EXTENSIONS OF THE GROSS STAR THEOREM

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1. BACKGROUND. The Gross star theorem ([3]; [5], p. 276) asserts that each element $z(w)$ of the inverse of a function $w = \phi(z)$, meromorphic for $|z| < \infty$, can be continued to infinity along almost all rays from the center of the element. This has been generalized [4]: first, by replacement of the rays by very general families of "parallel curves"; and second, by replacement of the class of inverses of meromorphic functions by a considerably broader class. The generalizations depended on the following theorem concerning schlicht functions ([4], p. 4):

THEOREM I. *Let* $t = h(s)$ *be lower semi-continuous for* $0 < s < 1$, *where* $0 < b \leq h(s) \leq +\infty$. *Let* $w = \psi(\sigma)$ $(\sigma = s + it)$ *be schlicht in the domain* G: $0 < s < 1$, $-b < t < h(s)$; *and let*

$$\lim_{t \to h(s)} \psi(s + it) = 0, \quad h(s) < \infty \tag{1}$$

for each s *in a subset* E *of* (0,1). *Then* E *has measure* 0.

In the same paper ([4], p. 20) the following theorem was proved:

THEOREM II. *Let* E *be a closed set of capacity zero on* $|z| = 1$. *Then there exists a schlicht function* $w = \Phi(z)$ *in* $|z| < 1$ *such that* $\lim_{z \to z_0} \Phi(z) = \infty$ *for each* z_0 *in* E, *while* $\lim_{z \to z_1} \Phi(z)$ *is finite for* $|z_1| = 1$ *and* z_1 *not in* E.

2. TWO THEOREMS ON SCHLICHT FUNCTIONS.

THEOREM 1. *Let* B *be a closed countable subset of the extended plane. In Theorem* I *let* (1) *be replaced by the condition*

$$\lim_{n \to \infty} \psi(s + it_n) \in B, \quad h(s) < \infty \tag{1'}$$

for every sequence $t_n \to h(s)$, *whenever* $\lim \psi(s + it_n)$ *exists or is* ∞. *Then the conclusion that* E *has measure zero remains valid.*

Proof. The limits in (1') are the "cluster values" of ψ on the segment $s = \text{const.}$, $-b < t < h(s)$, as t approaches the boundary $h(s)$. These cluster values must form a closed connected set. However, B is totally disconnected. Hence, for each s in E the limit in (1') exists for all sequences $t_n \to h(s)$; that is, $\lim_{t \to h(s)} \psi(s + it)$ exists or is ∞ and is an element b_k of B $(k = 1, 2, \cdots)$. For each b_k, let E_k be the subset of E for which the limit equals b_k. Then E_k has measure zero, by Theorem I, so that $E = \bigcup_k E_k$ has measure zero.

Remark 1. It is natural to conjecture that the theorem remains true if B is an arbitrary totally disconnected closed set. It is certainly false in this generality; for we can choose E to be closed, of positive measure and totally disconnected, h(s) to be 1 and ψ to be the identity. However, it may remain true if B has linear measure zero or if B has capacity zero. The following theorem is a result in this direction.

THEOREM 2. *In Theorem* I *let the function* $z = \psi(\sigma)$ *satisfy the additional hypothesis*: $|\psi(\sigma)| < 1$ *in* G. *Let* B *be a closed subset of* $|z| = 1$ *having capacity*

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zero. Let the condition (1) *be replaced by the following:*

$$(1'') \qquad \lim_{n\to\infty} \psi(s + it_n) \in B, \ h(s) < \infty$$

for every sequence $t_n \to h(s)$, *whenever* $\lim \psi(s + it_n)$ *exists. Then the conclusion that* E *has measure zero remains valid.*

Proof. As in the proof of Theorem 1, we conclude that for s in E $\lim_{t\to h(s)} \psi(s + it)$ exists and is an element of B. Now it follows from Theorem II that we can choose a function $\zeta = \Phi(z)$ which is schlicht for $|z| < 1$ and for which $\Phi(z) \to 0$ as z approaches a point of B. Hence, $\psi_1(\sigma) = \Phi(\psi(\sigma))$ is schlicht in G and $\psi_1(s + it) \to 0$ as $t \to h(s)$ for s in E. Therefore, by Theorem I, E has measure zero.

3. CONTINUATION OF THE INVERSE OF A MEROMORPHIC FUNCTION ALONG A FAMILY OF CURVES. Let $w = \phi(z)$ be meromorphic in a domain D of the z-plane or, more generally, of a Riemann surface R. By local inversion of $w = \phi(z)$ at each noncritical point z_0 we obtain a functional element $z(w)$; at a critical point z_0 we obtain an algebraic functional element (that is, a series in $(w - w_0)^{1/n}$ or in $(1/w)^{1/n}$). These are the "internal elements" ([1], p. 100) of the inverse function. They are all analytic continuations of each other, but do not necessarily form all such continuations; for $\phi(z)$ may itself be continuable. For this reason we call the set of these elements the *restricted inverse* of $\phi(z)$.

Let $z(w)$ be a functional element of the restricted inverse, with center at w_0 and with $z(w_0)$ noncritical. Let γ_w: $w = w(t)$ $(0 \leq t < a)$ be a continuous path in the extended plane with $w(0) = w_0$. Then there exists a $t_1 \leq a$ and a unique continuation of $z(w)$ by regular elements of the restricted inverse along γ_w, for $0 \leq t < t_1$. This continuation assigns to each t a value $z = \psi(t)$ in D such that $\phi(\psi(t)) = w(t)$. The function $\psi(t)$ is continuous and represents a path γ_z in the z-plane.

Suppose that continuation is possible for $0 \leq t < t_1$ but no farther, without leaving the restricted inverse. If $t_1 < a$, there are then two possibilities: as $t \to t_1$, $\psi(t)$ approaches a critical point of $\phi(z)$; or, as $t \to t_1$, $\psi(t)$ approaches the boundary of D—that is, for every compact subset A of D, $\psi(t)$ remains outside A for t sufficiently close to t_1. In the latter case, $\psi(t)$ must approach a unique component of the boundary; since $\phi(\psi(t)) \to w(t_1)$ as $t \to t_1$, γ_z is an asymptotic path for $\phi(z)$.

We now define a family of paths γ_w as follows: As in Theorem I, let $t = h(s)$ be lower semi-continuous for $0 < s < 1$, and let $0 < b \leq h(s) < +\infty$; let G be the domain $-b < t < h(s)$, $0 < s < 1$; and let $w = f(\sigma)$ $(\sigma = s + it)$ be meromorphic in G. Then, for each fixed s, $w = f(s + it)$ defines a path γ_w^s on which t is the parameter for $0 \leq t < h(s)$. We assume there exists one regular element $z(w)$ of the restricted inverse of $\phi(z)$ such that $z[f(\sigma)]$ is defined for $0 < s < 1$, $|t| < b$. We also assume that $z(w)$ can be continued along each path γ_w^s for $0 \leq t < h_1(s)$ and no farther. This continuation defines $\psi(\sigma)$ as a meromorphic function for $0 < t < h_1(s)$, $0 < s < 1$, and $\phi(\psi(\sigma)) = f(\sigma)$; the curve $z = \psi(s + it)$ is for each fixed s a path γ_z^s in D and, if $h_1(s) < h(s)$, γ_z^s either approaches a critical point as $t \to h_1(s)$ or is an asymptotic path for $\phi(z)$ relative to some boundary component.

If $f(\sigma)$ is of form $\alpha\sigma + \beta$ and $h(s) \equiv +\infty$, then we are studying continuation of the inverse of $\phi(z)$ along the parallel lines γ_w^s; if $f(\sigma)$ is an exponential function, we are studying continuation of the inverse along rays radiating from a point, as in the Gross star theorem.

4. MEROMORPHIC FUNCTIONS WITH A COUNTABLE NUMBER OF SINGULARITIES.

THEOREM 3. *In the notations of Section* 3, *let* D *be the extended* z*-plane minus a countable closed set* B. *Then* $h_1(s) = h(s)$ *for almost all* s $(0 < s < 1)$; *that is, the inverse can be continued indefinitely along almost all curves* γ_w^s.

Proof. Where $h_1(s) < h(s)$ and the path γ_z^s does not approach a critical point, it must approach B. Thus $\psi(\sigma)$ satisfies a boundary condition as in Theorem 1. The proof is completed exactly as the proof of Theorem 2 in [4], with the new Theorem 1 replacing Theorem 1 of [4].

Remark 2. In accordance with Remark 1, I conjecture that the theorem remains true for more general sets B, also, that it remains valid when ϕ is meromorphic on a Riemann surface R with "null boundary" in an appropriate sense. If R has a parabolic universal covering surface, this is indeed true, for this case can at once be reduced to Theorem 2 of [4].

5. FUNCTIONS HAVING AN ISOLATED SINGULARITY. When $\phi(z)$ is meromorphic in a region whose boundary is large (for example, in a region whose boundary contains an arc), then one cannot expect to be able to continue the inverse function in almost all directions, for the cluster values of ϕ on the arc form a barrier to continuation in a set of directions which in general has positive measure. However, one can state that continuation is possible almost everywhere except for such a barrier. The following theorem deals with an example of the sort of case which can be treated.

THEOREM 4. *In the notations of Section* 3, *let* D *be the domain* $0 < |z| < 1$. *Then wherever* $h_1(s) < h(s)$, *either* (a) $|\psi(s + it)| \to 1$ *as* $t \to h_1(s)$, *or* (b) $\psi(\sigma)$ *converges to* 0 *or to a critical point of* $\phi(z)$ *as* $t \to h_1(s)$. *The case* (b) *arises only for a set* E *of values of* s *of measure zero.*

Proof. Since, for $h_1(s) < h(s)$, the path γ_z^s must approach a critical point or a unique boundary component, (a) and (b) are the only two possibilities. The proof of the last assertion is the same as that for Theorem 2 in [4], with "approaches infinity" replaced by "approaches zero."

This theorem permits of wide generalization. For example, the domain $0 < |z| < 1$ can be replaced by an arbitrary domain D obtained from a domain D' by removing a countable closed set B, if in case (a) approach to $|z| = 1$ is replaced by approach to the boundary of D' and in case (b) approach to zero is replaced by approach to a point of B.

6. FUNCTIONS MEROMORPHIC IN THE UNIT CIRCLE.

THEOREM 5. *In the notations of Section* 3, *let* D *be the domain* $|z| < 1$. *Let* B *be a closed subset of* $|z| = 1$ *having capacity zero. Let* E *be a subset of* (0, 1) *such that, for each* s *in* E, $h_1(s) < h(s)$ *and the sequence* $\psi(s + it_n)$ *has all its limit points in* B *for every sequence* t_n *converging to* $h_1(s)$. *Then* E *has measure zero; that is, the set of paths* γ_w^s *on which unlimited continuation fails because* γ_z^s *approaches* B *has measure zero.*

Proof. Again the proof of Theorem 2 of [4] can be repeated, with Theorem 2 above replacing Theorem 1 of [4].

Discussion. If we assume that $\phi(z)$ is bounded or, more generally, of bounded type, then $\lim_{t \to h_1(s)} \psi(s + it)$ exists wherever $h_1(s) < h(s)$. For if $\psi(s + it)$ does not converge to a critical point of $\phi(z)$, it must tend to an arc α on $|z| = 1$. By the Fatou theorem, $\phi(z) \to f(s + ih_1(s))$ radially almost everywhere on α; this is impossible by the Riesz-Nevanlinna theorem ([5], p. 197), unless ϕ is constant (cf. [1], pp. 94-95). Hence, each γ_z^s has a unique endpoint, wherever $h_1(s) < h(s)$. The same conclusion holds for all s $(0 < s < 1)$, if we assume that for each s the function $f(s + it)$ has a limit, finite or ∞, as t approaches $h(s)$. For then the path γ_w^s is defined in the *closed* interval $0 \leq t \leq h(s)$, and the corresponding path γ_z^s must approach a regular point (possibly a pole) or a critical point of $\phi(z)$, or else approach the boundary $|z| = 1$. For example, $f(\sigma)$ could be linear and $h(s) \equiv +\infty$. Theorem 5 asserts that each class B of endpoints of capacity zero is mapped by ϕ onto a set which, as viewed from a point of the Riemann surface of the restricted inverse along the curves γ_w^s, appears as a set of measure zero. If ϕ is itself schlicht, this has even more concrete meaning. The theorem can be regarded as a complement to a theorem of Nevanlinna and Frostman ([5], p. 198; [2], p. 97).

If no restriction is made on $\phi(z)$, we cannot expect existence of a unique endpoint for each γ_z^s, for γ_z^s can even spiral towards $|z| = 1$ ([6]). Let one such spiral γ_z^s exist; then every γ_z^s which approaches $|z| = 1$ must also be a spiral. In this case, if there are two spirals $\gamma_z^{s_1}$ and $\gamma_z^{s_2}$, then the two bound a simply connected domain D_0 for which $|z| = 1$ appears as a prime end. If we map D_0 conformally onto the disc $|\zeta| < 1$ by $z = H(\zeta)$, the prime end becomes a single point ζ_0. The function $\phi_1(\zeta) = \phi(H(\zeta))$ is then meromorphic in D_0. In the previous reasoning, we can now replace $w = \phi(z)$ by $w = \phi_1(\zeta)$. The paths γ_w^s now correspond to paths γ_ζ^s in $|\zeta| < 1$, and for $s_1 < s < s_2$ those paths which approach $|\zeta| = 1$, as t approaches $h_1(s)$, must approach ζ_0. We can now apply Theorem 5 to conclude that $h_1(s) < h(s)$ only for a set of measure 0, for $s_1 < s < s_2$. Similar remarks can be made if, for $s_1 < s < s_2$, all γ_z^s have the same arc α as set of limit points on $|z| = 1$; for such an arc again forms a single prime end.

In general, there are interesting relations between Theorem 5 and the paper of Collingwood and Cartwright [1].

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