

# COMMUTANTS OF SHIFTS ON BANACH SPACES

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Among the operators on an arbitrary complex Banach space, we consider a class of operators that can be represented as certain unilateral shift operators on a Banach space of sequences. We associate an analytic structure with each of these operators in such a way that each element of the Banach space may be expressed as an analytic function on a neighborhood in the spectrum of the operator. This identification enables us to view the commutant of the operator as an algebra of multiplications by bounded analytic functions, thus giving us a commutant theory similar to that for the unilateral shift on the sequence space  $\ell^2$  [5, Problem 116].

For the special case in which the shift operator is an isometry, sharper estimates on the size of the convergence set for the commutant are available than for the general case. We show that in this special case, the commutant can be identified with a subalgebra of the space  $H^\infty$  on the unit disk.

We apply the theory developed to a commutant problem of A. L. Shields and L. J. Wallen [12], and we make other applications to results on factorization of power series with coefficients in  $\ell^p$ , the spectrum of a shift, and a question of existence of roots. In the last section, we discuss special cases of an approximation theory for elements of the commutant.

In addition to the work of Shields and Wallen on commutant problems, we mention the work of R. Gellar [2], [3], which has some structural similarity to the present work. Gellar also utilizes power series and considers commutant problems. The main distinction, however, is that Gellar starts with a particular Schauder basis and considers weighted shifts with respect to that basis. These shifts are occasionally included in the collection of shifts considered in our work. Moreover, a large collection of the shifts considered here are not of the type considered by Gellar. A further distinction is that our definition of a shift is basis-free, and indeed we carry out our work independent of the existence of a Schauder basis.

We wish to thank Professor Allen L. Shields for sending a preprint of [12], and for a valuable conversation concerning that paper.

## 1. PRELIMINARIES

A standard definition is that a unilateral shift on a Hilbert space is an operator  $U$  for which there exists an orthonormal basis  $\{e_n\}_{n=1}^\infty$  such that  $Ue_n = e_{n+1}$  for  $n = 1, 2, 3, \dots$ . Unilateral shifts on Hilbert space have been extensively studied, and many of their properties are well known ([5, Chapter 14] and [6, Chapter 7]). Much of the utility of such an operator derives from its unitary equivalence to the operator  $S_z$  of multiplication by  $z$  on the Hilbert space of square-summable power series  $\sum_{n=0}^\infty a_n z^n$  (the classical Hardy space  $H^2$ ). Thus an inherent analytic structure is associated with the unilateral shift, and this structure facilitates the study of a

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number of problems, such as that of identifying the commutant of the unilateral shift on  $\ell^2$ .

It is known [6, p. 109] that an operator  $T$  on a Hilbert space  $X$  is a unilateral shift if and only if

( $\alpha$ )  $T$  is an isometry,

( $\beta$ )  $T$  has co-rank one,

( $\gamma$ ) if  $x \in X$  and  $x \neq 0$ , then  $x$  is not infinitely divisible by  $T$ . (A vector  $x \in X$  is *infinitely divisible* by  $T$  if for each nonnegative integer  $k$ , there exists a vector  $y \in X$  such that  $x = T^k y$ .)

The conditions ( $\alpha$ ), ( $\beta$ ), and ( $\gamma$ ) may be imposed on an operator on a Banach space, and they could serve as a definition of a shift operator on a Banach space. For many purposes, however, condition ( $\alpha$ ) is too restrictive, and it is desirable that we use the weakened form

( $\alpha'$ )  $T$  is injective and has closed range.

*Definition.* A bounded linear operator  $T$  on a complex Banach space  $X$  will be called a *shift isometry* provided conditions ( $\alpha$ ), ( $\beta$ ), and ( $\gamma$ ) are satisfied. If conditions ( $\alpha'$ ), ( $\beta$ ), and ( $\gamma$ ) are satisfied, the operator will be called a *shift*.

The use of the term "shift" for an operator satisfying ( $\alpha'$ ), ( $\beta$ ), and ( $\gamma$ ) is further justified by the following theorem.

**THEOREM 1.** *Suppose  $T$  is a shift on the Banach space  $X$ . Then there exists a Banach space  $X_S$  of complex-valued sequences such that  $X$  is isomorphic and isometric to  $X_S$ , and such that on  $X_S$  the operator  $T$  corresponds to the unilateral shift operator  $T_S$  defined by the condition*

$$T_S(a_0, a_1, \dots) = (0, a_0, a_1, \dots).$$

*Proof.* Conditions ( $\alpha'$ ) and ( $\beta$ ) imply the existence of an element  $x_0 \in X$  with  $\|x_0\| = 1$  such that  $X$  is the Banach space direct sum

$$X = (x_0) \oplus TX,$$

where  $(x_0)$  is the one-dimensional subspace containing  $x_0$ . Let  $x \in X$ . Then there exist a complex scalar  $a_0(x)$  and an element  $x_1 \in X$  such that  $x = a_0(x)x_0 + Tx_1$ . Similarly, there exist a scalar  $a_1(x)$  and a vector  $x_2 \in X$  such that  $x_1 = a_1(x)x_0 + Tx_2$ , so that  $x = a_0(x)x_0 + a_1(x)Tx_0 + T^2x_2$ . By induction, there exist unique sequences  $\{a_n(x)\}_{n=0}^\infty$  and  $\{x_n\}_{n=0}^\infty$  of scalars and vectors such that for each positive integer  $n$ ,

$$(1) \quad x = a_0(x)x_0 + a_1(x)Tx_0 + \dots + a_n(x)T^n x_0 + T^{n+1}x_{n+1}.$$

We let  $X_S$  denote the space of sequences  $\{a_n(x)\}_{n=0}^\infty$ . The mapping  $x \rightarrow \{a_n(x)\}_{n=0}^\infty$  is linear and maps  $X$  onto  $X_S$ . By condition ( $\gamma$ ), no vector other than the zero vector maps onto the zero sequence. Thus the correspondence is an isomorphism.

Let  $\|\{a_n(x)\}_{n=0}^\infty\|$  be defined as  $\|x\|$ . Then the two vector spaces are isometric, and  $X_S$  is a Banach space.

Equation (1) implies that

$$Tx = a_0(x)Tx_0 + a_1(x)T^2x_0 + \dots + a_n(x)T^{n+1}x_0 + T^{n+2}x_{n+1},$$

so that the corresponding sequence for  $Tx$  is  $\{0, a_0(x), a_1(x), \dots\}$ . Thus  $T$  corresponds to the unilateral shift operator  $T_S$  on  $X_S$ .

*Remarks.* 1. For convenience, the notation  $a_n(x)$  will frequently be shortened to  $a_n$ .

2. The formula (1) for  $x$  has the same structure as the familiar Taylor formula for a function. In fact, if  $T$  is the operator  $S_z$  of multiplication by  $z$  on the square-summable power series, with  $x_0 = 1$  (the constant function 1), then (1) becomes

$$(2) \quad f(z) = a_0 + a_1z + \dots + a_nz^n + z^{n+1}f_{n+1}(z),$$

and this is the usual Taylor formula for the function  $f$ . Equation (1) will be called the *Taylor formula* for  $x$ , and the coefficients  $a_n(x)$  will be called the *Taylor coefficients*.

3. A further consequence of the closed range of  $T$  is that for each nonnegative integer  $n$ , the linear functional  $x \rightarrow a_n(x)$  is continuous. This is easily proved by means of the open-map theorem.

*Examples.* (a) If  $X = \ell^p$  ( $1 \leq p \leq \infty$ ), then the shift operator

$$T(a_0, a_1, \dots) = (0, a_0, a_1, \dots)$$

is a shift isometry.

(b) If  $A$  is the disk algebra, in other words, the sup-norm algebra of functions that are continuous on the disk  $\{z: |z| \leq 1\}$  and analytic for  $|z| < 1$ , and if  $S_z$  is the operation of multiplication by  $z$  (the function  $\phi$  defined by  $\phi(z) = z$ ), then  $S_z$  is a shift isometry on  $A$ . In this example, the Taylor formula for a function  $f \in A$ , which is also given by equation (2), does not lead to a series converging to  $f$  on the disk  $\{z: |z| \leq 1\}$ . Not every function  $f \in A$  is given by a power series convergent for  $|z| \leq 1$ .

## 2. THE ANALYTIC STRUCTURE

In general, the partial sums occurring in the Taylor formula for  $x$  need not converge to  $x$  in the norm topology of  $X$ . However, we shall see that the sums are associated with an analytic structure. For this purpose, we employ a lemma that A. M. Gleason used to obtain an analytic structure in certain commutative Banach algebras.

LEMMA (Gleason [4, Prop. 1.5]). *Let  $X_1$  and  $X_2$  be complex Banach spaces, and let  $\mathcal{B}(X_1, X_2)$  be the set of all bounded linear operators from  $X_1$  into  $X_2$ . Suppose that  $A \in \mathcal{B}(X_1, X_2)$  and that  $A$  is bijective. Then*

$$\|A^{-1}\|^{-1} = \inf \{ \|U\| : U \in \mathcal{B}(X_1, X_2) \text{ and } A - U \text{ is not surjective} \}.$$

Again, let  $T$  be our shift operator on  $X$ . Since  $T$  is injective and has closed range, it has an inverse in  $\mathcal{B}(X, TX)$ , which we denote by  $T^{-1}$ . Following along lines similar to Gleason's, we define the norm in  $X \oplus \mathbb{C}$  (where  $\mathbb{C}$  is the field of complex numbers) as

$$(3) \quad \|x \oplus \beta\| = \max \{ \|x\|, \|T^{-1}\| |\beta| \},$$

and we define  $A_T: X \oplus \mathbb{C} \rightarrow X$  by

$$A_T(x \oplus \beta) = Tx + \beta x_0.$$

The significance of the factor  $\|T^{-1}\|$  in (3) will become apparent in the proof of Theorem 3. The operator  $A_T$  is easily seen to be injective, since  $T$  is injective. Since  $X = (x_0) \oplus TX$ , the operator  $A_T$  is also surjective, and thus the lemma applies to  $A_T$ .

For  $r > 0$ , let  $D(r)$  be the open disk in  $\mathbb{C}$  of radius  $r$ , centered at the origin. Also, if  $T$  is an operator on  $X$  and  $\lambda \in \mathbb{C}$ , let  $T_\lambda$  be the operator  $T - \lambda I$ .

**THEOREM 2.** *Let  $r(T) = \|A_T^{-1}\|^{-1}$ . Then for  $\lambda \in D(r(T))$ , the operator  $T_\lambda$  is also a shift on  $X$ , and*

$$(4) \quad X = (x_0) \oplus T_\lambda X.$$

Moreover, if  $x \in X$  and  $\{a_n\}_{n=0}^\infty$  is the sequence of Taylor coefficients for  $x$ , then  $\sum_{n=0}^\infty a_n \lambda^n$  converges, and the representation for  $x$  induced by (4) has the form

$$(5) \quad x = \left( \sum_{n=0}^\infty a_n \lambda^n \right) x_0 + T_\lambda y_\lambda,$$

for some  $y_\lambda \in X$ .

*Proof.* Suppose  $\lambda \in D(r(T))$  and  $U: X \oplus \mathbb{C} \rightarrow X$  is defined by

$$U(x \oplus \beta) = \lambda x, \quad \text{for } x \in X, \beta \in \mathbb{C}.$$

Then  $\|U\| \leq |\lambda| \sup \{\|x\|: \|x \oplus \beta\| = 1\} \leq |\lambda| < \|A_T^{-1}\|^{-1}$ , and by the lemma,  $A_T - U$  is surjective. Therefore

$$X = (A_T - U)(X \oplus \mathbb{C}) = \{T_\lambda x + \beta x_0 \mid x \in X \text{ and } \beta \in \mathbb{C}\},$$

and hence  $X = (x_0) \oplus T_\lambda X$ . It remains to show that the sum is a Banach-space direct sum.

Let us again consider the Taylor formula for  $x$ . Since  $x = A_T(x_1 \oplus a_0)$ , we have the inequality  $\|x_1 \oplus a_0\| \leq \|A_T^{-1}\| \|x\|$ . Thus

$$\|x_1\| \leq \|A_T^{-1}\| \|x\| \quad \text{and} \quad |a_0| \|T^{-1}\| \leq \|A_T^{-1}\| \|x\|.$$

Since  $x_n = A_T(x_{n+1} \oplus a_n)$ , it follows from an induction argument that

$$\|x_{n+1} \oplus a_n\| \leq \|A_T^{-1}\| \|x_n\| \leq \|A_T^{-1}\|^{n+1} \|x\|,$$

and hence

$$\|x_{n+1}\| \leq \|A_T^{-1}\|^{n+1} \|x\|$$

and

$$(6) \quad |a_n| \leq \|A_T^{-1}\|^{n+1} \|T^{-1}\|^{-1} \|x\|.$$

For  $\lambda \in D(r(T))$ , let  $\phi_\lambda: X \rightarrow \mathbb{C}$  be defined by the formula

$$(7) \quad \phi_\lambda(x) = \sum_{k=0}^{\infty} a_k \lambda^k .$$

Since  $\lambda \in D(r(T))$  implies that  $|\lambda|^n < \|A_T^{-1}\|^{-n}$ , the estimate (6) implies that the series (7) converges. For each  $\lambda \in D(r(T))$ ,  $\phi_\lambda$  is linear because  $x$  is a linear function of its Taylor coefficients. Also, it follows from the estimate (6) that  $\phi_\lambda$  is bounded for fixed  $\lambda$ .

Now  $\phi_\lambda(x_0) = 1$  for each  $\lambda \in D(r(T))$ , and on the other hand, if  $y \in T_\lambda X$ , then  $\phi_\lambda(y) = 0$ . To see this, suppose that  $y = T_\lambda z$  and that  $z$  has Taylor coefficients  $\{a_n\}_{n=0}^\infty$ . Then

$$\phi_\lambda(y) = \phi_\lambda(Tz - \lambda z) = \phi_\lambda(Tz) - \lambda \phi_\lambda(z) = \sum_{n=0}^{\infty} a_n \lambda^{n+1} - \lambda \sum_{n=0}^{\infty} a_n \lambda^n = 0 .$$

Also, if  $x \in X$  and  $\phi_\lambda(x) = 0$ , then  $x \in T_\lambda X$ . For if  $x \in X$ , then  $x$  has a representation  $x = b_0 x_0 + T_\lambda y$ ; therefore  $0 = \phi_\lambda(x) = b_0 \phi_\lambda(x_0) + \phi_\lambda(T_\lambda y) = b_0$ , and hence  $x = T_\lambda y$ .

Thus  $T_\lambda X = \{x: x \in X \text{ and } \phi_\lambda(x) = 0\}$ . Since  $\phi_\lambda$  is a bounded linear functional,  $T_\lambda X$  is closed,  $X$  is the Banach-space direct sum  $X = (x_0) \oplus T_\lambda X$ , and (5) holds.

Next we show that  $T_\lambda$  is injective, and this will complete the argument that  $T_\lambda$  has properties  $(\alpha')$  and  $(\beta)$ . Suppose  $T_\lambda y = 0$  and  $\lambda \neq 0$ . Using the Taylor formula for  $y$ , we have the relations

$$\lambda y = \lambda a_0 x_0 + \lambda a_1 T x_0 + \cdots + \lambda a_n T^n x_0 + T^{n+1} \lambda y_{n+1}$$

and

$$T y = a_0 T x_0 + a_1 T^2 x_0 + \cdots + a_n T^{n+1} x_0 + T^{n+2} y_{n+1} .$$

Because of the uniqueness of the Taylor coefficients for  $T y = \lambda y$ , we see that  $\lambda a_0 = 0$ , and  $\lambda a_n = a_{n-1}$  for each positive integer  $n$ . It follows that each  $a_n$  is 0, and hence that  $y = 0$ . Thus  $T_\lambda$  is injective.

It remains to show that  $T_\lambda$  has property  $(\gamma)$ , in other words, that if  $x \in X$  and  $x \neq 0$ , then  $x$  is not infinitely divisible by  $T$ . For this purpose we observe that we may apply the part of the proof already completed to the operator  $T_\lambda$  (which has the properties  $(\alpha')$  and  $(\beta)$ ) to conclude that for all  $\mu$  in some plane neighborhood of zero, the operator  $T_\lambda - \mu I$  has closed range, that

$$X = (x_0) \oplus (T_\lambda - \mu I) X ,$$

and that each  $x \in X$  has the representation

$$x = \left( \sum_{k=0}^{\infty} b_k \mu^k \right) x_0 + (T_\lambda - \mu I) y ,$$

where  $\{b_k\}_{k=0}^\infty$  is the sequence of Taylor coefficients of  $x$  relative to  $T_\lambda - \mu I$ . But  $T_\lambda - \mu I = T_{\lambda+\mu}$ , and  $x$  also has the representation

$$x = \sum_{k=0}^{\infty} a_k(\lambda + \mu)^k x_0 + T_{\lambda+\mu} w,$$

where  $\{a_k\}_{k=0}^{\infty}$  is the sequence of Taylor coefficients of  $x$  relative to  $T$ , and where  $w \in X$ . Thus

$$\sum_{k=0}^{\infty} b_k \mu^k = \sum_{k=0}^{\infty} a_k(\lambda + \mu)^k$$

for all  $\mu$  in a neighborhood of zero. It now follows that if  $x$  is infinitely divisible by  $T_{\lambda}$ , which implies that each  $b_k = 0$  for each  $k$ , then also  $a_k = 0$  for each  $k$  and hence that  $x = 0$ .

*Definition.* For each  $x \in X$ , let  $\tilde{x}$  be the function on  $D(r(T))$  defined by

$$\tilde{x}(z) = \sum_{k=0}^{\infty} a_k z^k,$$

the sequence  $\{a_k\}_{k=0}^{\infty}$  being the sequence of Taylor coefficients for  $x$ . Let  $X_F = \{\tilde{x} : x \in X\}$ , and call  $X_F$  the *function-space representation* of  $X$ .

**COROLLARY:** *The mapping  $x \rightarrow \tilde{x}$  is an isomorphism of  $X$  onto  $X_F$ . Under this mapping,  $T$  corresponds to the operator  $S_z$  of multiplication by  $z$ . For each  $\tilde{x} \in X_F$  and each  $\lambda \in D(r(T))$ , there is an  $\tilde{x}_1 \in X_F$  such that  $\tilde{x}$  admits the factorization*

$$(8) \quad \tilde{x}(z) - \tilde{x}(\lambda) = (z - \lambda) \tilde{x}_1(z) \quad (z \in D(r(T))).$$

*Proof.* The first two assertions are clear from the conclusion of Theorem 2 and the definition of  $X_F$ . The factorization (8) now follows from the corresponding decomposition

$$x = \tilde{x}(\lambda) x_0 + (T - \lambda I) x_1$$

for the element  $x \in X$ .

### 3. THE SPECTRUM OF A SHIFT OPERATOR

Some information on the spectrum of a shift operator comes directly from Theorem 2 as follows. By the *compression spectrum* of the operator  $T$ , we mean the complex numbers  $\lambda$  such that  $T - \lambda I$  does not have dense range. This is analogous to the definition of compression spectrum for a Hilbert space operator (see P. R. Halmos [5, p. 188]).

We have shown that for  $\lambda \in D(r(T))$ , the range of  $T - \lambda I$  has closed range of codimension one, and hence  $D(r(T))$  lies in the compression spectrum of the shift  $T$ .

A stronger result on the compression spectrum is available by way of the index theory for semi-Fredholm operators. Index theorems of T. Kato [8, Theorems 1 and 6], when applied to an operator satisfying axioms  $(\alpha')$  and  $(\beta)$ , yield the information that for each complex  $\lambda$  satisfying  $|\lambda| < \|T^{-1}\|^{-1}$ , the operator  $T - \lambda I$  also satisfies  $(\alpha')$  and  $(\beta)$ . In particular, for these values of  $\lambda$ ,  $T - \lambda I$  has closed range of codimension one, and the compression spectrum contains the set  $D(\|T^{-1}\|^{-1})$ .

For an arbitrary shift operator, the spectrum can be a fairly arbitrary set. For instance, if  $K$  is a compact subset of the plane with connected complement and containing a neighborhood of the origin, then  $K$  is the spectrum of some shift operator. Simply let  $X$  be the sup-norm algebra of functions that are continuous on  $K$  and analytic at interior points. Let  $T$  denote multiplication by  $z$ , that is, by the function  $\phi(z) = z$ . Then  $T$  is a shift on  $X$ , and by standard function-algebraic arguments, the spectrum of  $T$  is the range  $K$  of the function  $\phi$ . The main tool needed for the argument is Mergelyan's theorem [11, Theorem 20.5].

In the case of a shift isometry, however, the index theory referred to above provides the complete solution to the problem of finding the spectrum. By the formula for the spectral radius, the spectrum is contained in the closed unit disk  $\{\lambda: |\lambda| \leq 1\}$ , and the index theory shows that the compression spectrum consists of  $\{\lambda: |\lambda| < 1\}$ . Thus the spectrum is in fact the whole disk. This fact will also occur later as a consequence of Theorem 6.

*Note.* The referee points out that the spectrum of each noninvertible isometry  $T$  on any Banach space is the unit disk.

*Proof.* Let  $D = \{\lambda: |\lambda| < 1\}$ . If  $\lambda \in D$ , then  $T - \lambda I$  is bounded below. If  $\sigma(T) \cap D \neq D$ , then, since  $\sigma(T) \cap D \neq \emptyset$ , we see that  $\partial\sigma(T) \cap D \neq \emptyset$ . But if  $\lambda \in \partial\sigma(T) \cap D$ , then  $\lambda$  is an approximate eigenvalue, a contradiction.

#### 4. ORTHOGONAL DECOMPOSITIONS

The applications of the index theory made in the previous section suggest that equation (4) of Theorem 2 should perhaps be valid for any  $\lambda$  satisfying the condition  $|\lambda| < \|T^{-1}\|^{-1}$ . The values of  $\lambda$  for which the theorem is proved are at most this large, since

$$\|A_T^{-1}\| \geq \sup \{ \|x \oplus 0\| : \|A_T(x \oplus 0)\| = 1 \} = \sup \{ \|x\| : \|Tx\| = 1 \} = \|T^{-1}\| ,$$

and  $r(T) = \|A_T^{-1}\|^{-1} \leq \|T^{-1}\|^{-1}$ . Moreover, strict inequality may actually occur. In the case of the disk algebra mentioned in the introduction, it is easy to compute that  $r(T) = \|A_T^{-1}\|^{-1} = 1/2$ . On the other hand, the operator  $T$ , being multiplication by  $z$ , is an isometry, and therefore  $\|T^{-1}\|^{-1} = 1$ .

One should note that the value of  $r(T)$  also depends on the choice of  $x_0$  taken in the complement of the range of  $T$ , and that a poor choice of  $x_0$  can lead to an even smaller value of  $r(T)$ . In the case of a Hilbert space, an obvious best candidate for  $x_0$  is an element of norm 1 that is orthogonal to  $TX$ . This choice is indeed the best, as we shall see, and in certain cases the idea can be extended to a Banach space.

Following R. C. James [7], we say that two vectors  $x$  and  $y$  in a Banach space are *orthogonal* provided that for each scalar  $t$ ,

$$\|x\| \leq \|x + ty\| \quad \text{and} \quad \|y\| \leq \|y + tx\| .$$

The conditions of the following theorem can always be satisfied if  $X$  is a Hilbert space. In the case of the Banach spaces  $\ell^p$  ( $1 \leq p \leq \infty$ ) and the unilateral shift operator  $T$  defined by the equation

$$T(a_0, a_1, a_2, \dots) = (0, a_0, a_1, a_2, \dots),$$

the conditions are satisfied with the vector  $x_0 = (1, 0, 0, \dots)$ .

**THEOREM 3.** *Suppose that  $T$  is a shift on  $X$ , and that in the decomposition  $X = (x_0) \oplus TX$  the vector  $x_0$  is orthogonal to each vector in  $TX$ . Then the radius  $r(T)$  of the neighborhood  $D(r(T))$  in Theorem 2 is  $\|T^{-1}\|^{-1}$ .*

*Proof.* By Theorem 2,  $r(T) = \|A_T^{-1}\|^{-1}$ , and therefore we show that  $\|A_T^{-1}\| = \|T^{-1}\|$ .

As we mentioned in the discussion above, we always have the relations  $r(T) = \|A_T^{-1}\|^{-1} \leq \|T^{-1}\|^{-1}$ .

On the other hand, if  $\|A_T(x \oplus \beta)\| \leq 1$ , then  $\|Tx + \beta x_0\| \leq 1$ . Since  $x_0$  and  $Tx$  are orthogonal,  $Tx$  and  $\beta x_0$  are also orthogonal. Therefore  $\|Tx\| \leq 1$  and  $|\beta| \leq 1$ , and hence

$$\|x \oplus \beta\| = \max \{ \|x\|, \|T^{-1}\| |\beta| \} \leq \max \{ \|T^{-1}\|, \|T^{-1}\| |\beta| \} \leq \|T^{-1}\|.$$

This yields the inequality  $\|A_T^{-1}\| \leq \|T^{-1}\|$ , and therefore

$$r(T) = \|A_T^{-1}\|^{-1} \geq \|T^{-1}\|^{-1}.$$

We remark that it is in this calculation that we use the factor  $\|T^{-1}\|$  occurring in the expression for the norm of  $x \oplus \beta$ .

**COROLLARY.** *Corresponding to each power series  $\sum_{n=0}^{\infty} a_n z^n$  with coefficients in  $\ell^p$  ( $1 \leq p \leq \infty$ ) and each  $\lambda$  ( $|\lambda| < 1$ ), there is a power series  $\sum_{n=0}^{\infty} b_n z^n$  with coefficients also in  $\ell^p$ , such that for  $|z| < 1$ ,*

$$\sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} a_n \lambda^n = (z - \lambda) \sum_{n=0}^{\infty} b_n z^n.$$

*Proof.* We apply Theorem 3 and the corollary to Theorem 2 to the unilateral shift on the space  $\ell^p$  ( $1 \leq p \leq \infty$ ).

### 5. COMMUTANTS OF SHIFT OPERATORS

Suppose  $T$  is a shift operator on a Banach space  $X$ . We now consider the bounded linear operators  $S$  on  $X$  that commute with  $T$ , in other words, the elements of the commutant of  $T$ .

**THEOREM 4.** *Suppose  $S$  commutes with  $T$ . On the function space  $X_F$ ,  $S$  corresponds to an operator of multiplication by a bounded analytic function having sup-norm at most  $\|S\|$ .*

*Proof.* By the results of the previous section, each  $x \in X$  can be expressed as

$$x = \tilde{x}(\lambda) x_0 + T_\lambda y_\lambda,$$

for  $\lambda \in D(r(T))$ , where  $\tilde{x}(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$  and  $\{a_n\}_{n=0}^{\infty}$  is the sequence of Taylor coefficients for  $x$ . Let the expression for the element  $Sx_0$  be

$$Sx_0 = \phi(\lambda) x_0 + T_\lambda z_\lambda,$$



where  $\phi$  is analytic on  $D(r(T))$ . Then

$$\begin{aligned} Sx &= \tilde{x}(\lambda) Sx_0 + ST_\lambda y_\lambda = \tilde{x}(\lambda) [\phi(\lambda) x_0 + T_\lambda z_\lambda] + ST_\lambda y_\lambda \\ &= \phi(\lambda) \tilde{x}(\lambda) x_0 + T_\lambda [\tilde{x}(\lambda) z_\lambda + Sy_\lambda] \end{aligned}$$

(the last step is permissible since  $S$  must also commute with  $T_\lambda$ ). Thus

$$\tilde{Sx}(\lambda) = \phi(\lambda) \tilde{x}(\lambda),$$

and  $S$  corresponds to multiplication by the analytic function  $\phi$  on  $D(r(T))$ .

To compute the sup-norm of  $\phi$ , we follow an argument of Halmos [5, Problem 53]. For a fixed  $\lambda$ , the mapping  $x \rightarrow \tilde{x}(\lambda)$  is a bounded linear functional, and hence there exists a number  $K_\lambda > 0$  such that

$$|\phi^n(\lambda)\lambda| \leq K_\lambda \|S^n T x_0\| \leq K_\lambda \|S\|^n \|T\|.$$

For  $\lambda \neq 0$ , this yields the inequality

$$|\phi(\lambda)| \leq |\lambda|^{-1/n} K_\lambda^{1/n} \|S\| \|T\|^{1/n},$$

from which it follows that  $|\phi(\lambda)| \leq \|S\|$ . Since  $\phi$  is continuous on  $D(r(T))$ ,  $|\phi(0)| \leq \|S\|$  also.

**COROLLARY.** *The commutant of a shift operator is a commutative algebra.*

*Proof.* The commutant of an operator is always a subalgebra of the algebra of all bounded linear operators on the space. In the present case the commutativity follows from commutativity of operators of multiplication by bounded analytic functions.

In analyzing the commutant of the Cesàro operator  $C_0$  on  $\ell^2$ , or rather of  $I - C_0$ , Shields and Wallen [12] are led to consider the following spaces.

Let  $H$  be a Hilbert space whose elements are complex-valued functions defined on an open disk  $D = D(1)$ , with the usual addition of functions and multiplication by scalars. We assume that there are no points in  $D$  at which all the functions in  $H$  vanish. Consider four additional assumptions:

- (a) Point evaluations are bounded linear functionals on  $H$ , so that to each point  $\zeta \in D$  there corresponds a function  $k_\zeta \in H$  such that  $f(\zeta) = (f, k_\zeta)$  for all  $f \in H$ .
- (b) The operator  $S_z$  of multiplication by  $z$  maps  $H$  into itself and is a contraction ( $\|S_z f\| \leq \|f\|$  for all  $f \in H$ ).
- (c) The functions  $k_\zeta$  are simple eigenfunctions of the operator  $S_z^*$ .
- (d) The functions in  $H$  are analytic on  $D$ .

Shields and Wallen showed that the operators that commute with  $S_z$  are precisely the operators that are multiplications by elements of  $H^\infty$ . As part of their development, they show that if  $A(S_z)$  denotes the weakly closed algebra with identity generated by  $S_z$ , then the assumptions (a), (b), and (c) alone imply that

$$H^\infty \subset A(S_z) \subset \text{commutant of } S_z.$$

The question they raise is whether under these conditions  $H^\infty = A(S_z)$ . We are able to give a positive answer provided condition (c) is strengthened slightly and all of the functions are assumed to be continuous.

Let us consider more closely condition (c), which states that each  $k_\xi$  is a simple eigenfunction of  $S_z^*$ . If  $h$  is an eigenfunction of  $S_z^*$  corresponding to the eigenvalue  $\bar{\xi}$ , then  $S_z^* h = \bar{\xi} h$ ,

$$(S_z^* h, g) = (\bar{\xi} h, g) \text{ and } (h, S_{z-\xi} g) = 0 \quad \text{for all } g \in H,$$

and  $h$  is orthogonal to the range of  $S_{z-\xi}$ . Clearly, the converse is true, and thus condition (c) holds if and only if the range of  $S_{z-\xi}$  has a one-dimensional orthogonal complement. In order to apply Theorem 4, we need to know that the range of  $S_{z-\xi}$  is closed. We also need to know that  $S_{z-\xi}$  is injective, and this condition is clearly satisfied if each  $f \in H$  is continuous. Assuming that each  $f \in H$  is continuous, we impose a condition (c') that we obtain from (c) by also requiring that  $S_{z-\xi}$  have closed range.

(c') For each  $\xi \in D$ ,  $S_{z-\xi}$  satisfies conditions  $(\alpha')$  and  $(\beta)$  of a shift operator.

**THEOREM 5.** *Suppose  $H$  is a Hilbert space of continuous functions in  $D$  satisfying axioms (a), (b), and (c'). Then*

$$H^\infty = A(S_z) = \text{commutant of } S_z.$$

*Proof.* As we mentioned above, Shields and Wallen have shown that  $H^\infty \subset A(S_z) \subset \text{commutant of } S_z$ . They have also shown that an operator  $S$  in the commutant of  $S_z$  is multiplication by some bounded function  $\phi$ . We show that  $\phi$  is actually analytic on  $D$ , and this will complete the proof.

Even without the information whether  $S_{z-\xi}$  satisfies condition  $(\gamma)$  of a shift operator, we note that conditions  $(\alpha')$  and  $(\beta)$  alone imply that for all  $\lambda$  in some neighborhood  $D(r)$  of zero, there is a decomposition

$$(9) \quad f = \tilde{f}(\lambda) k_\xi + (S_{z-\xi} - \lambda I) g_\lambda,$$

where  $g_\lambda \in H$  and  $\tilde{f}$  is analytic on  $D(r)$ . Thus

$$Sf = \phi f = \tilde{f}(\lambda) \phi k_\xi + (S_{z-\xi} - \lambda I) \phi g_\lambda,$$

and upon evaluating  $Sf$  at  $z = \xi + \lambda$ , we obtain the relation

$$(10) \quad (Sf)(\xi + \lambda) = \tilde{f}(\lambda) \phi(\xi + \lambda) k_\xi(\xi + \lambda).$$

On the other hand, repeating part of the argument in the proof of Theorem 4, we see that

$$S k_\xi = \alpha(\lambda) k_\xi + (S_{z-\xi} - \lambda I) \beta_\lambda$$

for all  $\lambda \in D(r)$ , where  $\beta_\lambda \in H$  and  $\alpha$  is analytic on  $D(r)$ . Therefore

$$Sf = \tilde{f}(\lambda) \alpha(\lambda) k_\xi + (S_{z-\xi} - \lambda I) [Sg_\lambda + \beta_\lambda],$$

and

$$(11) \quad (Sf)(\xi + \lambda) = \tilde{f}(\lambda) \alpha(\lambda) k_\xi(\xi + \lambda).$$

For  $\lambda \in D(r)$ , it cannot happen that  $k_\xi(\xi + \lambda) = 0$ , for if it does, then by equation (9),

$$f(\xi + \lambda) = \tilde{f}(\lambda) k_\xi(\xi + \lambda) = 0$$

for all  $f \in H$ , and this is ruled out by our initial assumption about  $H$ . Equations (10) and (11) now combine to yield the equation

$$\tilde{f}(\lambda) \phi(\xi + \lambda) = \tilde{f}(\lambda) \alpha(\lambda),$$

and finally

$$\phi(\xi + \lambda) = \alpha(\lambda) \quad (\lambda \in D(r)).$$

Since  $\alpha$  is analytic on  $D(r)$ ,  $\phi$  is analytic on a neighborhood of  $\xi$ , and since  $\xi$  is arbitrary in  $D$ ,  $\phi$  is analytic on all of  $D$ .

### 6. COMMUTANTS OF SHIFT ISOMETRIES

Suppose now that  $X$  is a Banach space and that  $T$  is a shift isometry on  $X$ . For convenience, we use the function-space representation of  $X$ , in which  $X$  is a space of power series convergent on the disk  $D(r(T))$ , and on which  $T$  is the operator of multiplication by  $z$ . As shown in Section 3,  $D(r(T))$  may be a rather small subset of the spectrum of  $T$ , which itself is the whole disk  $D(1)$ . Of course,  $r(T) = 1$  if an orthogonal decomposition is available, as in Section 4; but such decompositions are not always available, as in the case of the disk algebra. In Theorem 4, it was proved that the commutant of  $T$  is an algebra of multiplications by analytic functions on  $D(r(T))$ , with absolute values bounded by the operator norms.

In the present section, we shall use Banach-algebraic techniques to analyze further the commutant of a shift isometry. The main result obtained (Theorem 6) is that the multiplying functions on  $D(r(T))$  are actually convergent on the whole disk  $\{\lambda: |\lambda| < 1\}$  (which is the whole compression spectrum of  $T$ ), and with absolute values remaining bounded by the operator norms. As an easy consequence of this description of the commutant, we obtain the nonexistence of  $n$ th roots of  $T - \lambda I$ , where  $|\lambda| < 1$ .

For relevant material on the Banach-algebraic results quoted in this section, we refer the reader to C. E. Rickart [9, Chapter III].

Let  $\mathcal{A}$  denote the commutant of  $T$ . By standard arguments,  $\mathcal{A}$  is a closed subalgebra of the algebra  $\mathcal{B}(X)$  of all bounded linear operators on  $X$ , with the usual operations and operator norm. Also,  $\mathcal{A}$  contains the identity operator  $I$  and the operator  $T$  itself. By Theorem 4, each operator  $S$  in  $\mathcal{A}$  is multiplication by the element  $S(1)$  in  $X$ , and by the corollary,  $\mathcal{A}$  is a commutative algebra.

LEMMA 1. *The principal ideal  $T\mathcal{A}$  is a maximal ideal in  $\mathcal{A}$ .*

*Proof.* For  $S \in \mathcal{A}$ , let  $S(1)$  have the power series expansion  $\sum_{n=0}^{\infty} s_n z^n$  in the function-space representation of  $X$ . The mapping  $S \rightarrow s_0$  is a multiplicative linear functional on  $\mathcal{A}$ , and therefore its null space  $M_0$  is a maximal ideal in  $\mathcal{A}$ . We show that  $M_0$  is the same as the principal ideal  $T\mathcal{A}$ .

Suppose  $S \in M_0$ . For each  $x \in X$ , the element  $Sx = S(1) \cdot x$  has a power-series expansion with first coefficient zero, and hence there is an element in  $X$ , which we denote by  $S_1 x$ , such that

$$Sx = TS_1 x \quad (x \in X).$$

Since  $T$  is injective,  $S_1 x$  is uniquely determined by  $x$ . It follows immediately that the operator  $S_1$  thus determined is linear on  $X$ . Also, since  $T$  is an isometry,  $\|S_1 x\| = \|T S_1 x\| = \|Sx\|$ , and hence  $\|S_1\| = \|S\|$ , and  $S_1$  is bounded. Since  $T S_1 T x = S(Tx) = T(Sx)$ , and therefore  $S_1 T x = Sx$ , we see that  $S_1 T x = T S_1 x$ ; therefore  $S_1$  commutes with  $T$ . Thus for each  $S \in M_0$ , there is an element  $S_1 \in \mathcal{A}$  such that  $S = T S_1$ , and thus  $M_0 \subset T\mathcal{A}$ . On the other hand, it is clear that  $T\mathcal{A} \subset M_0$ .

For  $S \in \mathcal{A}$ , let  $\hat{S}$  denote the Gelfand transform of  $S$ . Let  $\mathcal{M}$  denote the maximal ideal space of  $\mathcal{A}$ .

LEMMA 2. For each  $S \in \mathcal{A}$ ,  $\|TS\| = \|S\|$ .

*Proof.*  $\|TS\| = \sup \{ \|TSx\| : \|x\| = 1 \} = \sup \{ \|Sx\| : \|x\| = 1 \} = \|S\|$ .

LEMMA 3. The algebra  $\mathcal{A}$  is semisimple.

*Proof.* We show that if  $S \in \mathcal{A}$  and  $\hat{S} = 0$  on  $\mathcal{M}$ , then  $S = 0$ . Let  $S(1)$  have power-series representation  $\sum_{n=0}^{\infty} s_n z^n$ . If  $\hat{S} = 0$  on  $\mathcal{M}$ , then  $s_0 = 0$  and hence  $S = T S_1$ . Therefore  $\hat{S} = \hat{T} \hat{S}_1 = 0$  on  $\mathcal{M}$ . By the formula for the spectral radius,

$$\lim (\|T^n S_1^n\|)^{1/n} = \|\hat{T} \hat{S}_1\|_{\infty} = 0.$$

By Lemma 2,  $\|T^n S_1^n\| = \|S_1^n\|$ , and therefore  $\|\hat{S}_1\|_{\infty} = \lim (\|S_1^n\|)^{1/n} = 0$ . This implies that the first coefficient of  $S_1(1)$ , which is  $s_1$ , is zero.

By an easy induction argument,  $s_n = 0$  for each  $n$ , and hence  $S = 0$ . Thus  $\mathcal{A}$  is semisimple.

LEMMA 4. Let  $\Gamma$  be the Šilov boundary of  $\mathcal{A}$ . Then

$$m = \min \{ |\hat{T}(t)| : t \in \Gamma \} = 1.$$

*Proof.* Suppose that  $|\hat{T}|$  has its minimum value  $m$  on  $\Gamma$  at  $t_0$  and that  $m < 1$ . Let  $m'$  be chosen so that  $m < m' < 1$ . Let

$$U = \{ t : t \in \mathcal{M} \text{ and } |\hat{T}(t)| < m' \}.$$

Then  $U$  is a neighborhood of  $t_0 \in \mathcal{M}$ . Since  $t_0 \in \Gamma$ , there exists an  $S \in \mathcal{A}$  such that  $|\hat{S}| = 1 = \|\hat{S}\|_{\infty}$  somewhere in  $U$ , and  $|\hat{S}| < 1$  outside  $U$ . For some positive integer  $n$ ,  $\|\hat{S}^n \hat{T}\|_{\infty} < m'$ . By the formula for the spectral radius,

$$\lim (\|(S^n T)^k\|)^{1/k} = \|\hat{S}^n \hat{T}\|_{\infty} < m'.$$

By Lemma 2,  $\|(S^n T)^k\| = \|S^{nk}\|$ , and therefore

$$\|\hat{S}^n\|_{\infty} = \lim (\|S^{nk}\|)^{1/k} < m' < 1.$$

But  $\|\hat{S}^n\|_{\infty} = \|\hat{S}\|_{\infty}^n = 1$ , which is a contradiction. Thus  $m \geq 1$ .

On the other hand,  $m \leq \|\hat{T}\|_{\infty} \leq \|T\| = 1$ , and thus  $m = 1$ .

We now think of  $T$  as a multiplication operator on  $\mathcal{A}$ . We have proved that it is a shift isometry on  $\mathcal{A}$ , and that

$$\mathcal{A} = (I) \oplus T\mathcal{A}.$$

Each  $S \in \mathcal{A}$  has a sequence of Taylor coefficients relative to this new setting, but it is clear that the new sequence is the same as the sequence of Taylor coefficients of the elements  $S(1)$ . Again, let these coefficients be denoted by  $\{s_n\}_{n=0}^\infty$ .

The main result of Gleason's paper [4] yields the information that the maximal ideal  $M_0 = T\mathcal{A}$  is the center of an analytic disk in  $\mathcal{M}$ , and that for each  $M$  in this "disk,"

$$\hat{S}(M) = \sum_{n=0}^\infty s_n [\hat{T}(M)]^n.$$

The same information is obtainable from Theorem 2 of the present work. However, neither in Gleason's theorem nor in Theorem 2 is the optimal convergence set obtained, except in the case where the orthogonality condition of Theorem 3 is satisfied. (Again, consider multiplication by  $z$  on the disk algebra.)

The optimal convergence set in the maximal ideal space is obtained, however, in [1]. The arguments there are measure-theoretic, but apparently they do not admit extension from the context of a Banach algebra to an arbitrary Banach space (such an extension would yield a sharpening of Theorem 2 itself, in addition to the sharpening of Theorem 4 being developed here). The main result of [1], stated in the notation of the present development, is as follows.

**THEOREM.** *Suppose that  $\mathcal{A}$  is a commutative, semisimple Banach algebra with identity  $I$ , that  $\mathcal{M}$  is the maximal ideal space of  $\mathcal{A}$ , and that  $\Gamma$  is the Šilov boundary of  $\mathcal{A}$ . Suppose  $T \in \mathcal{A}$  and the principal ideal  $T\mathcal{A}$  is a maximal ideal  $M_0$  that is not an isolated point of  $\mathcal{M}$ . Let  $m = \min \{ |\hat{T}(t)| : t \in \Gamma \}$ . Then  $m > 0$ , and the set  $V = \{ M : |\hat{T}(M)| < m \}$  is an analytic disk in  $\mathcal{M}$ . For each  $S \in \mathcal{A}$ ,*

$$\hat{S}(M) = \sum_{n=0}^\infty s_n \hat{T}^n(M) \quad (M \in V),$$

the sequence  $\{s_n\}_{n=0}^\infty$  being the set of Taylor coefficients for  $S$ .

Applying this to the present situation, we obtain the following result on the commutant  $\mathcal{A}$  of  $T$ .

**THEOREM 6.** *Suppose that  $T$  is a shift isometry on the Banach space  $X$ , and that  $S$  is in the commutant of  $T$ . Then, on the function-space representation of  $X$ ,  $S$  corresponds to multiplication by an element of  $H^\infty$  of the unit disk, with values bounded by  $\|S\|$ .*

*Proof.* We have shown that  $\mathcal{A}$  is semisimple, that  $M_0 = T\mathcal{A}$  is a maximal ideal in  $\mathcal{A}$ , and that  $m = \min \{ |\hat{T}(t)| : t \in \Gamma \} = 1$ . We now need to show that  $M_0$  is not isolated in  $\mathcal{M}$ .

Suppose  $M_0$  is isolated in  $\mathcal{M}$ . By Šilov's Idempotency Theorem, there exists a nonzero idempotent  $E \in M_0$ . Let the element  $E(1)$  in  $X$  have series representation  $\sum_{n=0}^\infty e_n z^n$ , with  $e_0 = 0$ . Since  $E^2 = E$ ,

$$e_n = e_0 e_n + e_1 e_{n-1} + \dots + e_n e_0$$

for each positive integer  $n$ . It follows by an easy induction argument that  $e_n = 0$  for each  $n$ , and that  $E = 0$ . Thus we obtain a contradiction, and  $M_0$  is not isolated in  $\mathcal{M}$ .

By the previous theorem, if  $S \in \mathcal{A}$ , then for all  $M \in \mathcal{M}$  such that  $|\hat{T}(M)| < 1$ ,

$$\hat{S}(M) = \sum_{n=0}^{\infty} s_n \hat{T}^n(M).$$

This proves that the series  $\sum_{n=0}^{\infty} s_n z^n$  has radius of convergence at least 1. Because  $\mathcal{A}$  is a commutative Banach algebra,  $\|\hat{S}\|_{\infty} \leq \|S\|$ , so that  $\sum_{n=0}^{\infty} s_n z^n$  is the power-series representation of an  $H^{\infty}$ -function. But this power series, when restricted to  $D(r(T))$ , is the same as the one obtained in Theorem 4.

**COROLLARY.** *Suppose  $T$  is a shift isometry on a Banach space  $X$ . If  $\lambda$  is a complex scalar ( $|\lambda| < 1$ ), then  $T - \lambda I$  has no  $n$ th root in  $\mathcal{B}(X)$ .*

*Proof.* If  $S \in \mathcal{B}(X)$  and  $S^n = T - \lambda I$ , then  $S$  commutes with  $T - \lambda I$  and hence with  $T$ . By Theorem 6, the function  $\hat{S}$  lies in  $H^{\infty}$ . But this would imply that  $\hat{S}^n(z) = z - \lambda$ , in other words, that  $z - \lambda$  has an  $n$ th root in  $H^{\infty}$ , which of course is impossible.

*Remark.* The same technique also shows that for  $|\lambda| < 1$ ,  $T - \lambda I$  has no inverse in  $\mathcal{B}(X)$ , as mentioned in Section 3.

## 7. FURTHER DISCUSSION OF THE COMMUTANT

Let us again restrict our attention to the case of a shift isometry on a Banach space  $X$ . We have seen that the commutant corresponds to an algebra of multiplications by  $H^{\infty}$ -functions. The full algebra of  $H^{\infty}$ -functions emerges as the commutant in case  $X$  is the Hardy space  $H^p$  ( $1 \leq p \leq \infty$ ). On the other hand, if  $X$  is the disk algebra or the algebra of absolutely convergent power series with the usual norm, then the commutant consists of multiplications by elements of that algebra and cannot be as large as  $H^{\infty}$ . It is not difficult to see that each absolutely convergent power series always defines a function that occurs in our description of the commutant.

It is of some interest to know whether there is some topology under which polynomials in  $T$  are dense in the commutant. If  $X$  is the disk algebra or the algebra of absolutely convergent power series, then the commutant of  $T$  is the operator-norm closure of polynomials in  $T$ , as is easily seen. If  $X$  is an  $H^p$ -space ( $1 \leq p \leq \infty$ ), it turns out that the commutant, consisting of all multiplications by  $H^{\infty}$ -functions, is generated by  $T$  in the strong operator topology. This result is largely an application of Fejér's theorem for arithmetic means of partial sums of Fourier series. The proof is given for  $p = 2$  in [5, Problem 117], and it remains valid for  $1 \leq p < \infty$ . For the case of  $X = H^{\infty}$ , it is not apparent that  $T$  generates the commutant, even in the weak operator topology. However, the commutant (which again is  $H^{\infty}$ ) is generated by  $z$  in the weak-\* topology of  $L^{\infty}[0, 2\pi]$ .

One might expect that in the case of  $X = \ell^p$  ( $1 \leq p \leq \infty$ ), the commutant is at least generated by  $T$  in the weak operator topology. The cases  $p = 1$  and  $p = 2$  have occurred above in the form of the absolutely convergent power series and  $H^2$ , where the result is even stronger. For  $1 < p < \infty$  ( $p \neq 2$ ), the difficulty in using Fejér's theorem in a manner analogous to the case  $p = 2$  is the apparent absence of an analogue of Parseval's theorem. We can state some results about the shift on  $\ell^p$ , however. A few straightforward calculations (which we omit) show that if  $S$  is in the commutant, and  $h$  is the corresponding  $H^{\infty}$ -function as given in Theorem 6,

then the sequence of power series coefficients for  $h$  lies in both  $\ell^p$  and  $\ell^q$ , where  $1/p + 1/q = 1$ . Moreover, the  $\ell^p$ - and  $\ell^q$ -norms of this sequence are at most  $\|S\|$ . If  $X = \ell^1$  or  $\ell^\infty$ , the commutant corresponds to  $\ell^1$ . For  $1 < p < \infty$  ( $p \neq 2$ ), no complete description of the commutant seems to be available. One can conclude, however, that the commutant in these cases is a proper subset of  $H^\infty$ , and that it does not contain all of the functions in the disk algebra. It is known [10, p. 223] that for each  $\mu > 0$ , there exists a function in the disk algebra whose sequence of power series coefficients does not lie in  $\ell^{2-\mu}$ . Since  $1 < p < \infty$  and  $p \neq 2$ , the function just described cannot correspond to an element of the commutant, if  $\mu$  is small enough.

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