ON A CONJECTURE OF ERDÖS, HERZOG, AND PIRANIAN

Gerald R. Mac Lane

Let $P(z)$ be a polynomial whose zeros lie on $|z| = 1$ and such that $|P(0)| = 1$. It is easy to prove that there exists a rectifiable curve C in $|z| < 1$ which joins $z = 0$ to some zero of $P(z)$ on $|z| = 1$ and on which $|P(z)| \leq 1$. In a recent paper Erdös, Herzog, and Piranian [1] have given simple explicit examples of such polynomials for which C has to be crooked, i.e., for which the length of C has to exceed 1. One conjecture made is that there exists a finite constant L such that, for every polynomial of the class considered, there is a corresponding curve C of length at most L. The purpose of the present note is to prove that this is false. It is not difficult to show that for polynomials of a fixed degree n there exists a corresponding finite, best possible constant L_n with the property stated. By enforcing suitable sinuosities in C we shall prove that $L_n \to \infty$.

THEOREM 1. *Let* A *be a compact subset of* $|z| < 1$ *with the following property: there exists a simply-connected domain* Δ *such that* $0 \notin \Delta$ *and* $A \subset \Delta \subset \{|z| < 1\}$.

Then there exists an integer $n_0 = n_0(A)$ *such that for each* $n \geq n_0$ *there is a polynomial* $P_n(z)$ *with the properties:* 1) $P_n(z)$ *is of degree* n *and all of its zeros lie on* $|z| = 1$, 2) $P_n(0) = 1$, *and* 3) $|P_n(z)| > 2$ *for* $z \in A$.

Before proving Theorem 1 we note three different types of sets A, any one of which will serve to prove that $L_n \to \infty$.

Example 1. Let ρ, $0 < \rho \leq 1$, be a constant. Let A be the finite spiral

$$z = \rho(1 - e^{-t})e^{it}, \quad 2\pi \leq t \leq T.$$

From Theorem 1 it follows that $L_n > \rho(1 - e^{-2\pi})(T - 4\pi)$ for $n \geq n_0$. Since T is arbitrary, $L_n \to \infty$.

Example 2. Let N and α be given, where N is a natural number and $0 < \alpha < \pi$. Let $0 < r_1 < r_2 < \cdots < r_N < 1$. Let A consist of the interval $[r_1, r_N]$ of the real axis together with the N arcs

$$|z| = r_n, \quad 0 \leq \arg z \leq 2\pi - \alpha, \qquad 1 \leq n \leq N, \ n \text{ odd},$$

and

$$|z| = r_n, \quad \alpha \leq \arg z \leq 2\pi, \qquad 1 \leq n \leq N, \ n \text{ even}.$$

Example 3. Let N, ρ_1, ρ_2, and h be given, where N is a natural number, $0 < \rho_1 < \rho_2 < 1$, and $0 < 2h < \rho_2 - \rho_1$. Let A consist of the two circular arcs

$$\{|z| = \rho_1, \ \pi/N \leq \arg z \leq 2\pi\} \text{ and } \{|z| = \rho_2, \ 0 \leq \arg z \leq \pi\},$$

the interval $[\rho_1, \rho_2]$ of the real axis, and the N segments

$$\arg z = n\pi/N, \quad \rho_1 \leq |z| \leq \rho_2 - h, \qquad 1 \leq n \leq N, \ n \text{ odd},$$

Received by the editors October 16, 1954.

and

$$\arg z = n\pi/N, \ \rho_1 + h \leq |z| \leq \rho_2, \qquad 1 \leq n \leq N, \ n \text{ even}.$$

All three of these examples show that arbitrarily long parts of a curve C may be forced to lie in an arbitrarily chosen neighborhood of $z = 0$. Thus the growth of L_n is not due solely to the behavior of possible curves C as they approach $|z| = 1$. Similarly, in the construction of Erdös, Herzog and Piranian, C is devious near $z = 0$. Theorem 1 is a simple consequence of the following theorem which, except for one added conclusion, we proved in an earlier paper [2, Theorem IV].

THEOREM A. *Let* D *be a simply-connected domain in the* z*-plane, bounded by a rectifiable Jordan curve* Γ. *Let* $D_1, \cdots, D_q$ *be* q *mutually disjoint simply-connected subdomains of* D, *and let* $f_k(z)$ *be holomorphic and zero-free in* D_k, $1 \leq k \leq q$. *Then there exists a sequence of polynomials* $Q_n(z)$, *with all zeros on* Γ, *such that*

$$\lim_{r \to \infty} Q_n(z) = f_k(z), \qquad z \in D_k,$$

uniformly in each compact subset of D_k, $1 \leq k \leq q$. *Furthermore,* $Q_n(z)$ *is of degree* n.

To derive Theorem 1 from Theorem A we take $|z| < 1$ for D, and $q = 2$. We may assume that $0 \notin \Delta$. Let $D_1 = \Delta$ and let D_2 be a suitably small neighborhood of $z = 0$. Let $f_1(z) \equiv 3$ and $f_2(z) \equiv 1$, and set $P_n(z) = Q_n(z)/Q_n(0)$.

The added conclusion referred to above is the last sentence in Theorem A. To fill this gap we proceed as follows. We have from [2] merely a subsequence $\{Q_{n_i}\}$ of the desired sequence $\{Q_n\}$. Also proved [2, § 5], although not explicitly stated as a theorem, is the following special case. *Let* $g(z)$ *be holomorphic and zero-free in* $D \cup \Gamma$. *Then there exists a sequence of polynomials* $T_n(z)$ *such that* T_n *is of degree* n *with all zeros on* Γ *and* $T_n(z) \to g(z)$ *uniformly in every compact subset of* D. Now let $g(z) \equiv 1$. For $2n_i \leq m \leq 2n_{i+1} - 1$, we define

$$Q_m^*(z) = Q_{n_i}(z) T_{m-n_i}(z).$$

It is clear that the sequence so defined does the job.

Finally, we note that Theorem A allows Theorem 1 and its corollary that $L_n \to \infty$ to be generalized to domains bounded by rectifiable Jordan curves.

REFERENCES

1. P. Erdös, F. Herzog, and G. Piranian, *On polynomials whose zeros lie on the unit circle*, submitted for publication in Duke Math. J.

2. G. R. Mac Lane, *Polynomials with zeros on a rectifiable Jordan curve*, Duke Math. J. 16 (1949), 461-477.

The Rice Institute