_1}^{-2s} p_{\nu_1+1}^{-\alpha_{\nu_1+1}s} \cdots p_r^{-\alpha_r s} \cdot (1 - 2^{-s})^{-1} \zeta_3(2s)$$

$$= \frac{1}{\nu_1!} \sum_{(1)}{}'' p_{\nu_1+1}^{-\alpha_{\nu_1+1}s} \cdots p_r^{-\alpha_r s} (1 - 2^{-s})^{-1} P_1^{\nu_1}(2s)\zeta_3(2s) + \cdots,$$

where the remaining terms involve lower powers of $P_1(2s)$, where $3 \leq \alpha_{\nu_1+1} \leq \cdots \leq \alpha_r$, and where the function

$$2^{1/2}\chi(s) = \frac{1}{\nu_1!}(1 - 2^{-s})^{-1} \sum_{(1)}{}'' p_{\nu_1+1}^{-\alpha_{\nu_1+1}s} \cdots p_r^{-\alpha_r s}$$

is regular for $\sigma > 1/3$. This time g has the expansion

$$g(s) = \chi(1/2)(s - 1/2)^{-1/2} \log^{\nu_1}(s - 1/2)^{-1} + \cdots,$$

where the remaining terms are of lower order at $s = 1/2$ than the term written. Thus if $\theta = \theta_{1/2}$, then

$$2\pi i\left\{\sum_{n\leq x+\delta x} b_n \log\frac{x+\delta x}{n} - \sum_{n\leq x} b_n \log x/n\right\}$$

$$\sim -\int_{\theta}^{1/2} \rho(x,\delta,s)s^{-2}\,X(1/2)(s-1/2)^{-1/2}\log^{\nu_1}(s-1/2)^{-1}\,ds$$

$$-\int_{1/2}^{\theta} \rho(x,\delta,s)s^{-2}\,X(1/2)(s-1/2)^{-1/2}\log^{\nu_1}(s-1/2)^{-1}ds,$$

where the first integral is taken along the upper edge of the cut, and the second along the lower edge. In going from the lower to the upper edge, $(s - 1/2)^{1/2}$ changes sign, and the logarithm increases its argument by $2\pi i$, so that if we put

$$S_1(x) = \sum_{n\leq x} b_n \log x/n,$$

then

$$S_1(x+\delta x) - S_1(x)$$

$$= -\frac{X(1/2)}{2\pi i}\int_{\theta}^{1/2} \rho(x,\delta,s)s^{-2}(s-1/2)^{-1/2}\{\log^{\nu_1}(s-1/2)^{-1}$$

$$+[\log(s-1/2)^{-1} + 2\pi i]^{\nu_1}\}\,ds$$

$$= \frac{X(1/2)}{2\pi}\int_{\theta}^{1/2} \rho(x,\delta,s)s^{-2}\{[\log(1/2-s)^{-1} - \pi i]^{\nu_1}$$

$$+[\log(1/2-s)^{-1} + \pi i]^{\nu_1}\}\,(1/2-s)^{-1/2}ds$$

$$\sim X(1/2)\pi^{-1}\int_{\theta}^{1/2} \rho(x,\delta,s)s^{-2}\,(1/2-s)^{-1/2}\log^{\nu_1}(1/2-s)^{-1}ds.$$

By means of the reductions of §2, this last expression is easily shown to be asymptotic to

$$4\,X(1/2)x^{1/2}\delta\,\pi^{-1}\int_0^{1/2-\theta} x^{-y}y^{-1/2}(1/2-y)(-\log y)^{\nu_1}\,dy,$$

and this in turn is asymptotic to

$$2\,\pi^{-1}\,\delta\,X(1/2)\,\Gamma(1/2)\cdot\frac{x^{1/2}\log_2^{\nu_1} x}{\log^{1/2} x} = 2\,\pi^{-1/2}\delta\,X(1/2)\cdot\frac{x^{1/2}\log_2^{\nu_1} x}{\log^{1/2} x}.$$

Thus, for suitably chosen $\delta = \delta(x)$, it follows that if $R(x|\alpha_{\nu_1+1}, \cdots, \alpha_r)$ is the number of integers less than or equal to x and of the form

$$(p_1 \cdots p_{\nu_1})^2 p_{\nu_1+1}^{\alpha_{\nu_1+1}} \cdots p_r^{\alpha_r},$$

then

$$R(x|\alpha_{{}_1+1}, \cdots, \alpha_r) \sim \frac{2(2^{1/2}+1)}{\pi^{1/2}\nu_1!} \sum_{(1)}'' \left(p_{\nu_1+1}^{\alpha_{\nu_1+1}} \cdots p_r^{\alpha_r}\right)^{-1/2} \cdot \frac{x^{1/2}\log_2^{\nu_1} x}{\log^{1/2} x}$$

and c) follows in the usual way.

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