ON SPECIAL *-REGULAR RINGS

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1. A ring R is *regular* if for every $a \in R$ there exists an $x \in R$ such that $axa = a$. A regular ring with an involution * is called **-regular* if $xx^* = 0$ implies $x = 0$. In this note, we study *-regular rings with unit 1 that possess some additional properties. Before specifying these properties, let us state some known facts about regular rings.

Two idempotents e and f of a ring R are *equivalent* (notation: $e \sim f$) if $e = xy$ and $f = yx$, where $x \in eRf$ and $y \in fRe$. For each element a of a *-regular ring R, we have the relations $aR = eR$ and $Ra = Rf$, where e and f are uniquely determined projections (self-adjoint idempotents), called the *left* and the *right projection* of a. Further, there exists a uniquely determined element $\bar{a}$ - the *relative inverse* of a - such that $a\bar{a} = e$ and $\bar{a}a = f$. The left and the right projections of any element a are equivalent [1].

A *-regular ring is *complete* (alternate terminology: a *regular Baer *-ring*) if the lattice of its projections is complete. Let $\{e_\alpha\}$ be any set of projections. By LUB e_α and GLB e_α we shall denote the least upper bound and the greatest lower bound of the set $\{e_\alpha\}$. A complete *-regular ring R is finite; that is, if a projection $e \in R$ is equivalent to 1, then $e = 1$. The *central cover* of a projection $e \in R$ is the smallest central projection g in R for which $ge = e$.

A regular ring is *abelian* if all its idempotents are central. An idempotent $e \in R$ is *abelian* if the subring eRe is abelian. A *-regular ring R is of *type* I if it has an abelian projection with central cover 1, and of *type* II if it does not possess nonzero abelian projections.

Every complete *-regular ring is a direct sum of a ring of type I and a ring of type II (see [1], [3]). A ring of type I is *n-homogeneous* (or of *type* I_n) if there exists a set of n mutually orthogonal equivalent abelian projections whose sum is 1. Every complete *-regular ring of type I is a special subdirect sum of homogeneous rings. The proof of this theorem can be found in [1], [2], [3]. Here we shall give a proof of the structure theorem based on the following known properties of complete *-regular rings of type I: (a) abelian projections with the same central cover are equivalent, and (b) if g is the central cover of a projection f, then there exists an abelian projection $e \leq f$ with central cover g.

First we show that in every complete *-regular ring of type I there exist a sequence of orthogonal abelian projections e_n and a decreasing sequence of central projections g_n with the following properties: The central cover of e_n is g_n. If $h_n = g_n - g_{n+1}$ and $h_n \neq 0$, then $h_n e_1, \cdots, h_n e_n$ are orthogonal equivalent abelian projections with the sum h_n. Thus the subring Rh_n is n-homogeneous.

The construction of these sequences is by induction on n. Since R is of type I, there exists an abelian projection e with central cover 1. We put $e_1 = e$ and $g_1 = 1$. Suppose now that the projections $e_1, \cdots, e_n$ and $g_1, \cdots, g_n$ with the required

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properties are already known. Take for g_{n+1} the central cover of

$$f = g_n(1 - e_1 - \cdots - e_n).$$

Clearly, $g_{n+1} \leq g_n$. By (b), there exists an abelian projection $e_{n+1} \leq f$ with central cover g_{n+1}. Write $h_n = g_n - g_{n+1}$. Since $g_{n+1} f = f$, we have the relation $h_n e_1 + \cdots + h_n e_n = h_n$. The summands on the left are orthogonal abelian projections with central cover h_n. Hence, by (a), the $h_n e_i$ are mutually equivalent. Furthermore, $e_{n+1} \leq f = g_n(1 - e_1 - \cdots - e_n)$ implies that e_{n+1} is orthogonal to $g_n e_1, \cdots, g_n e_n$. Since $g_{n+1} e_{n+1} = e_{n+1}$, e_{n+1} is orthogonal to $e_1, \cdots, e_n$. Hence e_{n+1} and g_{n+1} have all the required properties. It follows that we can construct the sequences $\{e_n\}$ and $\{g_n\}$.

Let $g = \text{GLB}\, g_n$. Since $g_n e_1, \cdots, g_n e_n$ are orthogonal abelian projections with central cover g_n, they are mutually equivalent. It follows that $g g_n e_i = g e_i$ $(i = 1, 2, \cdots, n)$ are orthogonal equivalent projections with central cover g. Since this holds for each n, we conclude that $\{ge_i\}$ is an infinite sequence of orthogonal equivalent projections. But a complete *-regular ring does not contain an infinite set of orthogonal equivalent projections different from 0. Hence $g = 0$.

If $h_n = g_n - g_{n+1}$, then $\text{LUB}\, h_n = 1$. The subring Rh_n is n-homogeneous if $h_n \neq 0$. Hence, we have proved the structure theorem for rings of type I in the following form.

If R *is a complete *-regular ring of type* I, *then there exists a sequence of orthogonal central projections* h_n, *with* $\text{LUB}\, h_n = 1$, *such that the subring* Rh_n *is* n-*homogeneous for each* $h_n \neq 0$.

2. The *-regular rings derived from finite W*-algebras have many special properties. We shall study the *-regular rings possessing some of these properties, and we shall investigate the interdependence of these properties, regarded as axioms. We shall consider five axioms:

(A) *The equality*

$$a_1 a_1^* + a_2 a_2^* + \cdots + a_n a_n^* = 0 \tag{1}$$

implies $a_1 = a_2 = \cdots = a_n = 0$ *for any number* n *of summands on the left.*

(A') *If* $aa^* + bb^* = 0$, *then* $a = b = 0$.

(B) *For all* $a, b \in R$, *there exists an element* $x \in R$ *such that*

$$xx^* = aa^* + bb^*. \tag{2}$$

(C) *If* $aa^* \in eRe$, *where* e *is any projection, then we can find an element* $x \in eRe$ *such that* $xx^* = aa^*$.

(D) *Every element of the form* aa^* *is the square of a self-adjoint element* u; *thus* $u^2 = aa^*$.

(E) *If* e *and* f *are equivalent projections, then there exists a* $u \in R$ *such that* $e = uu^*$ *and* $f = u^*u$.

These axioms hold in every C*-algebra.

Discussion of the axioms. 1. A *-regular ring satisfying Axiom A has characteristic 0 and can be regarded as an algebra over the field of rational numbers. The properties of such rings are studied in [4].

2. Axiom A implies Axiom A', but not conversely. In fact, let R be the field Z/(p) of residue classes modulo a prime p, where $p \equiv -1 \pmod 4$. Z/(p) is a *-regular ring, if we take the identity map as involution (that is, $x^* = x$). In this ring, the equality

$$aa^* + bb^* = a^2 + b^2 = 0$$

implies $a = b = 0$, so that Z/(p) satisfies Axiom A'. Since the equation $a^2 + b^2 + c^2 = 0$ has nontrivial solutions in Z/(p), Axiom A does not hold.

On the other hand, if a ring satisfies A' and B, then evidently it satisfies also A.

3. Axioms A and B are mutually independent. For let R be the field of complex numbers, and let the involution be the identity map. In this case equation (2) is of the form $a^2 + b^2 = x^2$ and always has a solution. Thus Axiom B holds. Since the equality $a^2 + b^2 = 0$ is satisfied for $a = bi$, Axiom A' does not hold.

On the other hand, take the field of rational complex numbers, where the involution is the complex conjugation. In this *-regular ring, Axiom A holds and B does not.

4. Let R be any ring with involution *. By S(R) we shall denote the set of all elements of the form aa^*; thus $S(R) = \{aa^* \mid a \in R\}$. If a *-regular ring satisfies Axiom C, then $S(eRe) = S(R) \cap eRe$.

5. I. Kaplansky introduced an axiom [2, Axiom SR] similar to Axiom D, but with the additional assumption that if u satisfies the condition $u^2 = aa^*$, it commutes with every element commuting with aa^*.

It is easy to see that Axiom D implies Axiom C. In fact, suppose that $aa^* \in eRe$, where e is a projection. By Axiom D, there exists a self-adjoint u such that $u^2 = aa^*$. It follows that $eu^2 = u^2$. Hence $(eu - u)(eu - u)^* = 0$, whence $eu = u$. Therefore, $aa^* = u^2 = uu^*$, where $u \in eRe$.

6. Two projections e and f are called **-equivalent* [2] (notation: $e \overset{*}{\sim} f$) if there exists a u such that $e = uu^*$ and $f = u^*u$. It is well known that *-equivalence implies ordinary equivalence, but not conversely. If $uu^* = e$, where e is a projection, then $u^*u = f$ is also a projection and $u \in eRf$.

3. Axioms C, D, and E are not independent. In fact, we have the following result.

THEOREM 1. *Axiom* D *implies Axiom* E, *and Axiom* E *implies Axiom* C.

*If a *-regular ring is finite and satisfies Axiom* C, *then it satisfies Axiom* E. *Hence, Axioms* C *and* E *are equivalent in finite rings.*

Proof. (i) D $\Rightarrow$ E (see [2]). Assume that Axiom D holds in a *-regular ring R. If $e \sim f$, there exist elements x and y such that

$$e = xy, \quad f = yx, \quad x \in eRf, \quad y \in fRe.$$

Choose a self-adjoint element v satisfying the condition $v^2 = yy^*$. Clearly, v commutes with yy^*. Since $fy = y$, we have the relation $fv^2 = v^2$, and therefore $(fv - v)(fv - v)^* = 0$. Hence $fv = vf = v$. Moreover,

$$(x^*x)(yy^*) = (yy^*)(x^*x) = f,$$

and if we write $u = xv$, we find that $uu^* = (xy)(y^*x^*) = e$ and

$$u^*u = vx^*xv = fvx^*xv = (x^*x)(yy^*)v(x^*x)v = (x^*x)v(yy^*)(x^*x)v$$
$$= (x^*x)v^2 = (x^*x)(yy^*) = f .$$

Hence $e \overset{*}{\sim} f$.

(ii) $E \Rightarrow C$. Suppose that Axiom E is valid in R. If a is such that $aa^* \in eRe$, where e is a projection, then $e(aa^*) = aa^*$. It follows that $ea = a$. Let e_1 and f_1 denote the left and the right projection of a. From $ea = a$ we deduce that $ee_1 = e_1$; thus $e_1 \leq e$. Since $e_1 \sim f_1$, it follows further from Axiom E that $e_1 \overset{*}{\sim} f_1$. Hence $e_1 = uu^*$ and $f_1 = u^*u$, where $u \in e_1Rf_1$. Put $v = au^*$. Then

$$vv^* = au^*ua^* = af_1a^* = aa^*,$$

since $af_1 = a$. The element $v = au^*$ belongs to the subring $e_1Re_1 \subset eRe$. Hence Axiom C holds in R.

(iii) $C \Rightarrow E$. Let R be a finite ring satisfying Axiom C. Let e and f be equivalent projections:

$$e = xy, \quad f = yx, \quad x \in eRf, \quad y \in fRe.$$

Since $yy^* \in fRf$, we can (by Axiom C) choose an element $v \in fRf$ such that $vv^* = yy^*$. Put $u = xv$. Then $uu^* = xyy^*x^* = e$. Further, $u^*u = v^*x^*xv = f_1$, where f_1 is a projection. It follows that $e \overset{*}{\sim} f_1$. Since $vf = v$, we see that $f_1f = f_1$. Hence $f_1 \leq f$. On the other hand, $f_1 \sim e \sim f$; thus $f_1 \sim f$. The finiteness of the ring R now implies $f_1 = f$. Hence $e \overset{*}{\sim} f$, so that Axiom E holds in R. The proof of Theorem 1 is now complete.

The dependence between Axioms A and B and Axioms C and D is more complicated.

LEMMA 1. *Let the *-regular ring* R *satisfy Axiom* E, *and let* $e \in R$ *be a projection such that there exists an equivalent orthogonal projection* $f \in R$. *Then Axioms* A' *and* B *hold in the subring* eRe.

Proof. Since Axiom E holds in R, and since $e \sim f$, there exists a $u \in eRf$ such that $e = uu^*$ and $f = u^*u$. The orthogonality of e and f implies

$$fu = feu = 0, \quad ue = ufe = 0, \quad u^2 = ufu = 0.$$

Now, let a and b be any elements of the subring eRe. If $z = a + bu$, then

$$zz^* = (a + bu)(a^* + u^*b^*) = aa^* + bb^* . \tag{3}$$

By Theorem 1, R satisfies also Axiom C. Hence, there exists an $x \in eRe$ such that $xx^* = zz^* = aa^* + bb^*$. Therefore, Axiom B holds in the subring eRe.

If $a, b \in eRe$ are such that $aa^* + bb^* = 0$, then we see from (3) that $z = a + bu = 0$. Multiplication by e on the right gives $a = 0$. Thus, Axiom A' is valid in the subring eRe.

Axioms A' and B imply that Axiom A holds in eRe, and similarly, in fRf.

THEOREM 2. *Let Axiom* E *hold in a *-regular ring* R. *If* $a_1, a_2, \cdots, a_n$ *satisfy equation* (1), *then the right projection of each element* a_k *is central and abelian.*

Proof. Let equation (1) hold. If all a_k are 0, there is nothing to prove. Suppose now, for instance, that $a_1 \neq 0$. Denote by e the right projection of a_1, and by $\bar{a}$ its relative inverse. If we multiply (1) by $\bar{a}$ on the left and by $\bar{a}^*$ on the right, we obtain the equation

$$e + b_2 b_2^* + \cdots + b_n b_n^* = 0 , \tag{4}$$

where $b_k = \bar{a} a_k$ $(k = 2, \cdots, n)$.

Take any element $x \in R$, and consider $ex(1 - e)$. If this product is not zero, then its right projection e_1 and its left projection f_1 are orthogonal, equivalent, and different from 0. If we multiply (4) by e_1 on both sides, we get the equation

$$e_1 + e_1 b_2 b_2^* e_1 + \cdots + e_1 b_n b_n^* e_1 = 0 . \tag{4*}$$

By assumption, Axiom E holds in R, hence also Axiom C. Since

$$(e_1 b_k)(e_1 b_k)^* \in e_1 R e_1 ,$$

we can choose $c_k \in e_1 R e_1$ so that $c_k c_k^* = e_1 b_k b_k^* e_1$. Thus (4*) can be written in the form $e_1 + c_2 c_2^* + \cdots + c_n c_n^* = 0$. By Lemma 1, Axiom A holds in the subring $e_1 R e_1$, because e_1 and f_1 are equivalent orthogonal projections. Hence $e_1 = c_1 = \cdots = c_n = 0$. This contradicts $e_1 \neq 0$. Consequently, $ex(1 - e) = 0$ for every $x \in R$. Similarly, $(1 - e)xe = 0$. Hence $ex = exe = xe$, and e is central.

Now take any projection $e' \leq e$. Multiplying both sides of (4) by e', we get the equation

$$e' + e' b_2 b_2^* e' + \cdots + e' b_n b_n^* e' = 0 .$$

As before, we conclude from (4) that e' is a central projection. Hence, e is central and abelian. This completes the proof of Theorem 2.

COROLLARY. *If a *-regular ring has no nonzero central abelian projections, then Axiom* E *implies Axiom* A.

Proof. Let Axiom E be valid in R. If $a_1, a_2, \cdots, a_n$ satisfy (1), then, by Theorem 2, the right projection of each a_k is central and abelian. If R has no nontrivial central abelian projections, then $a_k = 0$. Therefore, Axiom A holds in R.

4. Let R be a ring with involution *. We shall denote by P(R) the set of all elements of the form $a_1 a_1^* + a_2 a_2^* + \cdots + a_n a_n^*$, where $a_1, a_2, \cdots, a_n$ are any elements of R, their number $n \geq 1$ being arbitrary. The set P(R) is evidently closed under addition: $p_1, p_2 \in P(R)$ implies $p_1 + p_2 \in P(R)$. Furthermore, if $p \in P(R)$ and c is any element of R, then $cpc^* \in P(R)$. In fact, from $p = a_1 a_1^* + \cdots + a_n a_n^*$ we see that

$$cpc^* = (ca_1)(ca_1)^* + (ca_2)(ca_2)^* + \cdots + (ca_n)(ca_n)^* \in P(R) .$$

The set S(R) introduced above is a subset of P(R). Axiom B holds in R if and only if these two sets coincide.

LEMMA 2. *Let* $a_1, a_2, \cdots, a_m$ *be any elements of a ring* R *with involution* *, *and let* $b_1, b_2, \cdots, b_m$ *be such that* $b_1 b_1^* + b_2 b_2^* + \cdots + b_m b_m^* = 1 - p$, *where* $p \in P(R)$. *Then*

$$(5)\qquad \sum_{k=1}^{m} a_k a_k^* = \left(\sum_{k=1}^{m} a_k b_k^*\right)\left(\sum_{k=1}^{m} a_k b_k^*\right)^* + p',$$

where $p' \in P(R)$.

Proof. Put $u = \sum_{k=1}^{m} a_k b_k^*$. The sum

$$p_1 = \sum_{k=1}^{m} (a_k - ub_k)(a_k - ub_k)^*$$

may be written in the form

$$p_1 = \sum_{k=1}^{m} a_k a_k^* - 2uu^* + u\left(\sum_{k=1}^{m} b_k b_k^*\right)u^* = \sum_{k=1}^{m} a_k a_k^* - uu^* - upu^* .$$

Since $p \in P(R)$, the element upu^* is also in $P(R)$. If we write $p' = p_1 + upu^*$, where $p' \in P(R)$, then we obtain (5). Lemma 2 is thus proved.

LEMMA 3. *Let* R *be a *-regular ring, and let* e *and* f *be orthogonal projections such that Axioms* A' *and* B *hold in* eRe *and in* fRf. *Further, assume that the equation* $xx^* = aa^*$ *can be solved by an element* $x \in eRe$ *if* $aa^* \in eRe$, *and by an element* $x \in fRf$ *if* $aa^* \in fRf$. *Then Axioms* A' *and* B *hold in the subring* $(e+f)R(e+f)$.

Proof. Without loss of generality, we may assume that $e + f = 1$.

(i) Let $a, b \in R$ be such that $aa^* + bb^* = 0$. It follows that $eaa^*e + ebb^*e = 0$. By assumption, there exist elements $a_1, b_1 \in eRe$ such that

$$eaa^*e = a_1 a_1^* \qquad \text{and} \qquad ebb^*e = b_1 b_1^* .$$

Since Axiom A' holds in eRe, $a_1 a_1^* + b_1 b_1^* = 0$ implies $a_1 = b_1 = 0$. Thus $ea = eb = 0$. Similarly, we deduce that $fa = fb = 0$. Hence $a = b = 0$. Thus Axiom A' is valid in R.

(ii) Now, let $a, b \in R$ be arbitrary. The equation

$$(6)\qquad xx^* = aa^* + bb^*$$

is equivalent to the system

$$exx^*e = eaa^*e + ebb^*e, \qquad exx^*f = eaa^*f + ebb^*f, \qquad fxx^*f = faa^*f + fbb^*f .$$

By assumption, we can find elements $a_1, b_1 \in eRe$ and $a_2, b_2 \in fRf$ such that

$$eaa^*e = a_1 a_1^*, \qquad ebb^*e = b_1 b_1^*, \qquad faa^*f = a_2 a_2^*, \qquad fbb^*f = b_2 b_2^*.$$

Put

$$exe = x_1, \qquad exf = x_2, \qquad fxf = u;$$

then $ex = x_1 + x_2$. In this way we obtain the system

(7) $x_1x_1^* + x_2x_2^* = a_1a_1^* + b_1b_1^*, \quad x_2u^* = eaa^*f + ebb^*f, \quad uu^* = a_2a_2^* + b_2b_2^*.$

Since Axiom B is valid in fRf, there exists a $u \in fRf$ satisfying the third equation (7). Let g and h be the right and the left projection of u, and let $\bar{u} \in fRf$ be its relative inverse, so that $u\bar{u} = g$ and $\bar{u}u = h$. Since $u \in fRf$, we see that $gf = g$ and $hf = h$. Since $gu = u$, the third equation (7) gives the relation

$$(1 - g)faa^*f(1 - g) + (1 - g)fbb^*f(1 - g) = 0.$$

By Axiom A', it follows that $(1 - g)fa = (1 - g)fb = 0$, or $fa = ga$ and $fb = gb$. Write $x_2 = eaa^*\bar{u}^* + ebb^*\bar{u}^*$. Then

$$x_2u^* = eaa^*g + ebb^*g = eaa^*f + ebb^*f.$$

Hence, x_2 and u satisfy the second equation (7). Now we can apply Lemma 2 for $m = 2$, where we replace a_1, a_2 by ea and eb, and b_1, b_2 by $\bar{u}a$, $\bar{u}b$. In fact,

$$(\bar{u}a)(\bar{u}a)^* + (\bar{u}b)(\bar{u}b)^* = h = 1 - p,$$

where $p = 1 - h \in P(R)$. By (5) we see that $eaa^*e + ebb^*e = x_2x_2^* + p'$, where p' belongs to the set P(eRe). Because Axiom B holds in eRe, P(eRe) = S(eRe). This implies the existence of an element $x_1 \in eRe$ such that $p' = x_1x_1^*$. Therefore the first equation (7) is also satisfied. Let now $x = u + x_1 + x_2$. Since $x_1 \in eRe$, $x_2 \in fRf$, $u \in fRf$, and $ef = 0$, we have the equation

$$xx^* = uu^* + x_1x_1^* + x_2x_2^* + x_2u^* + ux_2^* = aa^* + bb^*.$$

Thus equation (6) has a solution. This completes the proof of Lemma 3.

5. It is well known that the ring R_n of all n-by-n matrices over a ring R is regular if and only if R is regular. Let R be *-regular, and let $X = (x_{ij})$ $(x_{ij} \in R)$ be any matrix. Put $X^* = (x_{ij}^*)^T$, where $(a_{ij})^T$ denotes the transpose of the matrix (a_{ij}). The map $X \to X^*$ is an involution in the ring R_n. In general, R_n is not *-regular, since $XX^* = 0$ does not imply $X = 0$. However, it is not difficult to see that R_n is *-regular for each n if and only if the ring R satisfies Axiom A.

THEOREM 3. *If Axioms* A' *and* B *hold in a *-regular ring* R, *then these axioms hold also in every matrix ring* R_n.

Proof. Since Axioms A' and B are valid in R, Axiom A is also valid. It follows that the matrix ring R_n is *-regular and that it satisfies Axiom A.

Since the ring R_1 is isomorphic with R, Theorem 3 is true for $n = 1$. Let us now suppose that Axiom B holds in R_m for some $m \geq 1$. We shall show that the axiom also holds in the matrix ring R_{m+1}. Let $E \in R_{m+1}$ be the matrix having 1's in the first m entries of the main diagonal, and 0's elsewhere. E is a projection, and $ER_{m+1}E$ is isomorphic to R_m. On the other hand, $FR_{m+1}F$ $(F = 1 - E)$ is isomorphic to R. Since, by the induction hypothesis, Axioms A' and B hold in R_m, we conclude by Lemma 3 that these axioms hold also in the ring R_{m+1} if the following additional conditions of this Lemma are satisfied: for each matrix $M \in R_{m+1}$, the equation $XX^* = MM^*$ is satisfied by a matrix $X \in ER_{m+1}E$ if $MM^* \in ER_{m+1}E$, and by a matrix $X \in FR_{m+1}F$ if $MM^* \in FR_{m+1}F$. We see in the following way that these conditions are fulfilled. Each element $M \in R_{m+1}$ can be written as 2-by-2 matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{8}$$

where A is an m-by-m matrix, $D \in R$, and B and C are m-by-1, respectively, 1-by-m matrices. Let $MM^* \in ER_{m+1}E$; then $EMM^* = MM^*$. This implies that $EM = M$. Since $E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, C and D are 0 in (8). It follows that

$$MM^* = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} AA^* + BB^* & 0 \\ 0 & 0 \end{pmatrix}.$$

Here we have denoted by B^* the 1-by-m matrix whose adjoint elements are those of B:

$$B = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, \qquad B^* = (b_1^*, \cdots, b_m^*).$$

Let D_1 be the m-by-m matrix having the elements $b_1, \cdots, b_m$ in the first column and 0's elsewhere. Evidently, $D_1 \in R_m$ and $BB^* = D_1 D_1^*$. By assumption, Axiom B is valid in R_m. Therefore, there exists a matrix $Y \in R_m$ such that $YY^* = AA^* + D_1 D_1^*$. Now we can write

$$MM^* = \begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix}^*.$$

If X denotes the first matrix on the right, then $MM^* = XX^*$, where $X \in ER_{m+1}E$.

Suppose now that $MM^* \in FR_{m+1}F$, so that $FM = M$. In this case, the matrix M has the form $M = \begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix}$, hence

$$MM^* = \begin{pmatrix} 0 & 0 \\ 0 & CC^* + DD^* \end{pmatrix}.$$

Here $CC^* + DD^* = c_1 c_1^* + \cdots + c_m c_m^* + dd^*$, where $c_1, \cdots, c_m, d \in R$. Since Axiom B holds in R, $c_1 c_1^* + \cdots + c_m c_m^* + dd^* = uu^*$ for some $u \in R$. Hence

$$MM^* = \begin{pmatrix} 0 & 0 \\ 0 & u \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & u \end{pmatrix}^* = XX^*,$$

where $X \in FR_{m+1}F$. We now conclude, by Lemma 3, that Axioms A' and B hold in the ring R_{m+1}. This completes the proof of Theorem 3.

THEOREM 4. *Let* R *be a complete *-regular ring without nonzero central abelian projections. Then Axiom* C *implies Axioms* A *and* B.

Proof. A complete *-regular ring is finite. Hence, by Theorem 1, Axiom C implies Axiom E. The ring R being without nonzero central abelian projections, we conclude by the corollary to Theorem 2 that Axiom A is valid in R.

Decompose R into a special subdirect sum of rings of types I_n and II. Let g and h_n ($n = 1, 2, \cdots$) be the corresponding central projections (here $h_n = 0$ if the component I_n is absent in R). Since there are no central abelian projections, $h_1 = 0$. Put

$$h' = g + \mathrm{LUB}\, h_{2m}, \quad h'' = \mathrm{LUB}\, h_{2m+1} \qquad (m = 1, 2, \cdots).$$

If $R' = Rh'$ and $R'' = Rh''$, then $R = R' \oplus R''$.

Consider first the summand R'. Since Rh_{2m} is 2m-homogeneous, we can find two orthogonal equivalent projections e_{2m} and f_{2m} with the sum h_{2m}. Also, $g = e' + f'$, where e' and f' are equivalent and orthogonal projections (see [1]). Here it is understood that $e_{2m} = f_{2m} = 0$ if $h_{2m} = 0$, and that $e' = f' = 0$ if $g = 0$. Let

$$e = e' + \mathrm{LUB}\, e_{2m}, \qquad f = f' + \mathrm{LUB}\, f_{2m}.$$

Then e and f are orthogonal equivalent projections with the sum $e + f = h'$. Axiom E being valid in R, we conclude by Lemma 1 that Axioms A and B hold in $eR'e$ and $fR'f$, and then, by Lemma 3, that these axioms hold also in R.

In each subring Rh_{2m+1} we choose four projections e_{2m+1}, f_{2m+1}, k_{2m+1}, t_{2m+1} in the following way: the first three are mutually orthogonal to the sum h_{2m+1}; further, $e_{2m+1} \sim f_{2m+1}$; and finally, k_{2m+1} and t_{2m+1} are equivalent orthogonal abelian projections with central cover h_{2m+1}. Denote by e, f, k, t the LUB of the sets of projections $\{e_{2m+1}\}$, $\{f_{2m+1}\}$, $\{k_{2m+1}\}$, $\{t_{2m+1}\}$. Then

$$e + f + k = h'', \quad e \sim f, \quad k \sim t, \quad ef = 0, \quad kt = 0.$$

By Lemma 1, Axioms A and B hold in the rings $eR''e$, $fR''f$, and $kR''k$. Hence, by Lemma 3, they hold also in the ring $(e + f + k)R''(e + f + k) = R''$. The proof of Theorem 4 is now complete.

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