GENERALIZED FOURIER INTEGRAL FORMULAS

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1. INTRODUCTION. The solution of certain types of boundary value problems in partial differential equations depends on a representation of a function $f(x)$, prescribed on the semi-infinite interval $x > 0$, in terms of the functions that satisfy the system

$$(1)\qquad \begin{aligned} &y''(x, \lambda) + \lambda^2 y(x, \lambda) = 0 \qquad (x > 0), \\ &hy(0, \lambda) - ky'(0, \lambda) + my''(0, \lambda) = 0, \end{aligned}$$

where $y(x, \lambda)$ is bounded for all x $(x \geq 0)$. Here h, k, and m denote non-negative constants at least one of which is positive, and λ is a paramenter. Since $y''(0, \lambda) = -\lambda^2 y(0, \lambda)$, the boundary condition above involves the parameter explicitly when $m \neq 0$.

The boundary value problem (1) with $m \neq 0$ arises, for example, in problems on diffusion of gases within electrodes of vacuum tubes to the surface where they enter the vacuum chamber. It also arises in problems on transverse displacements of membranes with certain forms of elastic support along an edge. Applications in which $m = 0$ while h and k are positive are treated in references [1], [2, pp. 206-210], and [3].

The characteristic functions of the system (1) are

$$(2)\qquad y(x, \lambda) = \cos[\lambda x - \alpha(\lambda)],$$

where λ is real and positive and where, for $k > 0$, $\alpha(\lambda)$ is defined by the conditions

$$(3)\qquad \tan \alpha(\lambda) = \frac{h - m\lambda^2}{k\lambda} \qquad \left(-\frac{\pi}{2} < \alpha(\lambda) < \frac{\pi}{2}\right).$$

In case $k = 0$ the characteristic functions are $y = \sin \lambda x$ and the representation of the function $f(x)$ is the classical Fourier sine integral formula. When $h = m = 0$, $y = \cos \lambda x$ and the representation is the Fourier cosine integral formula. In these two special cases the generalized formulas found in this paper reduce to those classical formulas. In establishing the generalized representation formula, then, the two cases $k = 0$ and $h = m = 0$ will be excluded from consideration.

2. THE TRIGONOMETRIC FUNCTIONS. Certain properties of functions associated with the characteristic functions (2) will be needed in the following sections. Their introduction here simplifies the presentation of the principal results.

The identities

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$$\cos(\lambda x-\alpha)\cos(\lambda\mu-\alpha) = \cos[\lambda(x+\mu)-\alpha]\cos\alpha + \sin\lambda x\sin\lambda\mu \tag{4}$$
$$= \sin[\lambda(x+\mu)-\alpha]\sin\alpha + \cos\lambda x\cos\lambda\mu$$

can be verified by elementary trigonometry.

When $m > 0$ and $k > 0$, consider the function

$$P(t, r) = \int_0^r \cos[\lambda t-\alpha(\lambda)]\cos\alpha(\lambda)\, d\lambda = \int_0^r y(t,\lambda)y(0,\lambda)\, d\lambda\ . \tag{5}$$

Let $H = h/m$ and $2K = k/m$; then, in view of equation (3),

$$P(t, r) = 2K\int_0^r \frac{2K\lambda^2\cos\lambda t + \lambda(H-\lambda^2)\sin\lambda t}{4K^2\lambda^2 + (H-\lambda^2)^2}\, d\lambda\ . \tag{6}$$

It is to be proved that $P(t, r) \to 0$ as $r \to \infty$, uniformly with respect to t for $t \geq x$, where x is any positive constant.

An alternate form of the integral (6), found by formally writing the Laplace transform of that integral with respect to t, leads to the formula

$$P(t, r) = K\int_t^\infty E(\tau-t)\sin r\tau\frac{d\tau}{\tau}\ , \tag{7}$$

where the function E is defined by the equations

$$BE(t) = e^{-Kt}\left[(K+B)e^{-Bt} - (K-B)e^{Bt}\right],\quad B = (K^2-H)^{1/2}$$

when $H \neq K^2$. When $H = K^2$,

$$E(t) = 2e^{-Kt}(1-Kt).$$

To verify that formulas (6) and (7) represent the same function $P(t, r)$ it is only necessary to show that both give the same function $\partial P/\partial r$ and the same initial value $P(t, 0) = 0$.

If $K^2 > H$ then B is real and $0 < B \leq K$; if $K^2 < H$ then B is a pure imaginary constant. Thus it is seen from the two formulas above for $E(t)$ that, whether B is zero or not, the function $E(t)$ is either a function of the form

$$F(t) = A\, e^{-(a+ib)t}(t+C),$$

where A, a, b, and C are constants, $a > 0$ and real, and b is real, or else $E(t)$ is the sum of two such functions. Now, by a partial integration,

$$\int_t^\infty F(\tau-t)\sin r\tau\frac{d\tau}{\tau} = \frac{1}{r}\left\{\frac{F(0)}{t}\cos rt + \int_t^\infty \cos r\tau\frac{\partial}{\partial\tau}\left[\frac{F(\tau-t)}{\tau}\right]d\tau\right\}.$$

When $t \geq x > 0$, the absolute value of the first term inside the braces does not exceed the constant $|F(0)|x^{-1}$. Since $t/\tau \leq 1$ and $1/t \leq 1/x$, it is easy to see that the absolute value of the integrand of the integral inside the braces does not exceed a constant multiple of $\exp[-a(\tau-t)]$, where the constant is independent of t. Thus the quantity in the braces is bounded uniformly with respect to t, and it follows that

$$|P(t, r)| < \frac{X_1}{r} \qquad (t \geq x > 0) \tag{8}$$

for some constant X_1 independent of t.

For $m = 0$ and $k > 0$ it will now be shown that the function

$$\begin{aligned} Q(t, r) &= \int_0^r \sin[\lambda t - \alpha(\lambda)] \sin \alpha(\lambda)\, d\lambda \\ &= h \int_0^r \frac{k\lambda \sin \lambda t - h \cos \lambda t}{h^2 + k^2\lambda^2}\, d\lambda \end{aligned} \tag{9}$$

tends to zero as $r \to \infty$, uniformly with respect to t for $t \geq x > 0$. By the same procedure used above to obtain formula (7), the following alternate form of the function (9) is found:

$$Q(t, r) = -\frac{h}{k} \int_t^\infty \exp\left[-\frac{h}{k}(\tau - t)\right] \sin r\tau \frac{d\tau}{\tau}. \tag{10}$$

Since the integral here has the same form as the integral in equation (7) when E(t) is replaced by a special case of the function F(t), it follows at once that a constant X_2 exists, independent of t, such that

$$|Q(t, r)| < \frac{X_2}{r} \qquad (t \geq x > 0). \tag{11}$$

3. FORMAL DEVELOPMENT. The linear representation of an arbitrary function f(x) prescribed on the semi-infinite interval, in terms of the characteristic functions $y(x, \lambda)$ where the values of the parameter λ range over all positive numbers, may be written in the form

$$f(x) = \int_0^\infty y(x, \lambda)\, g(\lambda)\, d\lambda. \tag{12}$$

A formal solution of this integral equation for the function $g(\lambda)$ will give the expansion formula that is to be established. The method of solution used here is analogous to the procedure with orthogonal functions when the eigenvalues λ are discrete. It has some advantages over the formal method used by Karush [1] in the special case $m = 0$.

Let both members of equation (12) be multiplied by y(x,t) and integrated with respect to x from 0 to r; then let the order of integration in the resulting iterated integral be changed. This leads to the equation

$$\int_0^\infty f(x) y(x, t)\, dx = \lim_{r \to \infty} \int_0^\infty g(\lambda) \int_0^r y(x, \lambda) y(x, t)\, dx\, d\lambda. \tag{13}$$

From either the Green's formula for the boundary value problem (1) or the direct integration of the product $y(x,\lambda)\ y(x,t)$ of trigonometric functions (2) it follows that, for $k > 0$,

$$(14)\quad 2\int_0^r y(x,\lambda)\, y(x,t)\, dx$$

$$= -\frac{2m}{k}\, y(0,t)\, y(0,\lambda) + \frac{\sin[(\lambda+t)r-\alpha(\lambda)-\alpha(t)]}{\lambda+t} + \frac{\sin[(\lambda-t)r-\alpha(\lambda)+\alpha(t)]}{\lambda-t}.$$

According to the Riemann-Lebesgue theorem in the theory of Fourier integrals,

$$\lim_{r\to\infty}\int_0^\infty G(\lambda)\sin[(\lambda\underline{+}t)r]\, d\lambda = \lim_{r\to\infty}\int_0^\infty G(\lambda)\cos[(\lambda\underline{+}t)r]d\lambda = 0$$

when the function $G(\lambda)$ is sectionally continuous on each finite interval $0 \leq \lambda \leq \lambda_0$ and absolutely integrable on the semi-infinite interval [5, pp. 14-15]. The function $(\lambda-t)^{-1}\sin[\alpha(\lambda)-\alpha(t)]$ that arises in equation (14) has a limit as $\lambda \to t$. Thus if $g(\lambda)$ satisfies the conditions just cited on $G(\lambda)$, it follows from equation (12) that

$$f(0) = \int_0^\infty y(0,\lambda)\, g(\lambda)\, d\lambda$$

and, from equations (13) and (14) and the Riemann-Lebesgue theorem, that

$$(15)\quad 2\int_0^\infty f(x)y(x,t)\, dx = -\frac{2m}{k} f(0)\, y(0,t) + \lim_{r\to\infty}\int_0^\infty g(\lambda)\cos[\alpha(\lambda)-\alpha(t)]\frac{\sin[(\lambda-t)r]}{\lambda-t}\, d\lambda .$$

Now suppose that, in addition to satisfying the conditions of sectional continuity and absolute integrability assumed above, the function $g(\lambda)$ has one-sided derivatives of the first order from the right and from the left at the point $\lambda = t$ $(t > 0)$. If $g(\lambda)$ is discontinuous at the point $\lambda = t$, let $g(t)$ represent the mean value of the limits $g(t+0)$ and $g(t-0)$. Then the limit that appears in equation (15) represents $\pi g(t)$, according to the primitive form of the classical Fourier integral formula or the extension of the Dirichlet integral formula to the unbounded interval [4, p. 90]. Thus

$$(16)\qquad g(t) = \frac{2}{\pi}\int_0^\infty f(x)\, y(x, t)\, dx + \frac{2m}{\pi k} f(0)\, y(0, t) \qquad (t > 0).$$

When $m > 0$, the representation (12) now becomes

$$f(x) = \frac{2}{\pi}\int_0^\infty y(x, \lambda)\int_0^\infty f(\mu)y(\mu, \lambda)d\mu d\lambda + \frac{2m}{\pi k} f(0)\int_0^\infty y(x, \lambda)y(0, \lambda)d\lambda.$$

The last integral on the right vanishes when $x > 0$ since it represents the limit, as $r \to \infty$, of the function $P(x, r)$ defined by equation (5) and satisfying condition (8). When $m = 0$, the final term in equation (16) vanishes. Thus for $m \geq 0$ the expansion formula becomes

$$(17)\qquad f(x) = \frac{2}{\pi}\int_0^\infty y(x, \lambda)\int_0^\infty f(\mu)y(\mu, \lambda)d\mu d\lambda \qquad (x > 0).$$

When the trigonometric form (2) of the function y is substituted here, the formula becomes

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\,[\lambda x-\alpha(\lambda)]\int_0^\infty f(\mu)\cos[\lambda\mu-\alpha(\lambda)]\,d\mu d\lambda \quad (x > 0). \tag{18}$$

This is the principal generalized Fourier integral formula that is to be established. When $h = m = 0$ and $k > 0$, it follows from equation (3) that $\alpha(\lambda) = 0$; then formula (18) reduces to the Fourier cosine integral formula. When $k = 0$, then $\alpha = \pm\,\pi/2$ and the formula reduces to the Fourier sine integral formula.

4. THEOREM 1. *Let* $f(x)$ *denote a function that is sectionally continuous on each finite interval* $0 \le x \le x_0$, *defined at each point of discontinuity* X *as the mean value of the two limits* $f(X+0)$ *and* $f(X-0)$, *and absolutely integrable on the semi-infinite interval* $x > 0$. *Then at each point* x $(x > 0)$ *at which the right- and left-hand derivatives of* $f(x)$ *exist, the generalized Fourier integral formula* (18) *represents the function* $f(x)$.

Under the conditions stated in the theorem the function is represented by its Fourier sine and cosine integral formulas, at the points specified in the theorem. In view of the identities (4) it follows that formula (18) represents $f(x)$ at those points, provided that either

$$\int_0^\infty\int_0^\infty f(\mu)\cos\,[\lambda(x+\mu)-\alpha(\lambda)]\,\cos\alpha(\lambda)\,d\mu\,d\lambda = 0 \tag{19}$$

or

$$\int_0^\infty\int_0^\infty f(\mu)\sin\,[\lambda(x+\mu)-\alpha(\lambda)]\,\sin\alpha(\lambda)\,d\mu\,d\lambda = 0\,. \tag{20}$$

Let each of the outer integrals here, with respect to λ, be written as the limit as $r \to \infty$ of the integral from 0 to r. The conditions on $f(\mu)$ insure the uniform convergence, with respect to λ, of each of the inner integrals and the validity of interchanging the order of integration of the definite and improper integrals. Thus the two conditions can be written

$$\lim_{r\to\infty}\int_0^\infty f(\mu)P(x+\mu, r)\,d\mu = 0 \qquad (x > 0)\,, \tag{21}$$

$$\lim_{r\to\infty}\int_0^\infty f(\mu)Q(x+\mu, r)\,d\mu = 0 \qquad (x > 0)\,, \tag{22}$$

where the functions P and Q are those defined by equations (5) and (9). Theorem 1 is true then if either of the conditions (21) or (22) is satisfied.

When $m > 0$ and $k > 0$ the function P satisfies condition (8). Hence for each fixed x $(x > 0)$ a constant X_1 exists such that

$$\left|\int_0^\infty f(\mu)P(x+\mu, r)\,d\mu\right| < \frac{X_1}{r}\int_0^\infty |f(\mu)|\,d\mu,$$

and from the absolute integrability of $f(\mu)$ it follows that condition (21) is satisfied.

When $m = 0$ and $k > 0$ it follows from the condition (11) on $Q(t, r)$, in the same manner, that condition (22) is satisfied. As noted earlier, the case $k = 0$ is included in the classical theory.

This completes the proof of Theorem 1.

5. REMARKS. The preceding proof is clearly valid under conditions other than those stated in the theorem. It depends on $f(x)$ being representable by its Fourier sine and cosine integral formulas and satisfying either of the conditions (19) or (20). In the proof above that one of those conditions is satisfied, it is sufficient that $f(x)$ be bounded and integrable on each finite interval and absolutely integrable on the infinite interval.

The formal development in Section 3 above involves conditions on the function $g(t)$. Those conditions are the same as the conditions imposed on $f(x)$ in the theorem. Under those conditions, the improper integral in equation (12) converges uniformly with respect to x $(x \geq 0)$, and equation (13) holds provided that the limit on the right-hand side exists. But for each positive t at which the one-sided derivatives of $g(t)$ exist, it was shown that the limit does exist and that $g(t)$ is represented by formula (16). When the expression (12) for $f(x)$ is substituted into (16) it follows that, when $k > 0$,

$$g(t) = \frac{2}{\pi}\int_0^\infty y(x, t)\int_0^\infty g(\lambda)y(x, \lambda)\,d\lambda\,dx + \frac{2m}{\pi k}\,y(0, t)\int_0^\infty g(\lambda)y(0, \lambda)\,d\lambda \tag{23}$$

for each value of t just described. The following expansion is therefore established.

THEOREM 2. *Let* $f(x)$ *satisfy the conditions stated in Theorem 1. Then when* $k > 0$, *for each positive value of* x *for which* $f(x)$ *has a derivative from the right and from the left, the function is represented by the formula*

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos[\lambda x - \alpha(x)]\int_0^\infty f(\mu)\cos[\lambda\mu - \alpha(\mu)]\,d\mu\,d\lambda \tag{24}$$

$$+ \frac{2m}{\pi k}\cos\alpha(x)\int_0^\infty f(\mu)\cos\alpha(\mu)\,d\mu.$$

When $h = m = 0$, then $\alpha(x) = 0$ and this formula reduces to the Fourier cosine integral formula. Thus, formula (24) is another generalization of the Fourier integral formula. It is a representation of $f(x)$ in terms of the functions $\cos[\lambda x - \alpha(x)]$ $(\lambda \geq 0)$, not in terms of the characteristic functions of the boundary value problem (1).

The special case of formula (18) when $m = 0$ was obtained formally in references [1] and [2]. The general case of that formula and the representation (24) together with the two expansion theorems are believed to represent new results.

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