ON ASYMMETRIC APPROXIMATIONS

by

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1. B. Segre [1] deduced the following theorem from his investigation of lattice points in certain non-convex domains:

Every irrational number ξ has infinitely many rational approximations u/v such that

(1) $$-\frac{\tau}{(1+4\tau)^{1/2}v^2} < \xi - \frac{u}{v} < \frac{1}{(1+4\tau)^{1/2}v^2}$$

where $\tau \geq 0$ is arbitrary.

C. D. Olds [2] gave a simple arithmetic proof for the case $\tau > 1$. N. Negoescu [3] used continued fractions to show that the inequality

(2) $$-\frac{\tau}{\alpha v^2} < \xi - \frac{u}{v} < \frac{1}{\alpha v^2}$$

has infinitely many solutions for $\tau \geq 0$ if $\alpha = \max((1+4\tau)^{\frac{1}{2}}, (\tau^2+4\tau)^{\frac{1}{2}})$, but as R. M. Robinson [5] pointed out, Segre's and Negoescu's theorems are equivalent, inasmuch as they are identical when $\tau \leq 1$, while for $\tau > 1$, Negoescu's theorem asserts the same property of ξ as does Segre's of $-\xi$, if τ is replaced by $1/\tau$ in (1). Recently, Negoescu [4] attempted to prove that of any three consecutive convergents of the continued fraction expansion of ξ, one at least satisfies (1) for arbitrary $\tau \geq 0$. It is shown here that this is true of one out of any five consecutive convergents; more precisely, at least one of the numbers p_{2n-1}/q_{2n-1}, p_{2n}/q_{2n}, p_{2n+1}/q_{2n-1} satisfies (2) with $\alpha = (1+4\tau)^{1/2}$, and one of the numbers p_{2n}/q_{2n}, p_{2n+1}/q_{2n+1}, p_{2n+2}/q_{2n+2} satisfies (2) with $\alpha = (\tau^2+4\tau)^{1/2}$. Here the p_k/q_k are convergents to ξ, n is an arbitrary positive integer, and $\tau > 0$ is arbitrary. Moreover, Negoescu's assertion is shown to be sometimes false.

For the special case $\tau = 1$ (or $c_1 = c_2$), the proof given in §2 simplifies considerably, and leads to a proof of the well-known theorem of Hurwitz whose perspicuity compares favorably with that given by A. Khintchine [6].

2. For $(p, q) = 1$, let $I(p/q)$ denote the closed interval

$$\left[\frac{p}{q} - \frac{1}{c_1 q^2},\ \frac{p}{q} + \frac{1}{c_2 q^2}\right]$$

where $c_1 > 0$, $c_2 > 0$. Let p/q and r/s be the two elements of the Farey sequence F_n such that $p/q < \xi < r/s$; thus $qr - ps = 1$. Then there is a gap between $I(p/q)$ and $I(r/s)$ only if

$$\frac{p}{q} + \frac{1}{c_2 q^2} < \frac{r}{s} - \frac{1}{c_1 s^2} .$$

This inequality is easily reduced to

$$\frac{s}{c_1 q} + \frac{q}{c_2 s} < 1 ,$$

or to $f(s/q) < 1$, where

$$f(x) = \frac{x}{c_1} + \frac{1}{c_2 x} .$$

Similarly, there is a gap between $I(p/q)$ and $I((p + r)/(q + s))$ exactly when

$$f\left(\frac{s + q}{q}\right) = f\left(1 + \frac{s}{q}\right) < 1 ,$$

and a gap between $I((p + r)/(q + s))$ and $I(r/s)$ exactly when

$$f\left(\frac{s}{q + s}\right) = f\left(\frac{s/q}{1 + s/q}\right) < 1.$$

Hence ξ certainly lies in one of the intervals $I(p/q)$, $I((p + r)/(q + s))$ or $I(r/s)$ if

$$\max\left(f\left(\frac{s}{q}\right),\ f\left(1 + \frac{s}{q}\right)\right) \geq 1$$

and

$$\max\left(f\left(\frac{s}{q}\right),\ f\left(\frac{s/q}{1 + s/q}\right)\right) \geq 1 ,$$

which is certainly the case if, for all $x > 0$,

$$\max(f(x),\ f(1+x)) \geq 1 \quad \text{and} \quad \max(f(x),\ f(\frac{x}{1+x})) \geq 1 .$$

It is easily verified that $g(x) = \max(f(x), f(1+x))$ is concave upward for positive x, and has its minimum value at $x = x_o$, where $f(x_o) = f(1+x_o)$. Putting $c_1/c_2 = c$, one obtains

$$x_o = \frac{c + (c_1^2 + 4c)^{1/2}}{2},\quad f(x_o) = \frac{(c_1^2 + 4c)^{1/2}}{c_1} = \frac{(c_1^2 + 4c_1c_2)^{1/2}}{c_1c_2},$$

so that $f(x_o) = 1$ only if $c_1 = 4c_2/(c_2^2 - 1)$. If $c_1^2 - 1 = 4\tau$, then $c_2 = (1+4\tau)^{\frac{1}{2}}$ and $c_1 = (1+4\tau)^{1/2}/\tau$, and we conclude that, if $p/q < \xi < (p+r)/(q+s)$, then for arbitrary $\tau > 0$ either

$$(3)\quad 0 < \xi - \frac{p}{q} \leq \frac{1}{(1+4\tau)^{1/2}\, q^2} \quad \text{or} \quad \frac{-\tau}{(1+4\tau)^{1/2}\,(q+s)^2} \leq \xi - \frac{p+r}{q+s} < 0 .$$

Similarly, the minimum of

$$h(x) = \max(f(x),\ f(\frac{x}{1+x})),$$

for $x > 0$, occurs at $x = x_1$, where $f(x_1) = f(x_1/(1+x_1))$, and

$$x_1 = \frac{-1 + (1+4c)^{1/2}}{2},\quad f(x_1) = \frac{(1+4c)^{1/2}}{c_1},$$

so that $f(x_1) = 1$ only if $c_2 = 4c_1/(c_1^2 - 1)$. If $c_1^2 - 1 = 4t$, then $c_1 = (1+4t)^{1/2}$ and $c_2 = (1+4t)^{1/2}/t$, and we conclude that, if $(p+r)/(q+s) < \xi < r/s$, then for arbitrary $\tau = 1/t > 0$ either

$$(4)\qquad 0 < \xi - \frac{p+r}{q+s} < \frac{1}{(\tau^2 + 4\tau)^{1/2}\,(q+s)^2}$$

or

$$\frac{-\tau}{(\tau^2 + 4\tau)^{1/2}\, s^2} \leq \xi - \frac{r}{s} < 0 .$$

The possibility of equality can easily be excluded in (3) and (4). For equality can hold in (3) only if $x_0 = s/q$; but it follows from the equation $f(x_0) = f(1 + x_0)$ that then c is rational, and since $f(x_0) = 1$ also c_1 is rational, so c_2 is rational, and finally ξ itself would have to be rational for either equality in (3) to hold. A similar argument applies to (4).

Thus we have shown that, whenever ξ lies in the "left half" of a Farey interval there corresponds a solution of the inequality

$$\frac{-\tau}{(1 + 4\tau)^{1/2} v^2} < \xi - \frac{u}{v} < \frac{1}{(1 + 4\tau)^{1/2} v^2},$$

and whenever ξ lies in the "right half" of a Farey interval there corresponds a solution of the inequality

$$\frac{-\tau}{(\tau^2 + 4\tau)^{1/2} v^2} < \xi - \frac{u}{v} < \frac{1}{(\tau^2 + 4\tau)^{1/2} v^2}.$$

Negoescu's form of Segre's theorem now follows upon noting that any ξ, being irrational, lies in the "left half" of infinitely many Farey intervals, and also in the "right half" of infinitely many Farey intervals, and that solutions corresponding to different intervals are distinct. (In the case $\tau = 0$, the theorem is implied by the fact that ξ lies in infinitely many left halves, since $p/q + 1/q^2 > (p + r)/(q + s)$.

In the course of the proof, we have obtained the following theorem, which may have independent interest: if F_n is an arbitrary Farey sequence and ξ lies between the adjacent elements p/q and r/s of F_n, then at least one of the numbers p/q, r/s and $(p + r)/(q + s)$ is a solution of the inequality

$$\frac{-\tau}{\beta v^2} < \xi - \frac{u}{v} < \frac{1}{\beta v^2},$$

where $\tau \geq 0$ and $\beta = \min ((1 + 4\tau)^{1/2}, (\tau^2 + 4\tau)^{1/2})$.

3. We now consider the case that p/q and r/s are successive convergents of ξ. Then ξ is in the left half of the interval if

$$\frac{p}{q} = \frac{p_{2n}}{q_{2n}}, \quad \frac{r}{s} = \frac{p_{2n-1}}{q_{2n-1}} \tag{5}$$

for some n, while ξ is in the right half if

$$(6) \qquad \frac{p}{q} = \frac{p_{2n}}{q_{2n}}, \quad \frac{r}{s} = \frac{p_{2n+1}}{q_{2n+1}}$$

for some n.

First consider the case (5), and suppose that $\xi \notin I(p/q)$ and $\xi \notin I(r/s)$, where now $c_1 = (1 + 4\tau)^{1/2}/\tau$ and $c_2 = (1 + 4\tau)^{1/2}$ as in the derivation of (3). Then $I(p/q)$ and $I((p + r)/(q + s))$ overlap, while $I(p/q)$ and $I(r/s)$ do not:

$$f\left(\frac{s}{q}\right) < 1, \quad f\left(1 + \frac{s}{q}\right) > 1 .$$

Since $f(x)$ is concave upward, $f(k + \frac{s}{q}) > 1$ for every positive integer k; this however is the condition that $I(p/q)$ and $I(\frac{kp + r}{kq + s})$ overlap. Hence if

$$\frac{p_{2n+1}}{q_{2n+1}} = \frac{a_{2n}p_{2n} + p_{2n-1}}{a_{2n}q_{2n} + q_{2n-1}} = \frac{a_{2n}\, p + r}{a_{2n}\, q + s}$$

is the next convergent after p/q (so that $\xi < p_{2n+1}/q_{2n+1}$), then $\xi \in I(p_{2n+1}/q_{2n+1})$. In a similar fashion, if (6) holds, $\xi \notin I(p/q)$, $\xi \notin I(r/s)$ and $c_1 = (\tau^2 + 4\tau)^{1/2}/\tau$, $c_2 = (\tau^2 + 4\tau)^{1/2}$, then $\xi \in I(p_{2n+2}/q_{2n+2})$.

To see that it is not true that one out of any three convergents to satisfies (1), for arbitrary $\tau > 0$, we show that none of the convergents

$$\frac{p_2}{q_2} = \frac{3}{2}, \quad \frac{p_3}{q_3} = \frac{5}{3}, \quad \frac{p_4}{q_4} = \frac{8}{5}$$

of the continued fraction expansion of 81/50 satisfies (1) for $0.84 < \tau < 0.9$, and deduce by continuity that the same is true, for $0.85 < \tau < 0.89$, of irrational numbers ξ sufficiently close to 81/50.

We have

$$\frac{81}{50} - \frac{3}{2} = \frac{12/25}{4},$$

and so if (1) is satisfied for $\xi = 81/50$, $u/v = 3/2$, it must be that

$$\frac{12}{25} < \frac{1}{(4\tau + 1)^{1/2}}, \quad \text{or} \quad \tau < \frac{(25/12)^2 - 1}{4} < 0.836 .$$

Similarly, if (1) is satisfied for $u/v = 8/5$, then since $81/50 - \frac{8}{5} = \frac{1/2}{25}$,

$$\frac{1}{2} < \frac{1}{(4\tau + 1)^{1/2}}, \quad \text{or} \quad \tau < 0.75.$$

Finally, if (1) is satisfied for $u/v = 5/3$, it must be that

$$-\frac{\tau}{(4\tau + 1)^{1/2}} < 3^2\left(\frac{81}{50} - \frac{5}{3}\right) = \frac{-21}{50},$$

which implies that $\tau > 0.901$.

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