THE CLUSTER SET OF THE PRODUCT OF TWO FUNCTIONS IN H^∞

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Let I denote the set of all inner functions in H^∞, where H^∞ is the Banach algebra of all bounded analytic functions on the open unit disk $D = \{z: |z| < 1\}$. Let I^* denote the set of all functions f(z) in H^∞ for which the cluster set $C(f, e^{i\theta})$ at each point $e^{i\theta}$ on the circumference $C = \{z: |z| = 1\}$ is either the closed unit disk $|w| < 1$ or else a single point of modulus 1. This class of functions has been investigated in several recent papers (see, for example, [2] and [5]). In particular, A. J. Lohwater and G. Piranian [5, Theorem 3] have shown that the class I^* contains an outer function.

In [3], the question was raised whether I^* is a semigroup under multiplication. After some preliminary considerations, we show that I^* is not closed under multiplication (Theorem 2). The technique we use to construct functions in $I^* - I$ leads to several surprising results. For example, we show that the norm of the product of two functions in I^* can be arbitrarily small. In the remainder of the paper, we discuss some of the consequences of Theorem 2 that underscore the differences between inner functions and functions in $I^* - I$.

We begin with a simple fact, which, for purposes of reference, we state without proof as a lemma.

LEMMA 1. *Let* $\{\lambda_n\}$ *be a sequence of nonzero numbers in* D. *If the series* $\sum_{n=1}^\infty |1 - \lambda_n|$ *converges, then*

$$\left|1 - \prod_{n=1}^{\infty} \lambda_n\right| \leq \sum_{n=1}^{\infty} |1 - \lambda_n|.$$

Our next lemma gives an inequality for a Blaschke product whose zeros converge rapidly to C. Let $\{a_n\}$ be a sequence of points in D such that $|a_n| = r_{2n-1}$ $(n = 1, 2, \cdots)$, where

$$r_n = 1 - 2^{-n^2}.$$

Since $\{a_n\}$ is a Blaschke sequence, we can form the associated Blaschke product

$$B(z) = \prod_{k=1}^{\infty} b(a_k, z),$$

where

$$b(a_k, z) = \frac{\bar{a}_k}{|a_k|} \frac{a_k - z}{1 - \bar{a}_k z}.$$

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We shall denote the nth partial product of $B(z)$ by $B_n(z)$.

LEMMA 2. *Let* $B(z)$ *be the Blaschke product constructed above. Then*

$$|B(r_{2n} e^{i\theta}) - B_n(e^{i\theta})| \le 2^{-3(n-1)} \qquad (n = 1, 2, \cdots)$$

for each point $e^{i\theta}$ *on* C.

For the proof of Lemma 2, we need two estimates. If $|z| \le r_{2n}$, then

$$|B_n(z) - B(z)| \le \left|1 - \prod_{k=n+1}^{\infty} b(a_k, z)\right| \le \sum_{k=n+1}^{\infty} |1 - b(a_k, z)|,$$

where the second inequality is a consequence of Lemma 1. Next, a familiar argument (see [4, p. 65], for example) shows that

$$\sum_{k=n+1}^{\infty} |1 - b(a_k, z)| \le \sum_{k=n+1}^{\infty} \frac{1 - |a_k|}{|a_k|} \cdot \frac{2}{1 - r_{2n}}.$$

Note that

$$\sum_{k=n+1}^{\infty} \frac{1 - |a_k|}{|a_k|} \frac{2}{1 - r_{2n}} \le 6 \sum_{k=n+1}^{\infty} \frac{1 - |a_k|}{1 - r_{2n}} \le 6 \cdot 2^{(2n)^2} \sum_{k=n+1}^{\infty} 2^{-(2k)^2} \le 6 \cdot 2^{-3n}.$$

Therefore we have established the inequality

(1) $$|B_n(z) - B(z)| \le 6 \cdot 2^{-3n} \qquad \text{for } |z| \le r_{2n}.$$

Assume now that $|z| \le 1$. Then

$$|B_n'(z)| = \left|\sum_{k=1}^{n} \frac{B_n(z)}{b(a_k, z)} b'(a_k, z)\right| \le \sum_{k=1}^{n} |b'(a_k, z)|.$$

But $\sum_{k=1}^{n} |b'(a_k, z)| = \sum_{k=1}^{n} \frac{1 - |a_k|^2}{|1 - \bar{a}_k z|^2}$, and hence

$$\sum_{k=1}^{n} |b'(a_k, z)| \le \sum_{k=1}^{n} \frac{1 + |a_k|}{1 - |a_k|} \le \sum_{k=1}^{n} \frac{2}{1 - |a_k|} \le 2n \frac{1}{1 - r_{2n-1}} \le 2^{(2n)^2 - 3n + 1}.$$

Therefore

$$\sup_{|z| \le n} |B_n'(z)| \le 2^{(2n)^2 - 3n + 1}.$$

To obtain our second estimate, we observe that

$$|B_n(r_{2n} e^{i\theta}) - B_n(e^{i\theta})| = \left| \int_{r_{2n}}^{1} B_n'(te^{i\theta})\, dt \right|$$

$$\leq (1 - r_{2n}) \sup_{|z|\leq 1} |B_n'(z)| \leq 2^{-(2n)^2} \cdot 2^{(2n)^2-3n+1} .$$

Hence,

$$|B_n(r_{2n} e^{i\theta}) - B_n(e^{i\theta})| \leq 2 \cdot 2^{-3n} . \tag{2}$$

Combining the relations (1) and (2), we obtain the inequality in Lemma 2.

Let $w = f(z)$ be a function in H^∞, and let S be a closed, connected set lying in $|z| < 1$ except for one point $e^{i\theta_0}$ on C. We say that a value w lies in the cluster set $C_S(f, e^{i\theta_0})$ if the sequence $\{z_n\}$ used in defining $C(f, e^{i\theta_0})$ is further restricted to lie in S. If S is the radius drawn to the point $e^{i\theta_0}$, we denote the resulting cluster set by $C_\rho(f, e^{i\theta_0})$ and call it the *radial cluster set of* f(z) *at the point* $e^{i\theta_0}$. Preliminaries aside, we now prove the following theorem.

THEOREM 1. *Let* $\{e^{i\theta_m}\}$ *be a countable set of points on* C. *Then there exists a Blaschke product* $w = B(z)$ *such that*

$$\{|w| = 1\} \subseteq C_\rho(B, e^{i\theta_m}) \qquad (m = 1, 2, \cdots).$$

To prove the theorem, it suffices to show that there exists a Blaschke product $w = B(z)$ such that at each point $e^{i\theta_m}$ the radial cluster set of B(z) contains a countable, dense subset of $\{|w| = 1\}$.

Let $\{e^{i\phi_k}\}$ be a countable dense set of points on $|w| = 1$. By means of the diagonal process, we can arrange the ordered pairs

$$(e^{i\theta_m}, e^{i\phi_k}) \qquad (m, k = 1, 2, \cdots)$$

into a sequence $S = S(n) = (e^{i\theta_{m_n}}, e^{i\phi_{k_n}})$ such that each ordered pair appears infinitely often in the sequence.

Next, as in Lemma 2, we construct a Blaschke product $w = B(z)$ whose zeros satisfy the condition

$$|a_n| = r_{2n-1} \qquad (n = 1, 2, \cdots;\ r_n = 1 - 2^{-n^2}).$$

We choose the argument of a_1 so that

$$B_1(e^{i\theta_1}) = \frac{\bar{a}_1}{|a_1|} \frac{a - e^{i\theta_1}}{1 - \bar{a}_1 e^{i\theta_1}} = e^{i\phi_1} ,$$

where the ordered pair $(e^{i\theta_1}, e^{i\phi_1})$ is the first term of the sequence S. Assuming that the arguments of $a_1, \cdots, a_{n-1}$ have been chosen, we select the argument of a_n so that

$$B_n(e^{i\theta_k}) = e^{i\phi_m},$$

where $B_n(z)$ is the nth partial product of $B(z)$, and where the ordered pair $(e^{i\theta_k}, e^{i\theta_m})$ is the nth term of the sequence S.

We now assert that the Blaschke product $w = B(z)$ constructed above has the required property. Let $(e^{i\theta_j}, e^{i\phi_k})$ be an arbitrary ordered pair in S. Then, by Lemma 2, the inequality

$$|B(r_{2n} e^{i\theta_j}) - e^{i\phi_k}| \leq 2^{-3(n-1)}$$

holds for infinitely many values of n. Consequently, $e^{i\phi_k} \in C_\rho(B, e^{i\theta_j})$, and the theorem is proved.

In the sequel, we shall use $\overline{E}$ and $m(E)$ to denote the closure of E and the measure of E. The proof of the main theorem (Theorem 2) of this paper is based on the following measure-theoretic result.

LEMMA 3. *There exists a sequence* $\{E_n\}$ *of pairwise disjoint, measurable subsets of the closed interval* $[0, 2\pi]$ *such that*

(i) $m(E_n) < 2\pi/2^{n+1}$ $(n = 1, 2, \cdots)$

and

(ii) *for each* n, *the set* E_n *is metrically dense on* $[0, 2\pi]$; *that is, for each subinterval* J *of* $[0, 2\pi]$, *the measure of* $J \cap E_n$ *is positive.*

We begin the proof of the lemma with an auxiliary construction. Let X denote the closed unit interval [0, 1]. Let K_1^1 be the open interval of length 1/4 centered at 1/2. Let K_2^1 and K_2^2 be the open intervals of length 4^{-2} centered at the midpoints of the two intervals whose union is $X - K_1^1$. Let K_3^1, K_3^2, K_3^3, and K_3^4 be the open intervals of length 4^{-3} centered at the midpoints of the four intervals whose union is $X - (K_2^1 \cup K_2^2)$. Proceeding in this way, we obtain the set

$$K = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{n-1}} K_n^j.$$

We remark that the construction of K is analogous to the construction of the complement of the Cantor set. It is evident that the subintervals $\{K_n^j\}$ are pairwise disjoint and that the measure of K is

$$m(K) = \frac{1}{4} \sum_{j=1}^{\infty} 2^{-j} = 1/2.$$

Now let J be some bounded open interval, and let ϕ_J be the order-preserving affine transformation of the open interval (0, 1) onto J. We define the set T(J) by the formula $T(J) = \phi_J(K)$, where K is the set constructed above. Since ϕ_J is an affine transformation, it follows that $m(T(J)) = \frac{1}{2} m(J)$ and $\overline{T(J)} = \overline{J}$. The definition of T may be extended to any bounded open set. If G is a bounded open set of the real line, then G can be written in a unique way as a union of pairwise disjoint intervals $\{J_n\}$. Hence we define

$$T(G) = \bigcup_{n=1}^{\infty} T(J_n).$$

Again we note that $m(T(G)) = \frac{1}{2} m(G)$ and $\overline{T(G)} = \overline{G}$.

We have now set up the necessary apparatus, and we proceed with the construction as follows. Let

$$G_0 = (0, 2\pi), \quad G_1 = T(G_0), \quad \cdots, \quad G_n = T(G_{n-1}), \quad \cdots.$$

Then it is clear that $G_n \subseteq G_{n-1}$, $\overline{G}_n = [0, 2\pi]$, and $m(G_n) = 2\pi/2^n$. Furthermore, every subinterval of G_n has length at most $2\pi/4^n$. If we set

$$F_1 = G_0 - G_1, \quad F_2 = G_1 - G_2, \quad \cdots, \quad F_n = G_{n-1} - G_n, \quad \cdots,$$

then $F_n \cap F_m = \emptyset$ for $n \neq m$, and $m(F_n) = 2\pi/2^n$.

We assert now that if V is any subinterval of $[0, 2\pi]$; then $m(V \cap F_n) > 0$ for sufficiently large n. To see this, select an n so large that $2\pi/4^{n-1} < |V|/2$, where $|V|$ denotes the length of V. Since the midpoint of V belongs to $\overline{G}_{n-1}$, and since each subinterval of G_{n-2} has length less than $|V|/2$, it follows that V contains a maximal subinterval J of G_{n-1}. Therefore

$$m(V \cap F_n) = m(V \cap (G_{n-1} - G_n)) > m(J \cap (G_{n-1} - G_n)).$$

But

$$m(J \cap (G_{n-1} - G_n)) = m(J - T(J)) = \frac{1}{2} m(J) > 0.$$

This shows that $m(V \cap F_n) > 0$, provided n is sufficiently large.

Let Z denote the set of all positive integers, and let ψ denote a one-to-one mapping from the Cartesian product $Z \times Z$ onto Z. If $\psi(j, k) = n$, then we set $F_n = G_{jk}$. Next we define

$$E_k = \bigcup_{j=1}^{\infty} G_{jk}.$$

The sets E_k $(k = 1, 2, \cdots)$ are then pairwise disjoint and metrically dense in $[0, 2\pi]$. Finally, we select a subsequence $\{E_n\}$ of $\{E_k\}$ such that $m(E_n) < 2\pi/2^{n+2}$. This concludes the proof of Lemma 3.

It is easy to verify that if f_1 and f_2 are two inner functions, then

$$\sup_{z \in D} |f_1(z) f_2(z)| = \|f_1 f_2\| = 1 .$$

The following example, which we present in the form of an existence theorem, shows that the norm of the product of two functions in $I^* - I$ need not be 1.

THEOREM 2. *There exist functions* $f_1(z)$ *and* $f_2(z)$ *in* I^* *such that* $\|f_1 f_2\| < 1$.

Let $\{E_n\}$ be a sequence of measurable subsets of C having the properties listed in Lemma 3. Let $\{s_n\}$ be a countable dense subset of $(-\infty, 0]$. We may assume that the sequence $\{s_n\}$ is ordered so that the series $\sum s_n^2 m(E_n)$ converges. Let

$$U_1(x) = \begin{cases} -1 & \text{if } x \notin \bigcup E_n , \\ s_n & \text{if } x \in E_n , \end{cases}$$

and

$$U_2(x) = \begin{cases} -1 & \text{if } x \notin \bigcup E_n , \\ s_n & \text{if } x \in E_n \text{ and } s_n < -2 , \\ -2 - s_n & \text{if } x \in E_n \text{ and } s_n \geq -2 . \end{cases}$$

Note that $U_1(x) + U_2(x) \leq -2$. Since the functions $U_1(x)$ and $U_2(x)$ are in L^2, they can be extended harmonically to the open unit disk D. Denote by $u_1(z)$ and $u_2(z)$ the bounded harmonic functions with boundary values U_1 and U_2, respectively. Let $v_k(z)$ $(k = 1, 2)$ be a harmonic conjugate to $u_k(z)$ $(k = 1, 2)$ in D. Then the functions

$$g_1 = e^{u_1 + iv_1} \quad \text{and} \quad g_2 = e^{u_2 + iv_2}$$

are bounded and analytic in D. It follows now from the construction of g_1 and g_2 that

$$\|g_1\| = \|g_2\| = 1 \quad \text{and} \quad \|g_1 g_2\| < 1 .$$

For each n, let K_n be a subset of E_n with the following properties:

(i) K_n is a countable dense subset of C, and

(ii) if $e^{i\theta} \in K_n$, then $\lim_{r \to 1} g_k(re^{i\theta}) = \exp\{u_k(e^{i\theta}) + iv_k(e^{i\theta})\}$ $(k = 1, 2)$.

The existence of the sets K_n is assured by Fatou's theorem and the assumption that each E_n is metrically dense on C. Now, if we set $E = \bigcup_{n=1}^{\infty} K_n$, then E is a countable, dense subset of C. By Theorem 1, there exists a Blaschke product $w = B(z)$ such that

$$\{|w| = 1\} \subseteq C_\rho(B, e^{i\theta})$$

for each $e^{i\theta}$ in E. Let $f_1(z) = B(z) g_1(z)$ and $f_2(z) = B(z) g_2(z)$. Since multiplication by $B(z)$ is an isometry, we see that

$$\|f_1\| = \|f_2\| = 1 \quad \text{and} \quad \|f_1 f_2\| < 1.$$

To complete the proof of the theorem, it suffices to show that the functions $f_1(z)$ and $f_2(z)$ belong to I^*. Let $e^{i\theta_0}$ be some point of C, and let $w_0 = r_0 e^{i\phi_0}$ be any point of the open unit disk $|w| < 1$. Since the sequence $\{s_n\}$ is dense on $(-\infty, 0]$, we can find a convergent subsequence of $\{s_n\}$ such that

$$\lim_{n \to \infty} e^{s_n} = r_0,$$

where for simplicity of notation $\{s_n\}$ denotes the subsequence. Now, corresponding to each s_n of this subsequence, we select a point $e^{i\theta_n}$ in $E_n \cap E$ such that

$$\lim_{n \to \infty} e^{i\theta_n} = e^{i\theta_0}.$$

This is possible, because for each n the set $E_n \cap E$ is dense on C. Thus we see that at every point of the sequence $\{e^{i\theta_n}\}$, where $e^{i\theta_n} \in E_n \cap E$, the radial limit of $g_1(z)$ exists and, moreover,

$$\lim_{r \to 1} |g_1(re^{i\theta_n})| = e^{s_n}.$$

But $f_1(z) = B(z)\, g_1(z)$; therefore, for each index n, we have the inclusion

$$\{|w| = e^{s_n}\} \subseteq C_\rho(f_1, e^{i\theta_n}).$$

Consequently, w_0 belongs to the radial boundary cluster set of $f_1(z)$ at $e^{i\theta_0}$, and *a fortiori* w_0 is in $C(f_1, e^{i\theta_0})$. (For the definition of the radial boundary cluster set, we refer the reader to [1, p. 98].) Since w_0 is an arbitrary point in $|w| < 1$, and since $C(f_1, e^{i\theta_0})$ is closed and connected, we conclude that the cluster set $C(f_1, e^{i\theta_0})$ is the closed unit disk $|w| \leq 1$. A similar argument shows that the cluster set of $f_2(z)$ at each point $e^{i\theta}$ on C is also the closed unit disk. Therefore, the functions $f_1(z)$ and $f_2(z)$ belong to I^*, and the theorem is proved.

The following result is an immediate consequence of Theorem 2.

COROLLARY 1. I^* *is not a semigroup under multiplication.*

If in the proof of Theorem 2 we alter slightly the definition of the function $U_2(x)$, we obtain the following surprising result.

COROLLARY 2. *Let* $f_1(z)$ *be the function constructed in Theorem* 2. *Then for each* $\varepsilon > 0$, *there exists a function* $f_2(z)$ *in* I^* *such that* $\|f_1 f_2\| < \varepsilon$.

Before we state the next corollary, we introduce the following definition. We shall call a function f(z) in H^∞ a *generalized divisor of zero* if there exists a sequence $\{h_n\}$ of functions in H^∞ such that

$$\inf \|h_n\| > 0 \quad \text{and} \quad \lim_{n \to \infty} \|fh_n\| = 0.$$

COROLLARY 3. *The functions constructed in Theorem* 2 *are generalized divisors of zero.*

Remark. In [2, p. 25], we constructed a function f(z) in $I^* - I$ whose set of omitted values in $|w| < 1$ has positive (logarithmic) capacity. On the other hand, it is well known [1, p. 35] that the set of values omitted by a nontrivial inner function in $|w| < 1$ is at most of capacity zero. In a recent private communication, Professor George Piranian has shown to one of the authors that the set of omitted values of a function f(z) in $I^* - I$ can consist of a single point. A minor modification of our arguments shows that a function in $I^* - I$ can assume every value of $|w| < 1$.

As a final application of Theorem 2, we consider the extreme points of the unit ball Σ

$$\Sigma = \{f \in H^\infty: \|f\| \leq 1\}$$

in H^∞. (See [4, p. 136] for the definition of extreme points.) It is known [4, p. 138] that every inner function in H^∞ is an extreme point of Σ. On the other hand, our next theorem shows that functions in $I^* - I$ need not be extremal.

THEOREM 3. *There exists a function in* $I^* - I$ *that is not an extreme point of* Σ.

We present an outline of the proof. First consider the sequence $\{s_n\}$ used in the proof of Theorem 2, and modify it so that it satisfies the growth condition

$$\log(1 - e^{s_n}) > -n .$$

Next we repeat *verbatim* the relevant parts of the proof of Theorem 2 to obtain the function $f_1(z)$ in $I^* - I$. It follows then from the relation

$$\int_{-\pi}^{\pi} \log(1 - |f_1(e^{i\theta})|)\, d\theta > -\infty$$

and a result of Hoffman [4, p. 138] that $f_1(z)$ is not an extreme point of Σ.

Theorem 2 shows that I^* is not a semigroup under multiplication.

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