SEMINORMAL OPERATORS

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A bounded linear operator T on a Hilbert space is called a *seminormal* operator if $T^*T - TT^* = D \geq 0$ or $D \leq 0$. Several authors, especially C. R. Putnam, J. G. Stampfli, and S. K. Berberian, have determined conditions that assure the normality of a seminormal operator. Let $\mathscr{B}(H)$ denote the algebra of all bounded operators on a Hilbert space H, and $\mathscr{K}$ the ideal of all compact operators. Let $\hat{T}$ be the image of T in $\mathscr{B}(H)/\mathscr{K}$, under the quotient map, and let $\sigma(\hat{T})$ be the spectrum of $\hat{T}$ in the C^*-algebra $\mathscr{B}(H)/\mathscr{K}$. In Section 1, we show that T is normal whenever T is a seminormal operator and $\sigma(\hat{T})$ consists of certain arcs and a countable set. This will imply that T is normal if it is seminormal and the spectrum of a compact perturbation of T lies on certain arcs plus a countable set. These results extend some results obtained by T. Yoshino [13], the author [4], and Stampfli [8] to [11].

In Section 2, we use the results of Section 1 to obtain several theorems giving algebraic conditions under which T is normal. If T is a seminormal operator such that $I - T^*T$ is compact and $i(T - \lambda I) = 0$ (i is the Fredholm index) for some λ with $|\lambda| \leq \|T\|^{-1}$, then T is normal. From this we derive conditions on the strong asymptotic behavior of T and T^* that imply the normality of a seminormal operator T. For a seminormal contraction for which the rank of $I - T^*T$ is finite, we present necessary and sufficient conditions on the asymptotic behavior of T and T^* that imply normality.

1. SPECTRAL CONDITIONS

The *Weyl spectrum* $\omega(T)$ of T is defined as $\bigcap \sigma(A + K)$, where the intersection is taken over all K in $\mathscr{K}$ [3].

Our results are based on the relations among $\sigma(T)$, $\omega(T)$, and $\sigma(\hat{T})$. Whenever H is infinite-dimensional, then $\sigma(\hat{T}) \subset \omega(T) \subset \sigma(T)$, and each of these sets is a nonempty, compact subset of the plane. An operator is said to satisfy Weyl's theorem if $\omega(T) = \sigma(T) - \pi_{00}(T)$, where $\pi_{00}(T)$ is the set of isolated eigenvalues of finite multiplicity. L. A. Coburn [3] has shown that hyponormal operators (that is, operators for which $T^*T - TT^* \geq 0$) satisfy Weyl's theorem, and S. K. Berberian has shown that seminormal operators satisfy Weyl's theorem [1].

Recall that an operator is called a *semi-Fredholm* [*Fredholm*] operator if its range R(T) is closed and its null space N(T) is finite-dimensional [if N(T) and $R(T)^\perp$ are finite-dimensional]. The semi-Fredholm [Fredholm] operators constitute an open set in $\mathscr{B}(H)$. We shall denote the set of Fredholm operators by $\mathscr{F}$. If T is a semi-Fredholm operator, the *index* of T is defined to be

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$$i(T) = \dim N(T) - \dim R(T)^{\perp}.$$

The index is a continuous map, and $i(T)$ is finite if and only if $T \in \mathcal{F}$. Finally, T is a Fredholm [semi-Fredholm] operator if and only if $\hat{T}$ has an inverse [left inverse] in $\mathcal{B}(H)/\mathcal{K}$. For many of the facts concerning the theory of Fredholm operators, we refer to [6].

Next we show a relationship between $\omega(T)$ and $\sigma(\hat{T})$. M. Schechter [7] has characterized $\omega(T)$ as

$$\{\lambda \mid (T - \lambda I) \notin \mathcal{F}\} \cup \{\lambda \mid (T - \lambda I) \in \mathcal{F} \text{ and } i(T - \lambda I) \neq 0\}.$$

From this characterization of $\omega(T)$ we obtain the following lemma.

LEMMA 1. *The boundary of* $\omega(T)$ *is contained in* $\sigma(\hat{T})$.

Proof. This lemma follows from the observation that

$$S = \{A \mid A \in \mathcal{F} \text{ and } i(A) \neq 0\}$$

is open in $\mathcal{B}(H)$ and the map $\lambda \to T - \lambda I$ is continuous from $\mathbb{C}$ to $\mathcal{B}(H)$; the set S is open, since it is the intersection of two open sets in $\mathcal{B}(H)$.

It follows that if $\omega(T)$ has empty interior (in particular, if $\sigma(T)$ has empty interior), then $\omega(T) = \sigma(\hat{T})$.

Our theorems will combine the results of J. Stampfli and the algebraic decomposition of an arbitrary operator into a normal and a completely nonnormal part. This decomposition is known, and we state it without proof in the following lemma [5].

LEMMA 2. *Let* T *be an operator on* H. *There exists a unique decomposition* $H = H_1 \oplus H_2$ *into reducing subspaces* H_1 *and* H_2 *of* T, *such that* $T_1 \equiv T \mid H_1$ *is normal and* $T_2 \equiv T \mid H_2$ *has no reducing subspace on which* T_2 *is normal.*

In this paper we shall consider only curves Γ in the plane that have the following two properties:

1) Γ has a continuous second derivative at every point,

2) Γ has a countable set of crossing points.

J. Stampfli has shown that if $\sigma(T)$ lies on such a curve Γ with only a finite set of crossing points and if T is hyponormal, then T is normal [11]. His result can be extended to the case of a countable number of crossing points, by means of a result of the author [4, Proposition 4].

Let Γ be as described above, that is, let a countable number of crossing points be allowed. The following proposition gives a quite general condition for normality of a seminormal operator.

PROPOSITION 1. *If* T *is a seminormal operator and* Γ *is a curve* (*as described above*) *such that* $\sigma(T) - \Gamma$ *is countable, then* T *is normal.*

Proof. Let $T = T_1 \oplus T_2$ be the decomposition of T, as in Lemma 2, into normal and completely nonnormal parts, and let $H = H_1 \oplus H_2$ be the corresponding decomposition of H. Assume that $H_2 \neq \{0\}$, contrary to the proposition. Without loss of generality, we can also assume that T is hyponormal.

If the operator T_2 has any isolated points in its spectrum, then T_2 has a reducing subspace on which it is normal [8]. Hence we can assume that $\sigma(T_2)$ has no isolated points.

In the complex plane there exists no nonempty countable set each of whose points is a limit point of the set; that is, there exists no nonempty, countable, perfect subset of the plane. Combining this with the observation that $\sigma(T_2)$ has no isolated points, we conclude that $\sigma(T_2)$ lies on the curve Γ. Furthermore, if $\sigma(T_2)$ contained any simple segments of the curve then, by [11, Theorem 1], T_2 would have a nontrivial reducing subspace on which it is normal. Therefore we may conclude that $\sigma(T_2) \subset \{\lambda \mid \lambda \text{ is a crossing point of } \Gamma\}$. This is a countable set, and by [4, Proposition 4], the operator T_2 must be normal. This contradicts our assumption that $H_2 \neq \{0\}$. Therefore the proposition is proved.

We say that a closed curve in the plane *encloses* a point λ if λ lies in a bounded component of the complement of the curve. We can restate Proposition 1 in terms of the Calkin spectrum of the operator T.

PROPOSITION 2. *Let* T *be a seminormal operator, and let* Γ *be a curve (as described above) such that the set* $\sigma(\hat{T}) - \Gamma$ *is countable. If every closed curve in* $\sigma(\hat{T})$ *encloses a point not in* $\sigma(T)$, *then* T *is normal.*

Proof. Since the spectral conditions are satisfied for T^* as well as for T, it suffices to assume that T is hyponormal. First we show that as a subset of the plane, $\omega(T)$ contains no interior. If $\omega(T)$ has an interior, then Lemma 1 implies that $\sigma(\hat{T})$ contains a closed curve, and each point enclosed by that curve is in $\omega(T)$. Since $\sigma(T) \supset \omega(T)$, such a situation is impossible, by our last hypothesis. Applying Lemma 1, we conclude that $\omega(T) = \sigma(\hat{T})$.

By Weyl's theorem for hyponormal operators,

$$\sigma(T) = \omega(T) \cup \pi_{00}(T) = \sigma(\hat{T}) \cup \pi_{00}(T) .$$

Since $\pi_{00}(T)$ is countable, T satisfies the hypotheses of Proposition 1, and therefore T is normal.

The next results are direct corollaries of Propositions 1 and 2; special cases of them are already known. The following is a slight generalization of [11, Theorem 1].

PROPOSITION 3. *Suppose* T *is seminormal and* $T = B + K$, *where* K *is compact. If there exists a curve* Γ *such that* $\sigma(B) - \Gamma$ *is countable, then* T *is normal.*

Proof. Since $\sigma(\hat{T}) \subset \sigma(B)$, T will satisfy the hypothesis of Proposition 2 if each closed curve in $\sigma(\hat{T})$ encloses a point not in $\sigma(T)$. However, $\omega(T) = \omega(B) \subset \sigma(B)$, hence each closed curve in $\omega(T)$ encloses an open set of points not in $\sigma(T)$. By Weyl's theorem for seminormal operators, we conclude that each closed curve in $\sigma(T)$ enclosed a point not in $\sigma(T)$, and therefore the same is true for each closed curve in $\sigma(\hat{T})$. Thus T is normal, by Proposition 2.

As a direct corollary of Proposition 3 we obtain [13, Theorem 1].

COROLLARY 1. *If* T *is a seminormal operator and* $T = N + K$, *where* K *is compact and* N *is quasinilpotent, then* T *is normal.*

Remark. After this paper was submitted, the author received a preprint from C. R. Putnam, *An inequality for the area of hyponormal spectra.* The propositions in Section 1 can be deduced from Putnam's beautiful inequality.

2. ALGEBRAIC CONDITIONS

In this section, we restrict our attention to nonspectral conditions on a seminormal operator that imply normality. A *contraction* T is an operator such that $\|T\| \leq 1$.

THEOREM 1. *Let* T *be a seminormal contraction on a Hilbert space* H, *and let* $I - T^*T$ *be compact. Then* T *is normal if and only if* $i(T - \lambda I) = 0$ *for some* λ *with* $|\lambda| < 1$. *If* T *is normal, then* $\sigma(T) \cap \{\lambda \mid |\lambda| < 1\} = \{\lambda_j\}_{j \in J}$ *is countable and*

$$T = T_0 \oplus \sum_{j \in J} \oplus \lambda_j I_j,$$

where $H = H_0 \oplus \sum_{j \in J} \oplus H_j$ *is the corresponding decomposition of* H *by reducing subspaces of* T, T_0 *is unitary, each* H_j *is finite-dimensional,* I_j *is the identity operator on* H_j, *and* $|\lambda_j| \to 1$.

Proof. If T is normal, then $N(T) = N(T^*)$. Since $I - T^*T$ is compact, T is a semi-Fredholm operator, and $i(T) = 0$.

Conversely, we shall show that under the conditions of the theorem T is normal. By Lemma 2 we may assume, without loss of generality, that T is completely nonnormal, and prove the theorem by showing a contradiction. First we show that $T - \lambda I$ is a semi-Fredholm operator whenever $|\lambda| < 1$. Define

$$S_\lambda = (1 - |\lambda|^2)^{1/2}(I - \lambda T)^{-1} \quad \text{and} \quad T_\lambda = (I - \bar{\lambda}T)^{-1}(T - \lambda I).$$

The relation $I - T_\lambda^* T_\lambda = S_\lambda^*(I - T^*T)S_\lambda$ is easily verified. The hypothesis that $I - T^*T$ is compact implies that $I - T_\lambda^* T_\lambda$ is compact. Since

$$T_\lambda = (I - \bar{\lambda}T)^{-1}(T - \lambda I)$$

is a semi-Fredholm operator and $I - \bar{\lambda}T$ is invertible, the operator $(I - \bar{\lambda}T)T_\lambda = T - \lambda I$ is also a semi-Fredholm operator.

If $i(T - \lambda I) = 0$ for some $|\lambda| < 1$, then either $T - \lambda I$ is invertible or

$$\infty > \dim N(T - \lambda I) = \dim N(T^* - \bar{\lambda} I) > 0.$$

Since we assume that T is completely nonnormal, we can show that the relation $\dim (T - \lambda I) = \dim N(T^* - \lambda I) > 0$ cannot occur. Either T or T^* is hyponormal, and hence either $T - \lambda I$ or $T^* - \bar{\lambda} I$ is hyponormal. Thus either $N(T - \lambda I) \subset N(T^* - \lambda I)$ or $N(T^* - \bar{\lambda} I) \subset N(T - \lambda I)$. In either case, we must get equality, so that $N(T - \lambda I) = N(T^* - \lambda I)$. If we let $M = N(T - \lambda I)$, then $M \neq \{0\}$, and M reduces T. This contradicts the fact that T is completely nonnormal. Hence we conclude that $\lambda \notin \sigma(T)$ whenever $i(T - \lambda I) = 0$.

Because $\hat{T}$ is an isometry, we know that either $\sigma(\hat{T}) = \{\lambda \mid |\lambda| \leq 1\}$ or $\sigma(\hat{T}) \subset \{\lambda \mid |\lambda| = 1\}$. Since there exist $\mu \notin \sigma(T)$ with $|\mu| < 1$, we have the inclusion $\sigma(\hat{T}) \subset \{\lambda \mid |\lambda| = 1\}$. Hence T satisfies the hypothesis of Proposition 2, and therefore T is normal. This contradicts our assumption that T is completely nonnormal, and the proof of the theorem is complete.

Remark 1. In general, the requirement that $i(T - \lambda I) = 0$ for some λ ($|\lambda| < 1$) is weaker than the condition that there exists a λ ($|\lambda| < 1$) such that $\lambda \notin \sigma(T)$.

However, in the third paragraph of the proof of Theorem 1 we show that under the hypothesis of the theorem, the two conditions are equivalent.

Remark 2. We can remove the condition that $\|T\| \leq 1$ if we modify the hypothesis by replacing the condition that $i(T - \lambda I) = 0$, for some $|\lambda| < 1$, with the same condition for some $|\lambda| < \|T\|^{-1}$.

Remark 3. The simple unilateral shift V is hyponormal, and $I - V^*V$ is compact but V is not normal. For each λ, $|\lambda| < 1$, $i(V - \lambda I) = -1$, and $\sigma(V) = \{\lambda \mid |\lambda| \leq 1\}$.

We shall call on operator T *quasi-isometric* if $I - T^*T$ is compact. Thus Theorem 1 gives necessary and sufficient conditions for a quasi-isometric seminormal operator to be normal.

B. Sz.-Nagy and C. Foiaş have introduced a classification of contraction operators that depends on the asymptotic behavior of the iterates of T and T^* [12, Chapter II, Section 4]. A contraction T belongs to class

$C_{0.}$ if $T^n \to 0$ strongly,

$C_{1.}$ if $T^n h \not\to 0$ for each $h \neq 0$,

$C_{.0}$ if $T^{*n} \to 0$ strongly, and

$C_{.1}$ if $T^{*n} \not\to 0$ for each $h \neq 0$.

Furthermore, one denotes by C_{ab} the operators in $C_{a.} \cap C_{.b}$.

COROLLARY 2. *If* T *is a quasi-isometric seminormal contraction in class* C_{11}, *then* T *is normal.*

Proof. Since $T^n h \not\to 0$ for each $h \neq 0$, we see that $N(T) = \{0\}$. Similarily, $N(T^*) = \{0\}$, and hence $i(T) = 0$. Therefore, by Theorem 1, T is normal.

Now we shall prove the same result under the assumption that $I - T^*T$ has finite rank and $T \in C_{00}$. For this, we need the following two lemmas.

LEMMA 3. *Let* T *be any operator; then* $I - TT^* = W(I - T^*T)W^* + P$, *where* W *is a partial isometry and* P *is the projection on* $N(T^*)$.

Proof. By the polar decomposition of an arbitrary operator T, we see that $T = W|T|$, where $|T| = (T^*T)^{1/2}$ and W is a partial isometry with initial domain $[(T^*T)H]$ and final domain $[TH]$. Then $T^* = |T| W^*$ and

$$I - TT^* = I - W|T|\,|T|W^* = W(I - |T|^2)W^* + I - WW^*$$
$$= W(I - T^*T)W^* + I - WW^*.$$

Now WW^* is the projection on $[TH]$, so that $I - WW^*$ is the projection on $(TH)^\perp = N(T^*)$. This completes the proof.

It will be convenient to use the symbol δ_T for the rank of $I - T^*T$.

LEMMA 4. *If* T *is any operator, then* $\delta_T + \dim N(T^*) = \delta_{T^*} + \dim N(T)$.

Proof. In Lemma 3, we showed that $I - TT^* = W(I - T^*T)W^* + P$, where W is the partial isometry between $[T^*TH]$ and $[TH]$, and where P is the projection on

$N(T^*)$. Since W is an isometry with final domain $[TH]$, it is clear that $W(I - T^*T)W^*H$ is in the orthogonal complement of PH. Therefore

$$\delta_{T^*} = \dim(I - TT^*)H = \dim(W(I - T^*T)W^*H + PH)$$

$$= \dim W(I - T^*T)W^*H + \dim PH .$$

Next we shall show that $\dim(W(I - T^*T)W^*H) + \dim N(T) = \delta_T$. Since W is a partial isometry and $(I - T^*T)W^*H$ is in its initial domain, we see that

$$\dim(W(I - T^*T)W^*H) = \dim(I - T^*T)W^*H .$$

Also, $\dim(I - T^*T)W^*H = \dim(I - T^*T)W^*WH$, since $[WH]^\perp = N(W^*)$. If we set

$$(I - T^*T)H = (I - T^*T)W^*WH + (I - T^*T)(I - W^*W)H ,$$

then the ranges of the two operators $(I - T^*T)(W^*W)$ and $(I - T^*T)(I - W^*W)$ are orthogonal; for the former is contained in $[T^*TH]$, and the latter is simply $N(T) = N(T^*T)$. Thus we can conclude that $\delta_T = \dim((I - T^*T)W^*H) + \dim N(T)$.

Now we can finish the proof of Lemma 4 by considering two cases, depending on whether $N(T)$ is finite- or infinite-dimensional. If $N(T)$ is infinite-dimensional, then $\dim(I - T^*T)H = \delta_T$ is infinite, and the equality is trivially satisfied. If $N(T)$ is finite-dimensional, then we may subtract $\dim N(T)$ from both sides of the equality $\delta_T = \dim((I - T^*T)W^*H) + \dim N(T)$, which we obtained above. Thus $\dim((I - T^*T)W^*H) = \delta_T - \dim N(T)$, and substituting this in the equality $\delta_{T^*} = \dim((I - T^*T)W^*H) + \dim N(T^*)$, also obtained above, we deduce the desired equality.

We can now give necessary and sufficient conditions for the normality of a seminormal contraction T with $\delta_T < \infty$.

THEOREM 2. *Let* T *be a seminormal contraction with* $\delta_T < \infty$. *Then* T *is normal if and only if* $T \in C_{00} \cup C_{11}$ *or* T *is the direct sum of two operators, each in* $C_{00} \cup C_{11}$.

Proof. If T is normal, then the decomposition given in Theorem 1 shows that T is the direct sum of two operators, one in C_{11} and the other in C_{00}.

Conversely, we have already seen by Corollary 2 that if $T \in C_{11}$, then T is normal. Thus we need only consider the case where $T \in C_{00}$. It follows from the theory of unitary dilations, and specifically from [12, Theorem II. 1.2 and Proposition I. 2.1], that $\delta_{T^*} = \delta_T$ whenever $T \in C_{00}$ and $\delta_T < \infty$. Lemma 4 implies that $\delta_T + \dim N(T^*) = \delta_{T^*} + \dim N(T)$. Since δ_{T^*} is finite, the rank of $I - TT^*$ is finite, and hence $\dim N(T^*)$ is finite. Subtracting $\dim N(T^*)$ and δ_{T^*} from both sides of the equation, we conclude that $0 = \delta_T - \delta_{T^*} = \dim N(T) - \dim N(T^*) = i(T)$. Therefore it follows from Theorem 1 that T is normal.

Now we present an example to show that the conclusion of Theorem 2 cannot be extended to operators with $\delta_T = \infty$. Let T be the unilateral weighted shift with weights ω_i (if H is a separable Hilbert space and $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for H, then T is the operator that maps e_i to $\omega_i e_{i+1}$). Let us choose for T the weights $\omega_i = i/(i + 1)$. J. Stampfli has shown that with these weights T is seminormal. In fact,

$$(T^*T - TT^*)e_i = \left(\frac{i}{i+1}\right)^2 - \left(\frac{i-1}{i}\right)^2 \geq 0$$

when $i > 1$ and $(T^*T - TT^*)e_1 = 1/4$. Since the infinite product $\prod_{i=1}^{\infty}\left(1 - \frac{1}{i+1}\right)$ does not converge, one can show that T belongs to the class $C_{0\cdot}$. All unilateral shifts belong to the class $C_{\cdot 0}$, and therefore T belongs to C_{00}. Furthermore, the rank of $I - T^*T$ is infinite, and T is not normal. In fact, the operator $I - T^*T$ belongs to the Hilbert-Schmidt class of compact operators. Thus we see that Theorem 2 cannot be extended to the case $\delta_T = \infty$, even when $I - T^*T$ is a Hilbert-Schmidt operator.

M. S. Brodskiĭ and M. S. Livšic have studied operators T whose imaginary part $\Im T = \frac{1}{2i}(T - T^*)$ is compact. If T is seminormal and $\Im T$ is compact, we can give the complete structure of T.

COROLLARY 3. *If* T *is a seminormal operator with compact imaginary part, then* T *is normal and*

$$T = T_0 \oplus \sum_{i \geq 1} \oplus \lambda_i I_i ,$$

where $H = H_0 \oplus \sum \oplus H_i$ *is the corresponding decomposition of* H *by reducing spaces of* T, T_0 *is selfadjoint, each* I_i *is the identity operator on* H_i, *no* λ_i *is real, and each* H_i $(i \geq 1)$ *is finite-dimensional. Furthermore, the only possible limit points of* $\{\lambda_i\}$ *are real.*

Proof. By Proposition 2, T is normal. Let $\{\lambda_i\}$ be the set of nonreal points in $\sigma(T)$. By Weyl's theorem and Lemma 1, we conclude that the only accumulation points of $\{\lambda_i\}$ are real and that for $\lambda_k \in \{\lambda_i\}$, the corresponding eigenspace H_k is finite-dimensional and reduces T. Since T is normal, $H_i \perp H_j$ if $i \neq j$. Suppose $K = \sum \oplus H_i$ and $H_0 = H \ominus K$. Then K and H_0 reduce T, and $T_0 = T \mid H_0$ is selfadjoint. Therefore $T = T_0 \oplus \sum \oplus \lambda_i I_i$.

The fact that in this corollary T is normal is [13, Theorem 3].

For the sake of completeness we present the following corollary of Proposition 1. An operator T is called *polynomially compact* if there exists a nonzero polynomial p such that p(T) is compact.

COROLLARY 4. *If* T *is a polynomially compact, seminormal operator, then* T *is normal.* T *has a decomposition*

$$T = T_1 \oplus \cdots \oplus T_k,$$

where $H = H_1 \oplus \cdots \oplus H_k$ *is the corresponding decomposition of* H, *and there exists a set* $\{\lambda_1, \cdots, \lambda_k\} \subset \sigma(T)$ *such that each* $T_i - \lambda_i I_i$ *is a compact normal operator.*

Proof. The author has given the structure of polynomially compact operators in [4]. In particular, $\sigma(T)$ is countable. By Proposition 1, we can conclude that T is normal. The structure of a polynomially compact, normal operator is given in [4, Theorem 2].

Remark. Let T be any polynomially compact operator, and let

$$p(z) = \prod_{i=1}^{k} (z - \lambda_i)^{n_i}$$

be the polynomial of minimum degree and with leading coefficient 1 such that $p(\hat{T})$ is compact. Then $\omega(T) = \sigma(\hat{T}) = \{\lambda_1, \cdots, \lambda_k\}$. S. K. Berberian [2] has pointed out that if T is normal, then $\omega(T)$ is a finite set if and only if T is polynomially compact.

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