for an n-gm in Wilder [13], we obtain (E) from (B) and (C). In the last section we shew how our homology invariants are connected with the concept of "avoidability" introduced by Wilder. Our results shew that (D) above includes certain results of White [20].

1.1. We continue to use essentially the notation of LTI, recalling that x is a fixed point of a locally compact subset M of Hilbert space, with neighborhoods U, V, ... It will be convenient also to add this: if $\overset{*}{H}$ is a finitely generated Abelian group then we shall write an isomorphism of the form

$$\overset{*}{H} \approx \overset{*}{M}(k) + \overset{*}{F}$$

without repeating that the R.H.S. consists of the direct sum of a module of rank k and a finite Abelian group. Further, the symbol F, with or without suffix, will always denote a compact subset of M.

2. ČECH AND VIETORIS HOMOLOGY. If X is any subset of M, we defined $\mathcal{H}_v^r(X)$ in LTI as the rth Vietoris homology group of X, where only cycles and homologies with compact carriers are considered. Similarly the Čech group $\mathcal{H}_c^r(X)$ is defined. Now if $F \subseteq F' \subseteq X$, there exist injection homomorphisms

$$i_N(F, F') : \mathcal{H}_N^r(F) \to \mathcal{H}_N^r(F'), \quad N = c, v$$

so that we have the direct limit, (see Wilder [13] p. 247)

$$\mathcal{H}_N^r(X) = \text{Dir Lim } \{\mathcal{H}_N^r(F), \ i_N(F, F')\},$$

where the F's run through all $F \subseteq X$.

Since F is a compact Hausdorff space, there exists an isomorphism

$$\zeta_F : \mathcal{H}_v^r(F) \approx \mathcal{H}_c^r(F),$$

by Begle ([14] p. 536). Given $[\alpha] \in \mathcal{H}_v^r(X)$, and $\alpha' \in [\alpha]$, then there exists $F \subseteq X$ such that $\alpha' \in \mathcal{H}_v^r(F)$. Define $\zeta_X[\alpha]$ to be $[\zeta_F \alpha']$. Then, since

$$\zeta_{F'} i_v(F, F') = i_c(F, F') \zeta_F ,$$

it can be verified that the definition of ζ_X is independent of the choice of $\alpha' \in [\alpha]$, and that

$$\zeta_X : \mathcal{H}_v^r(X) \approx \mathcal{H}_c^r(X) .$$

Now suppose that $X \subseteq Y \subseteq M$, and let j_N be the injections

2.1 $$j_N : \mathcal{H}_N^r(X) \to \mathcal{H}_N^r(Y), \quad N = c, v .$$

Then, since $\zeta_Y j_v = j_c \zeta_X$, we have

2.2 $$\mathcal{H}^r_v(X|Y) \approx \mathcal{H}^r_c(X|Y) \quad ,$$
where $\mathcal{H}^r_N(X|Y) = j_N \ \mathcal{H}^r_N(X)$. Hence, if $\mathfrak{C}^r_c(x)$ denotes the Čech analogue of $\mathfrak{C}^r_v(x)$, we have

2.3 $\mathfrak{C}^r_c(x)$ <u>exists if and only if</u> $\mathfrak{C}^r_v(x)$ <u>exists; and then</u>

$$\mathfrak{C}^r_v(x) \approx \mathfrak{C}^r_c(x) \quad .$$

Let G be the set of homology coefficients. We recall the following definitions (cf. LTI, 4.1).

2.4. DEFINITION. <u>The space</u> M <u>is</u> r - $lc_N(G)$ (N = v, c) <u>at</u> x <u>if and only if there exists a neighborhood function</u> $\lambda^r_N(U)$ <u>such that, given</u> U_1, U_2 <u>satisfying</u> $U_2 \subseteq \lambda^r_N(U_1)$, <u>then every Vietoris (Čech) r-cycle on</u> $\bar{U}_2$ <u>is</u> ~ 0 <u>on</u> $\bar{U}_1$. M <u>is then</u> $lc^r_N(G)$ <u>at</u> x <u>if and only if</u> M <u>is</u> p - $lc_N(G)$ <u>at</u> x, $0 \leq p \leq r$; <u>and</u> M <u>is</u> $lc^r_N(G)$ <u>if and only if it is</u> $lc^r_N(G)$ <u>at all its points</u>.

The definition requires that[2] $\mathcal{H}^r_N(\bar{U}_2|\bar{U}_1, G) = 0$. Hence, it follows from 2.2 that M is r - $lc_v(G)$ if and only if M is r - $lc_c(G)$. Čech [16] has proved

2.5. <u>If</u> M <u>is</u> $lc^r_c(I)$ <u>then it is</u> $lc^r_c(G)$ <u>for every discrete Abelian group</u> G.

We shall also need the following theorem, in which G is assumed to be either I or a field.

2.6. THEOREM. <u>Let</u> F <u>be a compact subset of a separable metric</u> $lc^r_v(G)$ <u>space, and let</u> $\alpha > 0$ <u>be given. Then there exists a finite set of</u> r-V-<u>cycles</u>, $\Gamma_1^r, \ldots, \Gamma_k^r$ <u>on</u> $U(F, \alpha)$ <u>such that every</u> r-V-<u>cycle</u> Γ^r <u>on</u> F <u>satisfies a homology of the form</u>

$$\Gamma^r \sim \sum_{i=1}^{k} g_i \Gamma^r_i \qquad \text{on} \quad U(F, \alpha), \ g_i \in G.$$

PROOF. When G = I, this is Newman ([10] Thm. 1). When G is a field, the theorem holds for Čech cycles, by Begle ([14] Corollary 2.3). Hence $\mathcal{H}^r_c(F|U(F, \alpha))$ is a vector space of dimension at most k, and so by 2.2 the same is true of $\mathcal{H}^r_v(F|U(F, \alpha))$. The theorem now follows.

3. THE "$\mathcal{D}$" GROUPS. Suppose that M is $lc^r_v(I)$, and that a finitely generated group $\mathfrak{C}^r_v(x, I)$ exists at x. With U_1, U_2 as in LTI, Definition 6.1, we have

$$\mathcal{H}^r_v(\bar{U}_2 - x|\bar{U}_1 - x) \approx \mathfrak{C}^r_v(x) \quad ,$$

where we suppress I for the moment. Let $[\Gamma_1^r], \ldots, [\Gamma_p^r]$ be a

[2] All complexes are taken augmented. (See LTI, Footnote 9).

basis for $\mathcal{H}_v^r(\bar{U}_2 - x \mid \bar{U}_1 - x)$, and for each $i (1 \leq i \leq p)$ choose $\Gamma_i^r \in [\Gamma_i^r]$ with compact carrier F_i. Let m be the first integer such that

$$V = U(x, 1/m) \Subset U_2 - \bigcup_{i=1}^{p} F_i .$$

Then, if Γ^r is an r-V-cycle on $\bar{V} - x$, there exists a homology of the form

$$\Gamma^r \sim \sum_{i=1}^{p} n_i \Gamma_i^r \quad \text{on } \bar{U}_1 - x,$$

and therefore

3.1. <u>Every</u> r-V-<u>cycle on</u> $\bar{V} - x$ <u>is homologous, on</u> $\bar{U}_1 - x$, <u>to an</u> r-V-<u>cycle on</u> $\bar{U}_2 - V$.

Given V' and W such that $x \in W \Subset V' \subseteq V$, let

$$\delta = \min (\rho(\bar{W}, \complement V'), \rho(\bar{U}_2, \complement U_1)),$$

where $\complement$ denotes complementation. Since $\bar{U}_2$ is compact, so is $\bar{U}_2 - V'$; and

$$\bar{U}_2 - V' \subseteq U(\bar{U}_2 - V', \delta) \subseteq \bar{U}_1 - W .$$

Since $\delta \geq 0$, then by 2.6, $\mathcal{H}_v^r(\bar{U}_2 - V' \mid \bar{U}_1 - W)$ is finitely generated, and so it is of the form (see 1.1):

$$\overset{*}{M}(k) + \overset{*}{F} .$$

If $W' \subseteq W$, then $\bar{U}_1 - W \subseteq \bar{U}_1 - W'$, and by the methods of the proof of LTI 6.6, there exist W_o, $k_o \leq k$, and $\overset{*}{F}_o$ such that

$$\mathcal{H}_v^r(\bar{U}_2 - V' \mid \bar{U}_1 - W') \approx \overset{*}{M}(k_o) + \overset{*}{F}_o .$$

when $W' \subseteq W_o$ We assert, moreover, that

3.2 $$j: \mathcal{H}_v^r(\bar{U}_2 - V' \mid \bar{U}_1 - W') \approx \mathcal{H}_v^r(\bar{U}_2 - V' \mid \bar{U}_1 - x) ,$$

where j is the injection homomorphism. For if Γ^r is an r-V-cycle on $\bar{U}_2 - V'$, such that $\Gamma^r \sim 0$ on $F' \subseteq \bar{U}_1 - x$, let P be such that $x \in P \Subset W_o - F'$. Then, by definition of W_o,

$$\mathcal{H}_v^r(\bar{U}_2 - V' \mid \bar{U}_1 - W') \approx \mathcal{H}_v^r(\bar{U}_2 - V' \mid \bar{U}_1 - P).$$

Since these groups are finitely generated and Abelian, it follows by LTI 5.3, that $\Gamma^r \sim 0$ on $\bar{U}_1 - W'$. Hence j is univalent, and since it is clearly "on", the isomorphism (3.2) is established.

Now V depends on U_1, U_2, while W_o depends on U_1, U_2, V'; write $V = V(U_1, U_2)$, $W_o = W_o(U_1, U_2, V')$. Define the neighborhood functions U_δ^r, $\delta^r(U)$, $\delta^r(U, U')$, $\delta^r(U, U', U'')$ to be, respectively,

$$U_\gamma^r , \ \gamma^r(U), \ V(U, U'), \ W_o(U, U', U'') .$$

Since $\Gamma_1^r, \ldots, \Gamma_p^r$ are on $\bar{U}_2 - V'$, then $\mathcal{H}_v^r(\bar{U}_2 - V' | \bar{U}_1 - x) \approx \mathfrak{C}_v^r(x)$; and therefore, by 3.2, we have proved:

3.3. If U_1, U_2, U_3, U_4 satisfy the relations

$$U_1 \subseteq U_\delta^r, \quad U_2 \subseteq \delta^r(U_1), \quad U_3 \subseteq \delta^r(U_1, U_2), \quad U_4 \subseteq \delta^r(U_1, U_2, U_3) ,$$

then

$$\mathcal{H}_v^r(\bar{U}_2 - U_3 | \bar{U}_1 - U_4) \approx \mathfrak{C}_v^r(x) .$$

A similar result is obtained when I is replaced by a field, and we are led to give the following definition, applicable to a general coefficient group G.

3.4. DEFINITION. M has the group $\mathcal{D}_v^r(x, G)$ at x if and only if there exist a neighborhood U_δ^r of x and functions

$$\delta^r(U), \quad \delta^r(U, U'), \quad \delta^r(U, U', U''),$$

such that, whenever U_1, U_2, U_3, U_4 satisfy the conditions

$$U_1 \subseteq U_\delta^r, \quad U_2 \subseteq \delta^r(U_1), \quad U_3 \subseteq \delta^r(U_1, U_2), \quad U_4 \subseteq \delta^r(U_1, U_2, U_3),$$

then the relation

$$\mathcal{H}_v^r(\bar{U}_2 - U_3 | \bar{U}_1 - U_4, G) \approx \mathcal{D}_v^r(x, G)$$

holds.

When G = I or a field we may state 3.3 as follows.

3.5. THEOREM. If M is $lc_v^r(G)$ and if[3] a finitely generated $\mathfrak{C}_v^r(x, G)$ exists at x, then $\mathcal{D}_v^r(x, G)$ exists also; and

$$\mathfrak{C}_v^r(x, G) \approx \mathcal{D}_v^r(x, G) .$$

The converse result will be investigated in Section 5. Note that 2.6 implies the following.

3.6. Suppose that M is $lc_v^r(G)$ and that $\mathcal{D}_v^r(x, G)$ exists. If G = I, then $\mathcal{D}_v^r(x, I)$ is a finitely generated group; if G is a field, then $\mathcal{D}_v^r(x, G)$ is a vector space over G of finite dimension.

Let $\mathcal{D}_c^r(x, G)$ denote the Čech analogue of 3.4. By 2.2, we have immediately from 3.5,

3.7. $\mathcal{D}_v^r(x, G)$ exists if and only if $\mathcal{D}_c^r(x, G)$ exists; and then $\mathcal{D}_v^r(x, G) \approx \mathcal{D}_c^r(x, G)$ (for every discrete G).

When G is a field, we can relate the "$\mathcal{D}$" groups to the local Betti numbers (see LTI 3.1) by the following theorem, whose geometrical content is that of a "Theorem of Alexander Type".

3.8. THEOREM. Let G be a field, and let M be $r - lc_c(G)$ and $(r+1) - lc_c(G)$ at x. Then there exists a finite local Betti number

[3] When speaking of vector spaces, we sometimes use "finitely generated" to mean "of finite dimension".

$p^{r+1}(x, G)$ <u>if and only if</u> $\mathcal{D}_c^r(x, G)$ <u>exists as a vector space over</u> G <u>of finite dimension</u> $d^r(x, G)$; <u>and then</u>

$$p^{r+1}(x, G) = d^r(x, G) .$$

PROOF. Let U_0 be any neighborhood of x, such that $\overline{U}_0$ is compact. Using the notation of 2.4, let U_1, U_2, U_3, U_4 be given to satisfy

(i) $$U_1 \subseteq \lambda_c^{r+1}(U_0),\ U_2 \subseteq \lambda_c^r(U_1),\ U_3 \Subset U_2,\ U_4 \Subset U_3 ;$$

we shall later impose further restrictions upon them. First, however, we shall prove that

$$\mathcal{H}_c^{r+1}(\overline{U}_0; \overline{U}_0 - U_3 \,|\, \overline{U}_0 - U_4) \approx \mathcal{H}_c^r(\overline{U}_2 - U_3 \,|\, \overline{U}_1 - U_4) ,$$

where the L.H.S. denotes the injection of $\mathcal{H}_c^{r+1}(\overline{U}_0, \overline{U}_0 - U_3)$ in $\mathcal{H}_c^{r+1}(\overline{U}_0, \overline{U}_0 - U_4)$. (Throughout the proof, we omit explicit mention of G).

Let $\overset{*}{U}, \overset{*}{V}, \ldots,$ denote finite open coverings of the compact set $\overline{U}_0$. By Wilder ([13] VII 1.10), each $\overset{*}{U}$ has a refinement $\overset{*}{V}$ such that if the nucleus[4] of a simplex of the nerve of $\overset{*}{V}$ meets both $\overline{U}_3$ and $\overline{U}_0 - \overline{U}_3$, then it meets $\mathcal{F}U_3$. Hence, the family Σ' of such $\overset{*}{V}$ is cofinal in the family Σ of all finite open coverings $\overset{*}{U}$ of $\overline{U}_0$. We may therefore always assume that all homology groups are taken with respect to Σ'. If $\overset{*}{U} \in \Sigma'$ and $C(\overset{*}{U})$ is a chain on $\overset{*}{U}$, let

$$C(\overset{*}{U}) = C_1(\overset{*}{U}) + C_2(\overset{*}{U}) ,$$

where $C_1(\overset{*}{U})$ consists of all cells of $C(\overset{*}{U})$ whose nuclei meet $\overline{U}_3$. In particular, if $\Gamma^{r+1} = \{\Gamma^{r+1}(\overset{*}{U})\}$ is a Čech cycle mod $\overline{U}_0 - U_3$ on $\overline{U}_0$, then

(ii) $$\partial(\Gamma^{r+1}(\overset{*}{U}) - \Gamma_2^{r+1}(\overset{*}{U})) = \partial\Gamma_1^{r+1}(\overset{*}{U}) ;$$

and since the L.H.S. consists of cells on[4] $\overset{*}{U} \wedge (\overline{U}_0 - \overline{U}_3)$ while the R.H.S. is on $\overset{*}{U} \wedge \overline{U}_3$, then each side is on $\mathcal{F}U_3$, by definition of Σ'. Let $\overset{*}{V} \in \Sigma'$ be a refinement of $\overset{*}{U}$, and let π denote projection from $\overset{*}{V}$ to $\overset{*}{U}$. Then, since Γ^{r+1} is an (r+1)-cycle mod $\overline{U}_0 - U_3$, we have an equation of the form

$$\Gamma^{r+1}(\overset{*}{U}) - \pi\Gamma^{r+1}(\overset{*}{V}) = \partial C^{r+2}(\overset{*}{U}) + C^{r+1}(\overset{*}{U}) ,$$

where $C^{r+2}(\overset{*}{U})$ is an (r+2)-chain on $\overset{*}{U} \wedge \overline{U}_0$, and $C^{r+1}(\overset{*}{U})$ is a chain on $\overset{*}{U} \wedge (\overline{U}_0 - U_3)$. Therefore

[4] For definitions of the various concepts in this proof, see Wilder [13], V §7.

$$\Gamma_1^{r+1}(\overset{*}{U}) - \pi\,\Gamma_1^{r+1}(\overset{*}{V})$$
$$= \partial C^{r+2}(\overset{*}{U}) + C^{r+1}(\overset{*}{U}) - \left(\Gamma_2^{r+1}(\overset{*}{U}) - \pi\,\Gamma_2^{r+1}(\overset{*}{V})\right),$$

so that $\{\Gamma_1^{r+1}(\overset{*}{U})\}$ is a cycle mod $\overline{U}_0 - U_3$ on $\overline{U}_3$. But

$$-\partial\left(\Gamma_2^{r+1}(\overset{*}{U}) - \pi\,\Gamma_2^{r+1}(\overset{*}{V}) - C_2^{r+1}(\overset{*}{U})\right)$$
$$= \partial\left(\Gamma_1^{r+1}(\overset{*}{U}) - \pi\,\Gamma_1^{r+1}(\overset{*}{V})\right) - \partial C_1^{r+1}(\overset{*}{U});$$

also, $\partial\Gamma_1^{r+1}$ is on $\mathcal{F}U_3$, and $C^{r+1}\wedge U_3$ is on $U_3 \cap (\overline{U}_0 - U_3) = \mathcal{F}U_3$; therefore each side is on $\mathcal{F}U_3$. Hence $\{\Gamma_1^{r+1}(\overset{*}{V})\}$ is a cycle mod $\mathcal{F}U_3$ on $\overline{U}_3$. Since $\pi\partial = \partial\pi$, we have

$$\partial\Gamma_1^{r+1}(\overset{*}{U}) - \pi\,\partial\Gamma_1^{r+1}(\overset{*}{V})$$
$$= \partial C_1^{r+1}(\overset{*}{U}) + [\,\partial C_2^{r+1}(\overset{*}{U}) - \partial(\Gamma_2^{r+1}(\overset{*}{U}) - \pi\,\Gamma_2^{r+1}(\overset{*}{V}))].$$

The L.H.S. and the term in square brackets are on $\mathcal{F}U_3$; hence, so is $\partial C_1^{r+1}(\overset{*}{U})$. Therefore

$$\partial\Gamma_1^{r+1}(\overset{*}{U}) - \pi\,\partial\Gamma_1^{r+1}(\overset{*}{V}) = \partial\left(C^{r+1}(\overset{*}{U}) - \left(\Gamma_2^{r+1}(\overset{*}{U}) - \pi\,\Gamma_2^{r+1}(\overset{*}{V})\right)\right).$$

i.e., $\phi\Gamma^{r+1} = \{\partial\Gamma_1^{r+1}(\overset{*}{U})\} = \partial\Gamma_1^{r+1}$ is a Čech cycle on

$$\mathcal{F}U_3 \subseteq \overline{U}_2 - U_3 \quad .$$

Suppose that $\Gamma'^{r+1} = \{\Gamma'^{r+1}(\overset{*}{U})\}$ is a second $(r+1)$ - cycle mod $\overline{U}_0 - U_3$ on $\overline{U}_0$, and that $\Gamma^{r+1} \sim \Gamma'^{r+1}$ mod $\overline{U}_0 - U_4$ on $\overline{U}_0$. Then if $\overset{*}{U} \in \Sigma'$, an equation of the form

$$\Gamma'^{r+1}(\overset{*}{U}) - \Gamma^{r+1}(\overset{*}{U}) = \partial C^{r+1}(\overset{*}{U}) + C^{r+1}(\overset{*}{U})$$

holds, where

$$C^{r+2}(\overset{*}{U}) \text{ is on } \overset{*}{U}\wedge\overline{U}_0 \text{ and } C^{r+1}(\overset{*}{U}) \text{ is on } \overset{*}{U}\wedge(\overline{U}_0 - U_4).$$

Hence,

(iii) $$\partial\Gamma_1'^{r+1}(\overset{*}{U}) - \partial\Gamma_1^{r+1}(\overset{*}{U}) - \partial C_1^{r+1}(\overset{*}{U})$$
$$= \partial C_2^{r+1}(\overset{*}{U}) - \partial\left(\Gamma_2'^{r+1}(\overset{*}{U}) - \Gamma_2^{r+1}(\overset{*}{U})\right) .$$

By the usual argument, each side is on $\mathcal{F}U_3$, and so

(iv) $\partial\Gamma_1'^{r+1}(\overset{*}{U}) - \partial\Gamma_1^{r+1}(\overset{*}{U}) = \partial C_1^{r+1}(\overset{*}{U}) + $ (R.H.S. of (iii)) .

Since $C_1{}^{r+1}(\overset{*}{U})$ is on $\overset{*}{U}\wedge(\bar{U}_3 - U_4)$, the R.H.S. of (iv) is on

$$\bar{U}_3 - U_4 \subseteq \bar{U}_1 - U_4 .$$

Thus

$$\phi\Gamma^{r+1} \sim \phi\Gamma'^{r+1} \quad \text{on } \bar{U}_1 - U_4 .$$

Now ϕ is clearly additive and linear, and therefore the correspondence of cosets

$$\phi^*: [\Gamma^{r+1}] \to [\phi\Gamma^{r+1}]$$

defines a homomorphism

$$\phi^*: \mathcal{H}_c^{r+1}(\bar{U}_0, \bar{U}_0 - U_3 | \bar{U}_0 - U_4) \to \mathcal{H}_c^{r}(\bar{U}_2 - U_3 | \bar{U}_1 - U_4),$$

which we shall now prove to be an isomorphism.

ϕ^* <u>is univalent</u>. For, suppose that $\phi^*[\Gamma^{r+1}] = 0$. Then $\phi\Gamma^{r+1} = \partial\Gamma_1{}^{r+1} \sim 0$ on $\bar{U}_1 - U_4$. By Wilder ([13] VII 1.6), there exists an (absolute) (r+1)-cycle Γ_0^{r+1} on $\bar{U}_1$, such that

$$\Gamma_0{}^{r+1} \sim \Gamma_1{}^{r+1} \text{ mod } \bar{U}_1 - U_4 \text{ on } \bar{U}_1 .$$

But $U_1 = \lambda_c^{r+1}(U_0)$ by (i) above, so that $\Gamma_0{}^{r+1} \sim 0$ on U_0; hence

$$\Gamma_1{}^{r+1} \sim 0 \quad \text{mod } \bar{U}_1 - U_4 \text{ on } \bar{U}_0 .$$

Thus, for each $\overset{*}{U} \in \Sigma'$ we have an equation of the form

$$\Gamma_1{}^{r+1}(\overset{*}{U}) = \partial C^{r+2}(\overset{*}{U}) + C^{r+1}(\overset{*}{U}),$$

where $C^{r+2}(\overset{*}{U})$ is on $\overset{*}{U}\wedge\bar{U}_0$ and $C^{r+1}(\overset{*}{U})$ is on $\overset{*}{U}\wedge(\bar{U}_1 - U_4)$. Hence

$$\Gamma^{r+1}(\overset{*}{U}) = \Gamma_1{}^{r+1}(\overset{*}{U}) + \Gamma_2^{r+1}(\overset{*}{U}) = \partial C^{r+2}(\overset{*}{U}) + C^{r+1}(\overset{*}{U}) + \Gamma_2{}^{r+1}(\overset{*}{U}),$$

i.e.,

$$\Gamma^{r+1} = \{\Gamma^{r+1}(\overset{*}{U})\} \sim 0 \text{ mod } \bar{U}_1 - U_4 \text{ on } \bar{U}_0 ,$$

i.e., ϕ^* is univalent.

We now prove that ϕ^* is "on". Suppose that

$$\Gamma^r \in [\Gamma^r] \in \mathcal{H}_c^r(\bar{U}_2 - U_3 | \bar{U}_1 - U_4) .$$

Then Γ^r is on $\bar{U}_2 - U_3 \subseteq \bar{U}_2$; and since $U_2 \subseteq \lambda_c^r(U_1)$ by (i), $\Gamma^r \sim 0$ on $\bar{U}_1$. By Wilder ([13] VII 1.3], there exists a cycle Γ^{r+1} mod $\bar{U}_2 - U_3$ on $\bar{U}_1$ such that $\partial\Gamma^{r+1} \sim \Gamma^r$ on $\bar{U}_2 - U_3$. Therefore, for each $\overset{*}{U} \in \Sigma'$,

$$\partial\Gamma^{r+1}(\overset{*}{U}) \sim \Gamma^r(\overset{*}{U}) \quad \text{on } \overset{*}{U}\wedge(\bar{U}_2 - U_3).$$

But $\partial\Gamma^{r+1}(\overset{*}{U}) - \partial\Gamma_1^{r+1}(\overset{*}{U}) = \partial\Gamma_2^{r+1}(\overset{*}{U})$ (by (ii) above), i.e.,

$$\partial\Gamma^{r+1}(\overset{*}{U}) \sim \partial\Gamma_1^{r+1}(\overset{*}{U}) \text{ on } \overset{*}{U} \wedge (\bar{U}_1 - U_3) ,$$

and so $\Gamma^r \sim \phi\, \Gamma^{r+1}$ on $\bar{U}_1 - U_3$. This proves that ϕ^* is "on" as asserted, and therefore

$$\phi^*: \mathcal{H}_c^{r+1}(U; \bar{U}_0 - U_3 | \bar{U}_0 - U_4) \approx \mathcal{H}_c^r(\bar{U}_2 - U_3 | \bar{U}_1 - U_4) .$$

By the method of LTI 3.2, et. seq. we may therefore write

(v) $$\mathcal{H}_c^{r+1}(M; M - U_3 | M - U_4) \approx \mathcal{H}_c^r(\bar{U}_2 - U_3 | \bar{U}_1 - U_4) .$$

Now suppose that $\mathcal{D}_c^r(x, G)$ exists and is a vector space of dimension $d^r(x, G)$. Besides imposing the conditions of (i) above on

$$U_1, U_2, U_3, U_4 ,$$

let us suppose further that they satisfy the conditions of Definition 3.4, and that U_1, U_2 are the first suitable sets of the form $U(x, 1/n)$. Then we may assume that the R.H.S. in (v) above has dimension $d^r(x, G)$. Define the functions U_κ^{r+1} , $\kappa^{r+1}(U)$ of LTI 3.1 to be, respectively,

$$\delta^r(U_1, U_2), \quad \delta^r(U_1, U_2, U) .$$

Then, since the L.H.S. of (v) is of constant dimension $d^r(x, G)$, it follows that the local Betti number $p^{r+1}(x, G)$ exists at x , and is equal to $d^r(x, G)$, as required.

Conversely, suppose that $p^{r+1}(x, G)$ exists at x and is finite, so that LTI 3.1 is satisfied. If to the conditions of (i) above we add the further conditions that $U_0 = U_\kappa^{r+1}$, $U_3 \subseteq U_\kappa^{r+1}$, and $U_4 \subseteq \kappa^{r+1}(U_3)$, we may assume that the L.H.S. of (v) has dimension $p^{r+1}(x, G)$. Let $U_\delta^r = \lambda_c^{r+1}(U_\kappa^{r+1})$ and $\delta^r(U) = \lambda_c^r(U)$, let $\delta^r(U, U')$ be the first set of the form $U(x, 1/n) \in U'$, and let $\delta^r(U, U', U'') = \kappa^{r+1}(U'')$. Then, since the R.H.S. has now the constant dimension $p^{r+1}(x, G)$, the conditions of Definition 3.4 are satisfied. Therefore $\mathcal{D}_c^r(x, G)$ exists at x , and $d^r(x, G) = p^{r+1}(x, G)$ as required. This completes the proof.

3.9. REMARK. In the proof, we have used the "Existence Lemmas" of Wilder ([13] VII §1). Together, these lemmas state that the Čech homology sequence, with coefficients in a field, is exact at the places we require. Hence our proof of (v) above is valid for every set of coefficients whose homology sequence has the same exactness property.

4. A LEMMA. In order to use integer coefficients, we shall have to prove a lemma concerning Vietoris cycles. First, we give the following useful definition of local connectivity.

4.1. DEFINITION (Begle [1] Def. 1.1.). M is r-lc at x if and only if, given U and $\epsilon > 0$, there exist V and $\eta = \eta^r(U, \epsilon) > 0$ such that every r-cycle on $\bar{V}[\eta]$ is ~ 0 on $\bar{U}[\epsilon]$. M is lc^p at x if and only if it is r-lc at x, $0 \leq r \leq p$; and M is lc^p if and only if it is lc^p at each of its points.

If the coefficients form a ring with a unit, Begle proves in Theorem 3.1, op. cit., that M is lc^r if and only if it is lc^r_v. (Although he assumes M to be compact, his proof requires only obvious modifications when M is locally compact.)

4.2. With U as in 4.1, let the first satisfactory V of the form $U(x, 1/m)$ be denoted by $\lambda^r(U)$. Then our lemma is as follows.

4.3. LEMMA. Let M be both r-lc and (r+1)-lc at x. If U, U_1, U_2, W are neighborhoods such that $U_1 \subseteq \lambda^{r+1}(U)$, $U_2 \subseteq \lambda^r(U)_1$, and $W \Subset U_2$, then for every r-V-cycle Γ^r on $\bar{U}_2 - W$ there exists an r-V-cycle $\psi\Gamma^r$ on $\mathcal{F}W$ such that $\Gamma^r \sim \psi\Gamma^r$ on $\bar{U}_1 - W$.

PROOF. Let $\Gamma^r = \{\Gamma^r(\epsilon_m)\}$ be an r-V-cycle on $\bar{U}_2 - W$. The idea of the proof is this: we shall construct an r-V-cycle $\{\Gamma'(\epsilon'_m)\}$ such that, for each m, $\Gamma^r(\epsilon_m) \sim \Gamma'(\epsilon'_m)$ on $\bar{U}_1 - W$, and $\Gamma'(\epsilon'_m)$ is on $U(\mathcal{F}W, \delta_m)$, $\delta_m \to 0$. We shall then project each $\Gamma'(\epsilon'_m)$ into $\Gamma''(\epsilon''_m)$ (say), on $\mathcal{F}W$, where $\Gamma'(\epsilon'_m) \sim \Gamma''(\epsilon''_m)$ on $\bar{U}_1$-W and $\{\Gamma''(\epsilon''_m)\}$ is also an r-V-cycle, namely the required $\psi\Gamma^r$. The details are as follows.

Let $\eta^r(\epsilon) = \eta^r(U_1, \epsilon)$, $\eta^{r+1}(\epsilon) = \eta^{r+1}(U, \epsilon)$ as in 4.1. The cycle $\Gamma^r = \{\Gamma^r(\epsilon_m)\}$ is an r-V-cycle on $\bar{U}_2 - W$, so that for each $m \geq 1$ we have an equation of the form[5]

(i) $\Gamma^r(\epsilon_m) - \Gamma^r(\epsilon_{m+1}) = \partial D^{r+1}(\delta_m)$ on $\bar{U}_2 - W$; $\epsilon_m \to 0$, $\delta_m \to 0$.

By choosing a subsequence, if necessary, we may suppose that

$$\epsilon_{m+1} \leq \epsilon_m < \eta^r(\delta_m),$$

$$\delta_{m+1} < \eta^{r+1}(\delta_m) \quad (< \delta_m),$$

$$\delta_m < \delta_0/2^{m+1}, \quad \delta_1 < \eta^{r+1}(\delta_0),$$

$$\delta_0 < \alpha = \min(\rho(x, \mathcal{C}W), \rho(\bar{W}, \mathcal{C}U_2)) .$$

Since $U_2 \subseteq \lambda^r(U_1)$, we have for each $m \geq 1$ an equation[5]

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[5] C, D, . . . , with or without suffixes, denote general (finite) chains with coefficients in the group understood. $C^r(\epsilon)$ denotes a chain of dimension r, and of mesh $< \epsilon$. The notation is more fully explained in LTI, 7.7.

(ii) $$\Gamma^r(\epsilon_m) = \partial C^{r+1}(\delta_m) \text{ on } \bar{U}_1 ,$$

so that $C^{r+1}(\delta_{m+1}) - C^{r+1}(\delta_m) + D^{r+1}(\delta_m)$ is an $(r+1)$ - cycle on $\bar{U}_1[\delta_m]$. But $U_1 \subseteq \lambda^{r+1}(U)$, and hence

(iii) $$C^{r+1}(\delta_{m+1}) - C^{r+1}(\delta_m) + D^{r+1}(\delta_m) = \partial C^{r+2}(\delta_{m-1}) \text{ on } \bar{U} .$$

For each $n \geq 1$, define W_n to be $U(W, \alpha/2^n)$, $W_0 = U_2$. Also, let

(iv) $$C^{r+2}(\delta_m) = C^{r+2}_{1,n}(\delta_m) + C^{r+2}_{2,n}(\delta_m) ,$$
$$C^{r+1}(\delta_m) = C^{r+1}_{1,n}(\delta_m) + C^{r+1}_{1,n}(\delta_m) ,$$

where in each equation the suffix "2" denotes the part consisting of all cells whose vertices are all on $\bar{W}_n$. Then, by (ii),

(v) $$\Gamma^r(\epsilon_m) - \partial C^{r+1}_{1,n}(\delta_m) = \partial C^{r+1}_{2,n}(\delta_m) ,$$

and both sides must be on $(\bar{U}_1 - W) \cup U(\mathcal{F} W_n, \delta_m)$. If $m \geq n$, then

$$\delta_m \leq \delta_n < \alpha/2^{n+1},$$

and therefore

(vi) $$\Gamma^r(\epsilon_m) - \partial C^{r+1}_{2,n}(\delta_m) = \partial C^{r+1}_{1,n}(\delta_m) \text{ on } \bar{U}_1 - W.$$

From (iii)

$$C^{r+1}_{2,n}(\delta_{m+1}) - C^{r+1}_{2,n}(\delta_m) - \partial C^{r+2}_{2,n}(\delta_{m-1})$$

$$= \partial C^{r+2}_{1,n}(\delta_{m-1}) - [D^{r+1}(\delta_m) + C^{r+1}_{1,n}(\delta_{m+1}) - C^{r+1}_{1,n}(\delta_m)]$$

$$= \partial C^{r+2}_{1,n}(\delta_{m-1}) - C^*_n(\delta_m), \text{ say,}$$

Since the L.H.S. is on $\bar{W}_n$, so is the R.H.S. But the R.H.S. consists of cells on the boundary of $C^{r+2}_{1,n}(\delta_{m-1})$, and therefore it is on

$$U(\mathcal{F} W_n, \delta_{m-1}) .$$

Hence,

(vii) $$\partial C^{r+1}_{2,n}(\delta_{m+1}) - \partial C^{r+1}_{2,n}(\delta_m) = \partial [\partial C^{r+2}_{1,n}(\delta_{m-1}) - C^*_n(\delta_m)]$$

$$\text{on } U(\mathcal{F} W_n, \delta_{m-1}) .$$

If $m \geq n + 1$, then $\delta_{m-1} \leq \delta_n < \alpha/2^{n+1}$; let

$$\Gamma^r_n = \{ \Gamma_n(\eta_n)\} = \{ \partial C^{r+1}_{2,n}(\delta_m), \ m \geq n + 1 \}.$$

Then we have proved that Γ_n^r is an r-V-cycle on $\overline{W}_n - W_{n+1}$. From (iv),

$$C^{r+1}(\delta_m) = C_{1,n}^{r+1}(\delta_m) + C_{2,n}^{r+1}(\delta_m) = C_{1,n+1}^{r+1}(\delta_m) + C_{2,n+1}^{r+1}(\delta_m),$$

whence

$$C_{2,n}^{r+1}(\delta_m) - C_{2,n+1}^{r+1}(\delta_m) = C_{1,n+1}^{r+1}(\delta_m) - C_{1,n}^{r+1}(\delta_m).$$

If $m \geq n+1$, both sides are on $\overline{W}_n - W_{n+2}$, and so

(viii) $$\partial C_{2,n}^{r+1}(\delta_m) - \partial C_{2,n+1}^{r+1}(\delta_m) = \partial(C_{1,n+1}^{r+1}(\delta_m) - C_{1,n}^{r+1}(\delta_m))$$

on $\overline{W}_n - W_{n+2}$. Therefore $\Gamma_n^r \sim \Gamma_{n+1}^r$ on $\overline{W}_n - W_{n+2}$, while from (vi) $\Gamma^r \sim \Gamma_n^r$ on $\overline{U}_1 - W$. Hence, for each n and all m, we have equations of the form

(ix) $$\Gamma^r(\epsilon_m) - \Gamma_n^r(\eta_m) = \partial C(\zeta_m) \text{ on } \overline{U}_1 - W,$$

(x) $$\Gamma_n^r(\eta_m) - \Gamma_{n+1}^r(\eta_m) = \partial C'(\zeta'_m) \text{ on } \overline{W}_n - W_{n+2},$$

where

$$\lim \zeta_m = \lim \zeta'_m = 0.$$

For each $x \in \overline{U}_1 - W$, choose a point of $\mathcal{F}W$ nearest to x and denote it by $\psi'(x)$. Then, for all $\epsilon > 0$, ψ' defines a transformation ψ of the chains $C(\epsilon)$ on $\overline{U}_1 - W$ into the chains $C(\epsilon')$ on $\mathcal{F}W$, where $\epsilon' \leq 2\delta + \epsilon$ if $C(\epsilon)$ is a chain on $U(\mathcal{F}W, \delta)$. It is easily verified that ψ is simplicial and commutes with ∂. Together with (ix) and (x), the theory of projection prisms then gives equations of the form

(xi) $$\psi\Gamma_m^r(\eta_m) - \psi\Gamma_{m+1}^r(\eta_{m+1}) = \partial C(\beta_m) \text{ on } \mathcal{F}W,$$

(xii) $$\Gamma_m^r(\eta_m) - \psi\Gamma_m^r(\eta_m) = \partial C(\beta'_m) \text{ on } \overline{W}_m - W_m;$$

where $\lim \beta_m = \lim \beta'_m = 0$ since $\lim \alpha/2^{m+1} = \lim \eta_m = 0$.

Hence by (xi), $\psi\Gamma^r = \{\psi\Gamma_m^r(\eta_m)\}$ is an r-V-cycle on $\mathcal{F}W$. By induction on r, $n \leq r \leq m$, (x) and (xii) give

$$\Gamma^r(\epsilon_m) - \psi\Gamma_m^r(\eta_m) = \partial C(\beta''_m) \text{ on } \overline{U}_1 - W,$$

where $\lim \beta''_m = 0$. Therefore

$$\Gamma^r \sim \psi\Gamma^r \text{ on } \overline{U}_1 - W,$$

and $\psi\Gamma^r$ is the cycle we require.

From equation (iii) of the last proof, it can be shewn that the sequence $\{C^{r+1}(\delta_m)\}$ of (ii) is an (r+1)-V-cycle mod $\bar{U}_2 - W$ on $\bar{U}_1$. Thus, with the help of similar techniques, the "Existence Lemmas" of Wilder (see 3.9 above) can be replaced by weaker statements, for any suitable discrete G, in order to prove the "$\dot{G}$" version of Theorem 3.8.

In the last two proofs, we have depended greatly on the process of dividing a chain into two portions - each being still a chain. This process is the geometric counterpart of the Excision property, and we therefore cannot expect strictly analogous homotopy versions of the proofs (see LTI Section 3).

5. "SOURCE AND SINK" THEORY. Roughly speaking, Lemma 4.3 states that, under the given conditions, every r-V-cycle sufficiently near to x is homologous, on a subset of M - x, to an r-V-cycle even nearer to x. On the other hand, 3.1 states that, under the conditions given there, every r-V-cycle sufficiently near to x is homologous, on a subset of M - x, to an r-V-cycle <u>further away</u>. To use a hydrodynamical picture, for the moment: x acts in the first case as a sink for r-V-cycles, and in the second case as a source. We now study the source property further. Throughout, I denotes the group of integers; Q denotes the field of rationals. The following lemma will be useful.

5.1. LEMMA. <u>Let F, F', F'' be compact sets in an $lc_V^p(I)$ separable metric space, and let $\alpha, \beta > 0$ be such that</u>

$$F \subseteq U(F,\alpha) \subseteq F' \subseteq U(F',\beta) \subseteq F'' .$$

<u>Suppose that</u>[6]

$$\mathcal{H}_V^r(F|F',I) \approx \mathcal{H}_V^r(F|F'',I) \approx \overset{*}{M}(k) + \overset{*}{F} .$$

<u>Then the vector space</u> $\mathcal{H}_V^r(F|F'',Q)$ <u>is of dimension</u> k.

PROOF. The proof is divided into five parts.

(A) It will be convenient to let F_α denote[7] $\mathcal{K}U(F,\alpha)$. The proof of Newman [10] Theorem 1 shews that; if G = I or Q, then for every $\delta > 0$ there exists a sequence $\{\epsilon_n\}(n \geq 0)$, depending on F and δ, such that $0 < \epsilon_{n+1} < \epsilon_n/3$, for all n, and such that the following three conditions are fulfilled:

(i) If Z^r is an r-cycle over G on $F[\epsilon_1]$, then Z^r is the first element of an r-V-cycle $\Gamma^r = \{Z^r(\epsilon_{n+1})\}$ over G, on F_δ. Denote Γ^r by $\nu(Z^r)$, $\nu = \nu(G)$. Then, also

$$\nu(Z_1^r + Z_2^r) = \nu(Z_1^r) + \nu(Z_2^r) .$$

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[6] See 1.1.

[7] $\mathcal{K}X$ = closure of X.

(ii) F has a finite open covering by sets of diameter less than $\epsilon_4/6$ with nerve $K(\subseteq F[\epsilon_1])$, such that, if $Z_1^r, \ldots, Z_k^r$ is a homology basis for[8] $H^r(K,G)$, then every r-V-cycle Γ^r over G on F satisfies a homology of the form

$$\Gamma^r \sim \sum_{i=1}^{k} g_i\, \nu(Z_i^r) \text{ on } F_\delta \ ; \ g_i \in G.$$

(iii) A sufficient condition that two r-V cycles $\Gamma_i^r = \{Z_{in}^r\}$ on F $(i = 1,2)$ be homologous on F_δ (over G) is that for all[2] $n \geq 0$,

(a) $$Z_{in}^r - Z_{i,n+1}^r = \partial C_i^{r+1}(\epsilon_{n+2}) \text{ on } U(F,\eta_n),$$

where

$$\eta_n = \sum_{m=0}^{n} \epsilon_m \text{ and } n \geq 0,$$

(b) $$Z_1^r(\epsilon_0) - Z_2^r(\epsilon_0) = \partial C^{r+1}(\epsilon_2) \text{ on } U(F,\epsilon_0).$$

(B) The function $\nu = \nu(G)$ in (i) defines a homomorphism of $H^r(K,G)$ in $\mathcal{H}_v^r(F',G)$. For if $1 \leq i \leq p$ and $Z'^r \sim Z^r$ on K, then since $\epsilon_4/3 = \text{mesh } k < \epsilon_2$, we may apply (iii) with $\delta = \alpha$, to give $\nu Z'^r \sim \nu Z^r$ on $F_\alpha \subseteq F'$. Hence we may without ambiguity define $\nu'[Z^r]$ to be $[\nu Z^r]$ and extend ν', by the additivity of ν in (i), to be the required homomorphism

$$\nu'(G) = \nu' : H^r(K,G) \to \mathcal{H}_v^r(F',G).$$

Now, if Γ^r is an r-V-cycle on F, there holds a relation of the form $\Gamma^r \sim \sum_{i=1}^{k} g_i\, \nu(Z_i^r)$ on $F_\alpha \subseteq F'$, whence

(iv) $$[\Gamma^r] = [\sum_{i=1}^{k} g_i \nu(Z_i^r)] = \nu'[\sum_{i=1}^{k} g_i Z_i^r] .$$

Let i be the injection of $\mathcal{H}_v^r(F|F',G)$ in $\mathcal{H}_v^r(F',G)$. Then by (iv)

$$\nu' H^r(K,G) = i\, \mathcal{H}_v^r(F|F',G).$$

(C) Since K is finite, we may express $H^r(F',I)$ as a direct sum (see 1.1):

$$H^r(K,I) = \overset{*}{M}(p) + \overset{*}{P}.$$

Let a homology basis of $H^r(K,I)$ be the set of cycles

$$Z_1^r, \ldots, Z_p^r, Z_{p+1}^r, \ldots, Z_n^r,$$

where $Z_1^r, \ldots, Z_p^r$ correspond to $\overset{*}{M}(p)$, and the rest to $\overset{*}{P}$. Then it

[8] H denotes the combinatorial group.

may easily be verified that $Z_1^r, \ldots, Z_p^r$ generate $H^r(K,G)$, since K is finite; and moreover than any homology

$$\sum_{i=1}^{p} r_i Z_i^r \sim 0 \text{ over } Q \text{ on } K, \quad r_i \in Q,$$

implies $r_i = 0$, $1 \leq i \leq p$.

(D) It follows from (B) that if N is the kernel of $\nu'(I)$, there exists an isomorphism

$$H^r(K,I)/N \approx \mathcal{H}_v^r(F|F',I),$$

by Noether's Theorem and the fact that i is univalent. We may assume that the basis $Z_1^r, \ldots, Z_n^r$ of $H^r(K,I)$ is chosen so that, for certain integers m_i, $1 \leq i \leq n$, the set $m_1 Z_1^r, \ldots, m_n Z_n^r$ is a basis for N. Since

$$H^r(K,I)/N \approx \overset{*}{M}(k) + \overset{*}{F}$$

by the hypothesis about $\mathcal{H}_v^r(F|F',I)$, we must have $m_i = 0$, $1 \leq i \leq k$, (assuming the Z_i^r suitably numbered).

(E) If $1 \leq i \leq k$ and $\nu_0 = \nu'(I)$, then $\nu_0(Z_i^r) \not\sim 0$ over Q on F'. For suppose on the contrary that $\nu_0(Z_1^r) \sim 0$ over Q. By (i), $\nu_0(Z_1^r)$ is a sequence of the form $\{Z^r(\epsilon_{n+1})\}$; hence for each n there exists a δ_n, $\lim \delta_n = 0$, and a chain $C^{r+1}(\delta_n)$ on $F_\alpha[\delta_n]$, such that $Z^r(\epsilon_{n+1}) = \partial C^{r+1}(\delta_n)$.

In (i) - (iii) above, replace F by F', and δ by β ; let $\{\epsilon'_n\}$ be the corresponding ϵ-sequence. Let α be the common denominator of all coefficients in $C^{r+1}(\delta_0)$. Since $Z_1^r(\epsilon_1)$ is non-zero and has coefficients in I, $\alpha \neq 0$. Let $\{\delta'_n\}$ be a sub-sequence of $\{\delta_n\}$ such that $\delta' < \epsilon'_{n+2}$ for each n; and let $\Gamma' = \{Z^r(\epsilon'_{n+2})\}$ be the corresponding subsequence of $\{Z^r(\epsilon_{n+1})\}$. In (iii), replace G, F, δ, by I, F', β, and take Γ_1^r and Γ_2^r to be $\alpha\Gamma'$ and zero, respectively. Then $\alpha\Gamma' \sim 0$ over I on $F'_\beta \subseteq F''$, and so $\alpha\nu_0(Z_1^r) \sim 0$ over I on F''. But by hypothesis

$$\mathcal{H}_v^r(F|F',I) \approx \mathcal{H}_v^r(F|F'',I) \approx \overset{*}{M}(k) + \overset{*}{F},$$

whence $\alpha\nu_0(Z_1^r) \sim 0$ over I on F', by LTI, 5.3. Now

$$\alpha\nu_0(Z_1^r) \sim \nu_0(\alpha Z_1^r),$$

so that $[\alpha Z_1^r] \in N$; therefore $\alpha = 0$, since $m_1 = 0$. This contradiction establishes the assertion that $\nu_0(Z_i^r) \not\sim 0$ over Q on F', $1 \leq i \leq k$. A similar argument shews that any relation of the form

$$\sum_{i=1}^{k} r_i \nu_0 (Z_i^r) \sim 0 \text{ over } Q \text{ on } F', \ r_i \in Q,$$

implies $r_i = 0$, $1 \leq i \leq k$. Thus, the set $\nu_0(Z_1^r), \ldots, \nu_0(Z_k^r)$ is linearly independent relative to homologies over Q on F'. By (C), $Z_1^r, \ldots, Z_k^r$ form a homology basis for $H^r(K, Q)$.

Since M is $lc^r(I)$, it is $lc^r(Q)$, by 2.5, and we may therefore suppose that $\nu(I)$ and $\nu(Q)$ agree when applied to $Z_1^r, \ldots, Z_k^r$.

Applying (ii) with $G = Q$, we conclude that $\nu_0(Z_1^r), \ldots, \nu_0(Z_k^r)$ form a homology basis for $\mathcal{H}_v^r(F|F', Q)$.

Hence, the vector space $\mathcal{H}_v^r(F|F', Q)$ is of dimension k, as required.

5.2. COROLLARY. Let the r-V-cycles $\Gamma_1^r, \ldots, \Gamma_k^r, \Gamma_{k+1}^r, \ldots, \Gamma_n^r$ on F be any set of generating cycles for $\mathcal{H}_v^r(F|F', I)$, such that $\Gamma_1^r, \ldots, \Gamma_k^r$ correspond to $\overset{*}{M}(k)$. Then $\Gamma_1^r, \ldots, \Gamma_k^r$ are also a set of generating cycles for $\mathcal{H}_v^r(F|F', Q)$.

PROOF. From the first isomorphism in part (D) of the proof above, it follows that $\nu_0(Z_1^r), \ldots, \nu_0(Z_k^r)$ form a set of generating cycles of the direct summand of $\mathcal{H}_v^r(F|F', I)$ corresponding to $\overset{*}{M}(k)$. Hence there exists a unimodular matrix (u_{ij}) of integers such that

$$\Gamma_i \sim \sum_{j=1}^{k} u_{ij} \nu_0(Z_j^r) \text{ over I on } F', \ i = 1, \ldots, k.$$

Hence

$$\Gamma_i \sim \sum_{j=1}^{k} u_{ij} \nu_0(Z_j^r) \text{ over Q on } F', \ i = 1, \ldots, k,$$

and therefore, since $\nu_0(Z_1^r), \ldots, \nu_0(Z_k^r)$ form a set of generating cycles for $\mathcal{H}_v^r(F|F', Q)$, (by part (E) of the last proof), so do $\Gamma_1^r, \ldots, \Gamma_k^r$.

5.3. COROLLARY. If F is the whole ($lc^r(I)$) space, and if

$$\mathcal{H}_v^r(F, I) \approx \overset{*}{M}(k) + \overset{*}{F},$$

then dimension $\mathcal{H}_v^r(F, Q) = K$.

The "$\mathcal{D}$" groups have the advantage of being expressed in terms of compact sets. The "$\mathfrak{C}$" groups, on the other hand, have the advantages of being easier to define, and of being associated with the "source" property of 3.1. We now show that the groups are often interchangeable, and we require the following lemma. Let G be such that 3.6 holds.

5.4. LEMMA. Suppose that M is $lc_v^r(G)$ and that a group $\mathcal{D}_v^r(x, G)$ exists at x. When $U_1 = U_\delta^r$ and $U_2 \subsetneq \delta^r(U_1)$, suppose *that* $V = \delta^r(U_1, U_2)$ *is such that the "source" property* 3.1 *holds, viz: given a cycle* Γ^r *on* $\overline{V} - x$, *there exists a cycle* $\phi\Gamma^r$ *on* $\overline{U}_2 - V$ *such that*

$$\Gamma^r \sim \phi\Gamma^r \text{ on } \overline{U}_1 - x .$$

Then a group $\mathfrak{C}^r_v(x, G)$ *exists, and* $\mathfrak{C}^r_v(x, G) \approx \mathcal{D}^r(x, G)$.

PROOF: Given $U_2' \subseteq V$, choose U_4 such that

$$U_4 \Subset U_2' \cap \delta^r(U_1, U_2, V) .$$

By 3.6, $\mathcal{D}^r_v(x, G)$ is finitely generated, since M is $lc^r_v(G)$. Hence,

$$\mathcal{D}^r_v(x, G) \approx \mathcal{H}^r_v(\overline{U}_2 - V|\overline{U}_1 - U_4, G) \approx \mathcal{H}^r_v(\overline{U}_2 - V|\overline{U}_1 - x, G) ,$$

by the same argument as for 3.2. Let Γ^r_1 and Γ^r_2 be r-V- cycles on $\overline{U}_2' - x \subseteq \overline{V} - x$. Then by hypothesis there exist cycles $\phi\Gamma^r_1$ and $\phi\Gamma^r_2$ on $\overline{U}_2 - V$, such that $\Gamma^r_i \sim \phi\Gamma^r_i$ $(i = 1, 2)$ on $\overline{U}_1 - x$. Hence, if $\Gamma^r_1 \sim \Gamma^r_2$ then $\phi\Gamma^r_1 \sim \phi\Gamma^r_2$ on $\overline{U}_1 - x$, so that we have a homomorphism

$$\phi^* : \mathcal{H}^r_v(\overline{U}_2' - x|\overline{U}_1 - x, G) \to \mathcal{H}^r_v(\overline{U}_2 - V|\overline{U}_1 - x, G),$$

defined by $\phi^*[\Gamma^r] = [\phi\Gamma^r]$. If $\phi^*[\Gamma^r] = 0$, then

$$0 \sim \phi\Gamma^r \sim \Gamma^r \text{ on } \overline{U}_1 - x ,$$

so that ϕ^* is univalent. $\mathcal{H}^r_v(\overline{U}_2' - x|\overline{U}_1 - x, G)$ is therefore isomorphic to a subgroup of the finitely generated Abelian group $\mathcal{D}^r_v(x, G)$. Hence, by LTI, 6.6 (or by an obvious adaptation of it when G is a field), a group $\mathfrak{C}^r_v(x, G)$ exists and is isomorphic to a subgroup of $\mathcal{D}^r_v(x, G)$. But by 3.5 the existence of a finitely generated $\mathfrak{C}^r_v(x, G)$ implies that

$$\mathcal{D}^r_v(x, G) \approx \mathfrak{C}^r_v(x, G),$$

as required, since $\mathcal{D}^r_v(x, G)$ is obviously unique to within isomorphism.

We can now prove the following theorem *with* G *as above*.

5.5. THEOREM. *Let* M *be* $lc^r_v(G)$, *and let* M *be* (r+1)- lc (G) *at* x. *If a group* $\mathcal{D}^r_v(x, G)$ *exists, then so does a group* $\mathfrak{C}^r_v(x, G)$; *and the two are isomorphic*.

PROOF. Since M is $lc^r_v(G)$, in particular it is r-lc(G) at x by Begle ([1] Theorem 3.1). By hypothesis, it is also (r+1) - lc(G) at x. Let $\mathcal{D}^r_v(x, G)$ be given, with associated δ-functions, and let U_1', U_1, U_2 be arbitrary neighborhoods such that

$$U_1 \subseteq \lambda^{r+1}(U_\delta^r) \cap U_1' , \quad U_1 \subseteq U_\delta^r ,$$

$$U_2 \subseteq \lambda^r(U_1') \cap \lambda^r(U_1), \quad U_3 \subseteq \delta^r(U_1, U_2).$$

Let W be any neighborhood such that $W \subseteq U_3$ and let U_3', U_4, U_4' be arbitrary except that $U_4 \subseteq V \cap \delta^r(U_1, U_2, U_3)$ and $U_4' \Subset U_4$, $U_3' \Subset W$. Then by definition

$$\text{(i)} \qquad \begin{aligned} \mathcal{D}_v^r(x, G) &\approx \mathcal{H}_v^r(\bar{U}_2 - U_3 | \bar{U}_1 - U_4, G) \approx \mathcal{H}_v^r(\bar{U}_2 - U_3 | \bar{U}_1' - U_4', G) \\ &\approx \mathcal{H}_v^r(\bar{U}_2 - U_3' | \bar{U}_1 - U_4, G) \approx \mathcal{H}_v^r(\bar{U}_2 - W | \bar{U}_1 - U_4, G) . \end{aligned}$$

Given an r-V-cycle Γ^r on $\bar{U}_2 - U_3$, there exists, by 4.3, an r-V-cycle $\psi\Gamma^r$ on $\mathcal{J}W$ such that $\Gamma^r \sim \psi\Gamma^r$ on $\bar{U}_1 - W \subseteq \bar{U}_1 - U_4 \subseteq \bar{U}_1' - U_4'$. Since ψ is additive, we therefore have a homomorphism

$$\text{(ii)} \qquad \psi^*: \mathcal{H}_v^r(\bar{U}_2 - U_3 | \bar{U}_1 - U_4, G) \to \mathcal{H}_v^r(\mathcal{J}W | \bar{U}_1 - U_4, G) ,$$

defined by the correspondence $\psi^*[\Gamma^r] = [\psi\Gamma^r]$ between cosets. Moreover, if $\psi^*[\Gamma^r] = 0$, then $\psi\Gamma^r \sim 0$ on $\bar{U}_1 - U_4$, i.e., $[\Gamma^r] = 0$, and therefore ψ^* is univalent. Since $\mathcal{J}W \subseteq \bar{U}_2 - U_3$, it follows that:

$$\mathcal{H}_v^r(\mathcal{J}W | \bar{U}_1 - U_4, G) \subseteq \mathcal{H}_v^r(\bar{U}_2 - U_3' | \bar{U}_1 - U_4, G) \subseteq \mathcal{H}_v^r(\bar{U}_1 - U_4, G).$$

Thus from (i) we obtain

$$\text{(iii)} \qquad \begin{aligned} \mathcal{D}_v^r(x, G) &\approx \psi^* \mathcal{H}_v^r(\bar{U}_2 - U_3 | \bar{U}_1 - U_4, G) \\ &\subseteq \mathcal{H}_v^r(\mathcal{J}W | \bar{U}_1 - U_4, G) \\ &\subseteq \mathcal{H}_v^r(\bar{U}_2 - U_3' | \bar{U}_1 - U_4, G) \\ &\approx \mathcal{D}_v^r(x, G) . \end{aligned}$$

Suppose now that G is a field A. Then, by 3.6, $\mathcal{D}_v^r(x, A)$ is a vector space of finite dimension over A; and so it follows from (iii) that

$$\text{(iv)} \qquad \mathcal{H}_v^r(\mathcal{J}W | \bar{U}_1 - U_4, A) = \mathcal{H}_v^r(\bar{U}_2 - U_3 | \bar{U}_1 - U_4, A) ,$$

since a finite dimensional vector space has no proper subspace isomorphic to it. Hence ψ^* is "on", i.e.,

(v). <u>for every</u> r-V-<u>cycle</u> Γ^r <u>over</u> A <u>on</u> $\mathcal{J}W$, <u>there exists</u> <u>a cycle</u> $\theta\Gamma^r$ <u>over</u> A <u>on</u> $\bar{U}_2 - U_3$ <u>such that</u> $\Gamma^r \sim \theta\Gamma^r$ <u>over</u> A <u>on</u> $\bar{U}_1 - U_4$.

Next, let $G = I$. Then $\mathcal{D}_v^r(x, I)$ is finitely generated Abelian, by 3.5. A comparison of ranks and finite parts in (iii) therefore gives

(vi) $$\mathcal{H}_v^r(\mathcal{J}W \mid \bar{U}_1 - U_4, I) \approx \mathcal{D}_v^r(x, I) ,$$

and so the periodic parts of

$$\mathcal{H}_v^r(\mathcal{J}W \mid \bar{U}_1 - U_4, I) \quad \text{and} \quad \psi^* \mathcal{H}_v^r(\bar{U}_2 - U_3' \mid \bar{U}_1 - U_4, I)$$

are equal, since they are finite. If Γ^r is an r-V-cycle on $\bar{U}_1 - U_4$ such that there exists an integer $p \neq 0$ for which $p\Gamma^r \sim 0$ over I on $\bar{U}_1 - U_4$, let us call Γ^r "periodic" on $\bar{U}_1 - U_4$. Since

$$\bar{W} - U_3' \subseteq \bar{U}_1 - U_3 \quad ,$$

the remark after (vi) yields the following result:

(vii) <u>for every integral</u> r-V-<u>cycle</u> Γ^r <u>which is periodic on</u> $\mathcal{J}W$, <u>there exists an integral</u> r-V-<u>cycle</u> $\theta\Gamma^r$ <u>on</u> $\bar{U}_2 - U_3$ <u>such that</u> $\Gamma^r \sim \theta\Gamma^r$ <u>over</u> I <u>on</u> $\bar{U}_1 - U_2$.

If an r-V-cycle on $\bar{U}_1 - U_4$ is not periodic. let us call it "free" on $\bar{U}_1 - U_4$. To extend (vii) to the free cycles on $\bar{U}_2 - U_3'$, we proceed as follows.

Since $\psi^* \mathcal{H}_v^r(\bar{U}_2 - U_3 \mid \bar{U}_1 - U_4, I)$ is a subgroup of the finitely generated Abelian group $\mathcal{H}_v^r(\mathcal{J}W \mid \bar{U}_1 - U_4, I)$ and both are isomorphic to $\mathcal{D}_v^r(x, I)$, there exist cycles $\Gamma_1^r, \ldots, \Gamma_k^r, \Gamma_{k+1}^r, \ldots, \Gamma_n^r$ say, on $\mathcal{J}W$, and non-zero integers $m_1, \ldots, m_n$ such that

$\mathcal{H}_v^r(\mathcal{J}W \mid \bar{U}_1 - U_4, I)$ is generated by $[\Gamma_1^r], \ldots, [\Gamma_n^r]$,

$\psi^* \mathcal{H}_v^r(\bar{U}_2 - U_3 \mid \bar{U}_1 - U_4, I)$ is generated by $[m_1 \Gamma_1^r], \ldots, [m_n \Gamma_n^r]$,

where $\Gamma_1^r, \ldots, \Gamma_k^r$ are free on $\bar{U}_1 - U_4$ and the rest are periodic. For each $i = 1, \ldots, n$, there exists a cycle Δ_i^r on $\bar{U}_2 - U_3$ such that $[m_i \Gamma_i^r] = \psi^*[\Delta_i^r]$, i.e.,

(viii) $$m_i \Gamma_i^r = \psi\Delta_i^r \sim \Delta_i^r \text{ over I on } \bar{U}_1 - U_4 .$$

By applying Corollary 5.2, we may moreover assume that

(ix) $$\mathcal{H}_v^r(\mathcal{J}W \mid \bar{U}_1 - U_4, Q) \text{ is generated by } [\Gamma_1^r]', \ldots, [\Gamma_k^r]' \quad ,$$

and that

$$\psi^* \mathcal{H}_v^r(\bar{U}_2 - U_3 \mid \bar{U}_1 - U_4, Q) \text{ is generated by } [\Delta_1^r]', \ldots, [\Delta_k^r]' \quad ,$$

where the primes refer to cosets over Q . Since Q is a field, and $\mathcal{J}W \subseteq \bar{U}_2 - U_3'$ there exists by (v) a matrix (r_{ij}) of rationals such that

(x) $$\Gamma_i^r \sim \sum_{j=1}^{k} r_{ij} \Delta_j^r \quad \text{over } Q \text{ on } \bar{U}_1 - U_4,\ i = 1, \ldots, k.$$

From (viii) and the fact that $I \subseteq Q$, it follows that

$$\Gamma_i^r \sim \sum_{j=1}^{k} r_{ij} m_j \Gamma_j^r \quad \text{over } Q \text{ on } \bar{U}_1 - U_4,\ i = 1, \ldots, k.$$

But $\Gamma_1^r, \ldots, \Gamma_k^r$ are linearly independent relative to Q-homologies on $\bar{U}_1 - U_4$, by (ix); and therefore

$$r_{ij} m_j = \delta_{ij} \text{ (Kronecker delta)},\ i, j = 1, \ldots, k.$$

Hence, since $m_j \neq 0$, $j = 1, \ldots, k$, we must have

$$m_j = \pm 1, \quad j = 1, \ldots, k;$$

and

$$r_{ij} = \delta_{ij},\ i, j = 1, \ldots, k.$$

We may clearly take the m_j's to be $+1$, so that (viii) and (x) must be identical. More important to us, however, is the fact that both now reduce to

(xi) $$\Delta_i^r \sim \Gamma_i^r \quad \text{over } I \text{ on } \bar{U}_1 - U_4 .$$

so that ψ^* is "on". Since every integral r-V-cycle Γ^r on $\mathcal{F}W$ satisfies a homology of the form

$$\Gamma^r \sim \sum_{i=1}^{n} q_i \Gamma_i^r \quad \text{over } I\ (q_i \in I) \text{ on } \bar{U}_1 - U_4 ,$$

there exists by (vii) and (xi) an integral r-V-cycle $\theta\Gamma^r$ on $\bar{U}_2 - U_3$ such that

(xii) $$\Gamma^r = \psi(\theta\Gamma^r) \sim \theta\Gamma^r \text{ on } \bar{U}_1 - U_4 ,$$

and therefore on $\bar{U}_1 - x$.

Now let G be either a field or I, and let Γ'^r be any r-V-cycle over G on $F \subseteq \bar{U}_3 - x$. Let W be such that $x \in W \subseteq \bar{U}_3 - F$. Then $\Gamma'^r \sim \psi\Gamma'^r$ on $\bar{U}_1 - x$, where $\psi\Gamma'^r$ is on $\mathcal{F}W$. By (v) if G is a field, or by (xii) if $G = I$, there exists an r-V-cycle $\theta(\psi\Gamma'^r)$ over G on $\bar{U}_2 - U_3$, with the property that $\theta(\psi\Gamma'^r) \sim \psi\Gamma'^r$ on $\bar{U}_1 - U_4$. Hence $\Gamma'^r \sim \theta(\psi\Gamma'^r)$ on $\bar{U}_1 - U_4$. But $U_3 = \delta^r(U_1, U_2)$ by definition, so that, if we write $\phi\Gamma'^r = \theta(\psi\Gamma'^r)$, the conditions of Lemma 5.2 are satisfied. Therefore $\mathfrak{C}_V^r(x, G)$ exists at x, and

$$\mathfrak{C}_V^r(x, G) \approx \mathcal{D}_V^r(x, G) .$$

This completes the proof.

We do not know whether or not analogues of 3.5 and 5.5 hold for every group of coefficients.

6. MANIFOLDS. In this section we shall use our local invariants to define manifolds, and we shall then apply the previous results. We should expact a space M to be "smooth" by the standards of, say, the "$\mathfrak{C}$" groups, if and only if the "$\mathfrak{C}$" system cannot detect any difference between M and locally Euclidean space of the same dimension. Guided by Ex. (i) of LTI 6.5, we therefore make the following definitions, in which M is locally compact metric and G is the set of coefficients. The suffix "N" is to be taken to be "c" or "v" according as the Čech or Vietoris forms of the definition are required; and n is non-zero.

6.1. DEFINITION. M is an "n-dimensional[9] homology $\mathfrak{C}_N$ manifold over G," and denoted by $\mathcal{M}^n(\mathfrak{C}_N, G)$, if and only if

1. dim M = n ;
2. M is $lc^n_N(G)$;
3. at each[2] $x \in M$, $\mathfrak{C}^r_N(x, G) \approx \mathcal{H}^r_N(S^{n-1}, G)$, r = 0, 1, 2, ...

6.2. DEFINITION. M is an "n-dimensional homotopy $\mathfrak{C}_{\simeq}$ manifold," denoted by $\mathcal{M}^n(C_{\simeq})$, if and only if

1. dim M = n;
2. M is LC^n;
3. at each $x \in M$, $\mathfrak{C}^r_{\simeq}(x) \approx \pi^r(S^{n-1})$, r = 0, 1, 2, ...

In order to connect with Wilder's work on manifolds, we need the following result:

6.3. LEMMA. Let dim M $\leq$ n, let F be a compact subset of M, and let $\dim_l F$ denote the Lebesgue dimension of F. Then

$$\dim_l F \leq n \ .$$

PROOF: From the relation $F \subseteq M$ it follows, by Hurewicz-Wallman ([19] Thm. III, 1, p. 26), that dim F $\leq$ dim M. Hence, dim F $\leq$ n , and therefore, since F is compact, $\dim_l F \leq n$ (by Thm. V. 8, p. 67, op. cit.).

With I the group of integers, we then have

6.4. THEOREM. An $\mathcal{M}^n(\mathfrak{C}_{\simeq})$ is an $\mathcal{M}^n(\mathfrak{C}_v, I)$.

PROOF: If the locally compact space M is LC^n, then it is $lc^n_v(I)$, by Hurewicz ([18] Thm. 7). For all[2] r and n, $\pi^r(S^{n-1})$ is Abelian, so that by LTI, 7.4, 7.5, and 7.6, $\mathfrak{C}^r_v(x, I)$ exists at each $x \in M$ and is isomorphic to $C^r_{\simeq}(x)$, $0 \leq r \leq n-1$. But

[9] Menger-Urysohn dimension.

$$\mathfrak{C}^r_{\sim}(x) \approx \pi^r(S^{n-1}) \approx \mathcal{H}^r_v(S^{n-1}),\ 0 \leq r \leq n-1 .$$

Hence, it remains to shew that $\mathfrak{C}^r_v(x, I)$ exists and is zero when $r \geq n$.

Let U_1 be a neighborhood of x, with the property that $\bar{U}_1$ is compact. By 6.3, $\dim_1(\bar{U}_1) \leq n$, and so every Čech $(n+1)$-cycle on $\bar{U}_1$ is trivial. Hence M is $(r+1)$ - $lc_c(I)$ at x, and therefore M is $lc^p_c(I)$ at x for all $p \geq 0$. Hence, by 2.2, M is $lc^p_v(I)$, and therefore $lc^p(I)$ at x, (see 41). Suppose that $U_1 \subseteq \lambda^{r+1}(M)$ and $U_2 \subseteq \lambda^n(U_1)$ (where the λ's depend on x, of course). Then, if Γ^n is any n-V-cycle on $F \subseteq \bar{U}_2 - x$, let W be a neighborhood of x such that $W \Subset \bar{U}_2 - F$. By 4.3, there exists an n-V-cycle $\psi\Gamma^n$ on $\mathcal{F}W$, such that $\Gamma^n \sim \psi\Gamma^n$ on $\bar{U}_1 - W \subseteq \bar{U}_1 - x$. Since $\dim M \leq n$, we may assume $\dim \mathcal{F}W \leq n$. Hence $\dim_1 \mathcal{F}W \leq n$, by 6.3, and therefore the Čech group $\mathcal{H}^r_c(\mathcal{F}W)$ is zero. But $\mathcal{H}^r_c(\mathcal{F}W) \approx \mathcal{H}^r_v(\mathcal{F}W)$, by 2.2, and so $\psi\Gamma^n \sim 0$ on $\mathcal{F}W \subseteq \bar{U}_1 - x$. Therefore $\Gamma^n \sim 0$ on $\bar{U}_1 - x$, so that $\mathfrak{C}^n_v(x, I)$ exists and is zero. For any $F \subseteq \bar{U}_1 - x$, $\dim_1 F \leq n$ since $\dim M \leq n$. Hence, $\mathcal{H}^r_c(F) = 0$ for all $r > n$, and therefore we may use 2.1 and 2.3 to get

$$\mathcal{H}^r_c(\bar{U}_1 - x) \approx \mathcal{H}^r_v(\bar{U}_1 - x) = 0 ,$$

whence $\mathfrak{C}^r_v(x, I)$ exists and is zero for all $r \geq n$, as required. This completes the proof.

6.5. Example (iii) of LTI, 6.5, shews that the converse of Theorem 6.4 is false.

6.6. DEFINITION. _Let us call M an $\mathcal{M}^n(\mathcal{D}_N, G)$ if and only if it satisfies the conditions of 6.1, but with "$\mathfrak{C}$" replaced throughout by "$\mathcal{D}$"_ .

From 2.3, 2.2 and 3.7, we have, for any discrete G,

6.8. THEOREM. M _is an_ $\mathcal{M}^n(\mathcal{D}_v, G)$ $[\mathcal{M}^n(\mathfrak{C}_v, G)]$ _if and only if it is an_ $\mathcal{M}^n(\mathcal{D}_c, G)$ $[\mathcal{M}^n(\mathfrak{C}_c, G)]$.

If G is a field or I , then by 3.5 and 5.5 (since an lc^n space of dimension $\leq n$ is lc^m for all $m > n$), we have

6.9. THEOREM. _An_ $\mathcal{M}^n(\mathfrak{C}_v, G)$ _is an_ $\mathcal{M}^n(\mathcal{D}_v, G)$ _and conversely._
Next, we have

6.10. THEOREM. _Every_ $\mathcal{M}^n(\mathcal{D}_v, I)$ is an $\mathcal{M}^n(\mathcal{D}_v, Q)$.

PROOF: If M is an $\mathcal{M}^n(\mathcal{D}_v, I)$, then it is $lc^n_v(I)$ by definition, and therefore $lc^n_v(Q)$ by 2.5. Let $x \in M$, and let r be a fixed integer, $0 \leq r \leq n$. Let U_1', U_1, U_2, U_3, U_4, U_4' be neighborhoods of x satisfying the following conditions (see 3.4):

$$U_1 \Subset U_1' \subseteq U_\delta^r \; ; \; U_2 \subseteq \delta^r(U_1) \cap \delta^r(U_1');$$

$$U_3 \subseteq \delta^r(U_1, U_2) \cap \delta^r(U_1', U_2);$$

$$U_4 \subseteq \delta^r(U_1, U_2, U_3) \cap \delta^r(U_1', U_2, U_3); \quad U_4' \Subset U_4 .$$

Then, by the definition of $\mathcal{D}_v^r(x, I)$ and of $\mathcal{M}^n(\mathcal{D}_v, I)$,

$$\mathcal{H}_v^r(\bar{U}_2 - U_3 | \bar{U}_1 - U_4, I) \approx \mathcal{H}_v^r(\bar{U}_2 - U_3 | \bar{U}_1' - U_4, I) \approx \mathcal{H}_v^r(S^{n-1}, I).$$

In 5.1, take F, F', F'' to be, respectively, $\bar{U}_2 - U_3$, $\bar{U}_1 - U_4$, $\bar{U}_1' - U_4'$ with $\alpha = \rho(\bar{U}_2 - U_3, \not{c}(\bar{U}_1 - U_4))$, $\beta = \rho(\bar{U}_1 - U_4, \not{c}(\bar{U}_1' - U_4'))$.

Then $\mathcal{H}_v^r(\bar{U}_2 - U_3 | \bar{U}_1 - U_4, Q) \approx \mathcal{H}_v^r(S^{n-1}, Q)$, and therefore $\mathcal{D}_v^r(x, Q)$ exists at x and is isomorphic to $\mathcal{H}_v^r(S^{n-1}, Q)$. Hence M is an $\mathcal{M}^n(\mathcal{D}_v, Q)$, as required.

6.11. <u>The converse is false</u>. For let P^{2n+1} be a $(2n+1)$-dimensional projective space, let $\hat{P}^{2n+1}$ be the cone of base P^{2n+1} and vertex q, and let $\hat{P}_o^{2n+1}$ be a copy of $\hat{P}^{2n+1}$ with vertex q_o. Let

$$M = \hat{P}^{2n+1} \cup \hat{P}_o^{2n+1} ,$$

where the bases of $\hat{P}^{2n+1}$ and $\hat{P}_o^{2n+1}$ are identified. Then M is compact and orientable over I and over Q. At all points except q and q_o M is locally Euclidean, and at q, q_o it is locally contractible.

It can be shewn that at q the groups $\mathcal{D}_v^r(q, G)$ exist for all G and $r = 0, 1, 2, \ldots$, and that

$$\mathcal{D}_v^r(q, G) \approx \mathcal{H}_v^r(P^{2n+1}, G) .$$

Similarly at q_o. Now[2], $\mathcal{H}_v^0(P^{2n+1}, I) = 0$. $\mathcal{H}_v^r(P^{2n+1}, I)$ is cyclic of order 2 if r is odd, and $\mathcal{H}_v^r(P^{2n+1}, I)$ is zero if r is even, while $\mathcal{H}_v^{2n+1}(P^{2n+1}, I)$ is cyclic infinite (see, e.g., Seifert-Threlfall [11]p.119). Hence, M is not an $\mathcal{M}^{2n+1}(\mathcal{D}_v, I)$. However, by 5.1, the above statements about $\mathcal{H}_v^r(P^{2n+1}, I)$ yield

$$\mathcal{H}_v^r(P^{2n+1}, Q) = 0, \quad r = 0, 1, \ldots, 2n;$$

$$\mathcal{H}_v^r(P^{2n+1}, Q) \approx Q ,$$

and therefore, for all $r \geq 0$,

$$\mathcal{D}_v^r(q, Q) \approx \mathcal{D}_v^r(q_0, Q) \approx \mathcal{H}_v^r(P^{2n+1}, Q) \approx \mathcal{H}_v^r(S^{2n+1}, Q) .$$

This proves that M is an $\mathcal{M}^{2n+1}(\mathcal{D}_v, Q)$, and the required counter-example is established.

Let us call M an $\mathcal{M}^n(p, G)$ if and only if it is an n-gm in the sense of Wilder ([13] VIII p. 284), and is separable metric. Then

6.12. THEOREM. <u>If</u> G <u>is a field, then</u> M <u>is an</u> $\mathcal{M}^n(\mathcal{D}_c, G)$ <u>if and only if</u> M <u>is an</u> $\mathcal{M}^n(p, G)$.

<u>Proof</u>: By definition, an $\mathcal{M}^n(\mathcal{D}_c, G)$ is $lc_c^n(G)$, and in particular $lc_c^o(G)$. Hence, if $x \in M$, it is easily seen that[2] $p^o(x, G) = 0$. If r is fixed, $0 < r < n$, then $\mathcal{D}_c^r(x, G)$ exists and is $\approx \mathcal{H}_c^r(S^{n-1}, G)$; it is therefore of finite dimension $d^r(x, G)$. From the fact that M is r-$lc_c(G)$ and $(r+1)$-$lc_c(G)$ at x, it follows by 3.8 that $p^{r+1}(x, G)$ exists at x and $p^{r+1}(x, G) = d^r(x, G)$. Hence M is an $\mathcal{M}^n(p, G)$. Using 3.8 in the opposite direction, we prove similarly that an $\mathcal{M}^n(p, G)$ is an $\mathcal{M}^n(\mathcal{D}_c, G)$. This completes the proof.

If M is compact, it is <u>orientable</u> over G if and only if $\mathcal{H}_v^n(M, G) \approx G$. We shall require further that M satisfy condition "D" of Wilder ([13] VIII 3, p. 250), viz:

6.13. <u>If</u> F <u>is a proper closed subset of</u> M, <u>then</u> $\mathcal{H}_c^n(F | M, G) = 0$.

For such a space M, we can prove a weak form of the Poincaré Duality Theorem, as follows. If $\mathcal{H}_v^r(M, I) \approx \overset{*}{M}(k) + \overset{*}{F}$, we define $p^r(M, I)$ to be k; if $\mathcal{H}_v^r(M, G)$ is a vector space over G, we define $p^r(M, G)$ to be dimension $(\mathcal{H}_v^r(M, G))$.

6.14. THEOREM. <u>If</u> M <u>is a compact and orientable</u> $\mathcal{M}^n(\mathcal{D}_v, I)$ <u>satisfying</u> 6.13, <u>then</u>

$$p^r(M, I) = p^{n-r}(M, I), \quad 0 < r < n ,$$

$$p^n(M, I) = p^o(M, I) + 1 = 1 .$$

<u>Proof:</u> By 5.1, $p^r(M, I) = p^r(M, Q)$, $0 \leq r \leq n$, and by 2.2, $p^r(M, Q)$ is equal to its Čech analogue $p^r(M, Q)_c$. Since M is an $\mathcal{M}^n(\mathcal{D}_v, I)$, it is an $\mathcal{M}^n(\mathcal{D}_v, Q)$ by 6.10, and therefore an $\mathcal{M}^n(\mathcal{D}_c, Q)$ by 6.8. Hence, M is an $\mathcal{M}^n(p, Q)$, by 6.12.

If F is a compact proper subset of M, then $\mathcal{H}_v^r(F | M, I) = 0$, by 6.13. Therefore, by 5.1 and 2.4, $\mathcal{H}_c^n(F | M, Q) = 0$. Hence, since Q is a field, we may apply, e.g., Wilder ([13] VIII 4.2, p. 253), to get

$$p_u^{n-r}(M, Q)_c = p_u^r(M, Q)_c, \quad 0 \leq r \leq n,$$

where the "u" shews that unaugmented complexes are to be taken. Now

$$p_u^r(M, Q)_c = p^r(M, Q)_c \text{ when } r \neq 0,$$

while

$$p_u^o(M, Q)_c = p^o(M, Q)_c + 1.$$

Therefore, using the remark at the beginning of the proof, we have, finally,

$$p^r(M, I) = p^{n-r}(M, I), \quad 0 < r < n,$$

and

$$p^n(M, I) = p^o(M, I) + 1 = 1,$$

as required.

6.15. COROLLARY. By 6.4, an $\mathcal{M}^n(\mathbb{C}_\sim)$ is an $\mathcal{M}^n(\mathbb{C}, I)$. Hence: The relations of 6.14 hold for every $\mathcal{M}^n(C_\sim)$.

We are unable to prove anything about the torsions of M, even when M is an $\mathcal{M}^n(C_\sim)$.

By 6.12, a locally compact $\mathcal{M}^n(\mathcal{D}_c, Q)$ is an $\mathcal{M}^n(p, Q)$. Hence, applying 6.10 and Wilder ([13] IX 1.2, 5.5), we have

6.16. THEOREM. A compact $\mathcal{M}^1(\mathcal{D}_v, I)$ is a 1-sphere.

6.17. THEOREM. An $\mathcal{M}^2(\mathcal{D}_v, I)$ is locally Euclidean.

From 6.5, we see that 6.17 has no higher-dimensional analogue. However, we have not been able to settle the following two problems.

6.18. Let M be an $\mathcal{M}^n(\mathbb{C}_v, I)$, such that, at each $x \in M$, M is 1-LC and $\mathbb{C}^1_\sim(x)$ exists and is zero. Is M an $\mathcal{M}^n(C_\sim)$?

6.19. Is every $\mathcal{M}^n(C_\sim)$ locally Euclidean?

7. AVOIDABILITY. If, in our locally compact space M, $\mathcal{D}_v^r(x)$ exists and is zero, then we are reminded of the situation where x is a "completely r-avoidable" point in the sense of Wilder ([13] VII 6.12). For convenience we recall the definition.

7.1. DEFINITION. The point $x \in M$ is said to be a completely r-avoidable point, if and only if, given U, there exist V and W such that $x \in W \Subset V \Subset U$, and such that every Čech r-cycle on $\mathcal{J}V$ is ~ 0 on $\bar{U} - W$.

The definition states that[2] $\mathcal{H}_c^r(\mathcal{J}V | \bar{U} - W) = 0$. Hence, by 2.4, $\mathcal{H}_v^r(\mathcal{J}V | \bar{U} - W) = 0$, and consequently the Čech and Vietoris forms of 7.1 are equivalent.

7.2. It is easily seen that $\mathcal{D}_v^r(x) = 0$ if and only if there exist neighborhood functions $\delta_o^r(U)$ and $\delta_o^r(U, U')$ at x, with the property that for each pair U_1 and U_2 satisfying the condition $U_2 \subseteq \delta_o^r(U_1)$, every r-V-cycle on[7] $\mathcal{K}(\delta_o^r(U_1)) - U_2$ bounds on $\bar{U}_1 - \delta_o^r(U_1, U_2)$. (Cf. LTI, 4.2).

7.3. THEOREM. If M is r-lc and (r+1)-lc at x, then $\mathcal{D}_v^r(x) = 0$ if and only if x is a completely r-avoidable point.

Proof: Suppose that $\mathcal{D}_v^r(x) = 0$. Given U_1, let $V = \delta_o^r(U_1)$; and given $U_2 \subseteq V_1$, let $V_2 = \delta_o^r(U_1, U_2)$. Then $\mathcal{H}_v^r(\bar{V}_1 - U_2 | \bar{U}_1 - V_2) = 0$ by hypothesis. But $\mathcal{F}V = \bar{V}_1 - V_1 \subseteq \bar{V}_1 - U_2$, so that $\mathcal{H}_v^r(\mathcal{F}V_1 | \bar{U}_1 - U_2) = 0$ also. Therefore, if U_2 is the first set of the form $U(x, 1/n) \subseteq V_1$, then U_1, V_1, V_2 are the U, V, W of 7.1. Hence, x is a completely r-avoidable point.

Conversely, if x is a completely r-avoidable point, define $\delta_o^r(U_1)$ to be $\lambda^r(\lambda^{r+1}(U_1)) = V_1$, and let $U_2 \subseteq V_1$ be given. Let V and W be obtained by putting $U = U_2$ in 7.1, and define $\delta_o^r(U_1, U_2)$ to be W. Then, if Γ^r is an r-V-cycle on $\bar{V}_1 - U_2 \subseteq \bar{V}_1 - V$, there exists by 4.3 an r-V-cycle $\psi\Gamma^r$ on $\mathcal{F}V$, with the property that $\Gamma^r \sim \psi\Gamma^r$ on $\bar{U}_1 - V$. By the avoidability property, $\psi\Gamma^r \sim 0$ on $\bar{U}_2 - W$, and therefore

$$\Gamma^r \sim 0 \text{ on } (\bar{U}_1 - V) \cup (\bar{U}_2 - W) \subseteq \bar{U}_1 - W .$$

Hence,

$$\mathcal{H}_v^r(\bar{V}_1 - U_2 | \bar{U}_1 - V_2) = 0 ,$$

i.e., $\mathcal{D}_v^r(x) = 0$, and the proof is complete.

The last theorem shews that the groups $\mathcal{D}_v^r(x)$ are, in a sense, a generalization of the concept of complete r-avoidability. A strict generalization would have retained the frontier set $\mathcal{F}V$, and the proof of 3.8 could easily have been modified to obtain the appropriate form of the theorem. However, sets of the form $\mathcal{F}V$ are particularly troublesome when homotopy is involved.

We shewed in 6.11 that when the coefficient set G is a field, the $\mathcal{M}^n(\mathcal{D}_v, G)$ spaces are identical with the manifolds $\mathcal{M}^n(p, G)$. These latter manifolds have been characterized in terms of avoidability when they are orientable and compact (White [20]); but our definition of an $\mathcal{M}^n(\mathcal{D}_v, G)$ has the advantage of being independent of the global properties of the space.

For any set of coefficients, we have the following result.

7.4. THEOREM. If $\mathcal{D}_v^r(x) = 0$, then $\mathfrak{C}_v^r(x) = 0$.

Proof: Let the neighborhood U_1 of x be given, and define $\gamma^r(U_1)$ to be $\delta_o^r(U_1)$ (see 7.2). We assert that $\gamma^r(U)$ is the function required for $\mathfrak{C}_V^r(x)$ to exist. For, given $U_2 \subseteq \gamma'^r(U_1)$ let Γ^r be an r-V-cycle on $F \subseteq \bar{U}_2 - x$. Then there exists U_3 such that $x \in U_3 \subseteq \bar{U}_2 - F$, i.e., $F \subseteq \bar{U}_2 - U_3$. Hence, by hypothesis, $\Gamma^r \sim 0$ on $U_1 - \delta_o^r(U_1, U_3)$, i.e., on a compact subset of $\bar{U}_1 - x$. Therefore

$$\mathcal{H}_V^r(\bar{U}_2 - x | \bar{U}_1 - x) = 0,$$

and so $\mathfrak{C}_V^r$ exists and is zero as required.

7.5. A sort of converse of 7.4 follows by putting $\mathfrak{C}_V^r(x) = 0$ in 3.5; we require G to be such that a theorem of the form 2.6 holds.

From now on, let $G = I$ or $G = Q$. If $\mathfrak{C}_V^r(x, G)$ exists and is finitely generated or of finite dimension, then property 3.1 holds. This property is reminiscent of the following definition (Wilder [13] IX 6.3).

7.6. DEFINITION. M <u>is locally</u> r-<u>avoidable at</u> x "<u>in the relative sense</u>" <u>if and only if there exists a neighborhood</u> P <u>of</u> x, <u>such that for each</u> $U \subset P$ <u>there exist</u> V <u>and</u> W <u>with</u> $x \in W \Subset V \Subset U$, <u>and with the property that every</u> r-V-<u>cycle on</u> $\mathcal{F}V$ <u>is</u> ~ 0 mod M-P <u>on</u> M-W.

Then we have

7.7. THEOREM. <u>Let</u> M <u>be</u> $lc_V^r(G)$. <u>If</u> $\mathfrak{C}_V^r(x, G)$ <u>exists and is finitely generated or of finite dimension, then</u> M <u>is locally</u> r-<u>avoidable at</u> x <u>in the relative sense</u> (<u>with respect to</u> G).

<u>Proof:</u> We prove the theorem when $G = I$, since the changes required when $G = Q$ will then be obvious. Take the neighborhoods U_1 and U_2 of 3.1 to be the first suitable sets of the form $U(x, 1/m)$. Then we assert that the "V" of 3.1 is the "P" required in 7.6. For let the neighborhoods U and V' be given, such that

$$x \in V' \Subset U \Subset P = V,$$

and let $\alpha = 1/2 \min [\rho(\mathcal{F}V, x), \rho(\mathcal{F}V, \mathcal{C}U)]$. Since M is locally compact, we may assume $\bar{U}$ to be compact, and so $\alpha > 0$. Hence, if $\mathcal{F}V = F^*$ and $F^*_\alpha = \mathcal{K}U(F, \alpha)$, then by 2.6 $\mathcal{H}_V^r(F^* | F^*_\alpha, I)$ is finitely generated, say by the elements $[\Gamma_1'^r], \ldots, [\Gamma_n'^r]$. Let the r-V-cycle $\Gamma_i'^r$ on F^* be a representative of $[\Gamma_i'^r]$, $1 \leq i \leq n$. Then, by the properties of V, there exists a compact set $F_i' \subseteq \bar{U}_2 - x$ such that, for suitable integers n_{ij} $(1 \leq j \leq k)$,

$$\Gamma_i'^r \sim \sum_{j=1}^{k} n_{ij} \Gamma_j^r \quad \text{on } F_i',$$

where the Γ_j^r are as in 3.1. Let W be the first set of the form

$U(x, 1/m)$ such that

$$W \in U(x, \alpha) - \bigcup_{i=1}^{n} F_i'$$

Then V' and W are the V and W required by 7.6. To prove this, let Γ^r be any r-V-cycle on F^*. Then there exist integers m_i, $1 \leq i \leq n$, such that

$$\Gamma^r \sim \sum_{i=1}^{n} m_i \Gamma_i'^r \quad \text{on } F_a^* ,$$

and therefore

$$\Gamma^r \sim \sum_{i=1}^{n} \sum_{j=1}^{k} m_i n_{ij} \Gamma_j^r \quad (= \Gamma^*, \text{ say})$$

on

$$\bigcup_{i=1}^{n} F_i' \cup F_a^* \subseteq U_2 - W \subseteq M - W .$$

But Γ^* is on

$$\bigcup_{i=1}^{k} F_i \subseteq \bar{U}_3 - V = \bar{U}_3 - P \subseteq M - P,$$

where the F_i are as in 3.1. Hence

$$\Gamma^r \sim 0 \mod M - P \text{ on } M - W ,$$

and therefore M is locally r-avoidable in the relative sense at x, with respect to G. This completes the proof.

A similar proof may be given by means of Čech cycles over a field. If G is a field, then by 6.8, 6.9, and 6.11 the $\mathfrak{M}^n(\mathfrak{C}_V, G)$'s, $\mathfrak{M}^n(\mathfrak{D}_c, G)$'s and $\mathfrak{M}^n(p, G)$'s are all identical; and so by 7.7, the n-gm's are at each point locally r-avoidable in the relative sense, if $r \neq n - 1$.

BIBLIOGRAPHY

14. E.G. Begle, Duality theorems for generalised manifolds, Amer. J. Math. vol. 67 (1945) pp. 59-70.

15. ___________, The Vietoris mapping theorem, Ann. of Math. vol. 51 (1950) pp. 534-543.

16. E. Čech. Sur la connexité locale d'ordre supérieure, Compositio Math. vol. 2 (1935) pp. 1-25.

17. H. B. Griffiths. Local topological invariants, Proc. London Math. Soc., ser. 3 vol. 3 (1953) pp. 350-367.

18. W. Hurewicz, Homotopie, Homologie, und lokaler Zusammenhang, Fund. Math. vol. 25 (1935) pp. 467-485.

19. ___________, and H. Wallman, Dimension Theory, Princeton, 1948.

20. P. A. White, On the equivalence between avoidability and co-local connectedness. An. da Acad. Brasileira de Ciencas vol. 19 (1947) pp. 143-151.

See also Footnote 1.

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