ON POLYNOMIAL RINGS OVER A HILBERT RING

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1. INTRODUCTION

Let K be a field with algebraic closure $\overline{K}$, let $\{X_i\}_1^n$ be a finite set of indeterminates over K, let f, $f_1, f_2, \cdots, f_r \in K[X_1, \cdots, X_n]$, and let A be the ideal of $K[X_1, \cdots, X_n]$ generated by $\{f_i\}_1^r$. Many equivalent forms of Hilbert's Nullstellensatz are known; among these equivalent statements are the following (see [7], [9, p. 19], [10], [12, pp. 5-6], and [13, pp. 164-167]).

(HN 1) *If* f *vanishes at each common zero of* $f_1, f_2, \cdots, f_r$ *over each extension field* L *of* K, *then* $f \in \sqrt{A}$.

(HN 2) *If* f *vanishes at each common zero of* $f_1, f_2, \cdots, f_r$ *over* $\overline{K}$, *then* $f \in \sqrt{A}$.

(HN 3) *If the polynomials* f_i *have no common zero over* $\overline{K}$, *then* $A = K[X_1, \cdots, X_n]$.

(HN 4) *If* $K[a_1, \cdots, a_n]$ *is a field, then each* a_i *is algebraic over* K.

(HN 5) $K[X_1, \cdots, X_n]/M$ *is algebraic over* K *for each maximal ideal* M *of* $K[X_1, \cdots, X_n]$.

(HN 6) *Each maximal ideal of* $\overline{K}[X_1, \cdots, X_n]$ *is of the form*

$$(X_1 - t_1, \cdots, X_n - t_n)$$

for some $t_1, \cdots, t_n \in \overline{K}$.

(HN 7) *Each proper prime ideal of* $K[X_1, \cdots, X_n]$ *is an intersection of maximal ideals of* $K[X_1, \cdots, X_n]$.

O. Goldman [7] and W. Krull [10] (see also [9, Chapter 1, Sections 1 to 3]) have developed an elegant approach to the Nullstellensatz, using the concept of a *Hilbert ring*, defined as follows (Goldman uses the term Hilbert ring; Krull's terminology is *Jacobson ring*). A commutative ring R with identity is a *Hilbert ring* if condition (HR 1) is satisfied in R.

(HR 1) *Each proper prime ideal of* R *is an intersection of maximal ideals of* R.

Again, several equivalent forms of (HR 1) are known, including the following (see [1, Chapter 5, Section 3, No. 4], [7], [9, Chapter 1, Sections 1 to 3], [10]).

(HR 2) *The Jacobson radical of* R/P *is zero for each proper prime ideal* P *of* R.

(HR 3) *If* P *is a nonmaximal proper prime ideal of* R, *then* P *is the intersection of a set of prime ideals of* R *properly containing* P; *equivalently, the pseudo-radical of* R/P *is zero for each nonmaximal proper prime ideal* P *of* R. (If S is a

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commutative ring, the *pseudoradical* of S is the intersection of the set of nonzero prime ideals of S; see [3].)

(HR 4) $R[X]$ *is a Hilbert ring.*

(HR 5) *For each maximal ideal* M *of* $R[X]$, $M \cap R$ *is a maximal ideal of* R.

For polynomial rings in infinitely many indeterminates, the situation concerning Hilbert's Nullstellensatz and Hilbert rings is quite different from the case of polynomial rings in finitely many indeterminates. For example, if K is a field and $\{X_\lambda\}$ is an infinite set of indeterminates over K, then $K[\{X_\lambda\}]$ need not be a Hilbert ring [10, Section 3], [1, Exercise 10, pp. 86-87], [11], and in fact, none of the conditions (HN 2) to (HN 7) need hold in $K[\{X_\lambda\}]$. The most general result of this paper is the following theorem (Theorem 3.3):

If R *is a commutative ring with identity and if* $\{X_\lambda\}_{\lambda \in \Lambda}$ *is an infinite set of indeterminates over* R, *then* $R[\{X_\lambda\}]$ *is a Hilbert ring if and only if* R *is a Hilbert ring and* $|\Lambda| < |R/M|$ *for each maximal ideal* M *of* R. (We use $|S|$ to denote the cardinality of the set S.)

The relation of the preceding theorem to the conditions (HN 1) to (HN 7) is summarized in Theorem 2.10 of Section 2.

2. POLYNOMIAL RINGS IN INFINITELY MANY INDETERMINATES

Throughout this section, R denotes a commutative ring with identity, $\{X_\lambda\}_{\lambda \in \Lambda}$ is a set of indeterminates over R, and $S = R[\{X_\lambda\}]$. We are concerned primarily (but not exclusively) with the case where Λ is infinite.

2.1. PROPOSITION. *If* M *is a prime ideal of* S *and if* $P = M \cap R$, *the following conditions are equivalent.*

a) S/M *is algebraic over* R/P.

b) $M \cap R[X_\lambda] \supset P[X_\lambda]$ *for each* λ *in* Λ.

Proof. a) $\to$ b): For each λ in Λ, we let $x_\lambda = X_\lambda + M$. By hypothesis, there exist elements $r_0, r_1, \cdots, r_n$ in R, with $r_n \notin P$, such that $\sum_0^n r_i x_\lambda^i = 0$. Hence $\sum_0^n r_i X_\lambda^i \in (M \cap R[X_\lambda]) - P[X_\lambda]$.

We note that $S/M \cong (R/P)[\{x_\lambda\}]$. Since R/P is an integral domain, it follows that S/M is algebraic over R/P if and only if each x_λ is algebraic over R/P [5, p. 489]. The implication b) $\to$ a) then follows if we reverse the steps in the proof that a) implies b).

2.2. PROPOSITION. *Let the notation be as in Proposition* 2.1. *Then* S/M *is integral over* R/P *if and only if* M *contains a monic polynomial in* $R[X_\lambda]$ *for each* λ *in* Λ.

Proof. We observe that $S/M = (R/P)[\{x_\lambda\}]$ is integral over R/P if and only if each x_λ is integral over R/P. Then Proposition 2.2 follows from the proof of Proposition 2.1.

2.3. COROLLARY. *If* R *is a field and if* M *is a maximal ideal of* S, *the following conditions are equivalent.*

a) S/M *is algebraic over* R.

b) $M \cap R[X_\lambda] \neq (0)$ *for each* λ *in* Λ.

2.4. COROLLARY. *If* R *is an algebraically closed field and if* M *is a maximal ideal of* S, *then the conditions*

a) $M = (\{X_\lambda - t_\lambda\})$ *for some elements* t_λ *in* R,

b) S/M *is algebraic over* R,

c) S/M = R

are equivalent.

Proof. The implications a) → b) and b) → c) are clear. If c) holds, then for each λ in Λ, there exists t_λ in R such that $X_\lambda \equiv t_\lambda \pmod{M}$. Hence $(\{X_\lambda - t_\lambda\}) \subseteq M$, and since $(\{X_\lambda - t_\lambda\})$ is maximal in S, $M = (\{X_\lambda - t_\lambda\})$.

2.5. THEOREM. *If* F *is a subfield of the field* K, *if* K/F *is transcendental, and if* S *is a subset of* K *such that* K = F[S], *then* $|S| = |K|$.

Proof. The set S contains a transcendence basis $B \neq \emptyset$ for K/F. Hence, if $s \in S - B$, there exists a nonzero polynomial $h_s(X)$ in F[B][X] such that $h_s(s) = 0$. We let d_s denote the leading coefficient of $h_s(S)$; then K = F[S] is integral over $F[B][\{1/d_s\}_{s \in S-B}]$. Therefore $F(B) = F[B \cup \{1/d_s\}]$. Moreover, $|K| = |F(B)|$, since K/F(B) is algebraic and F(B) is infinite [8, p. 143]. We prove that $|\{1/d_s\}_{s\in S-B}| \geq |F[B]|$; this inequality suffices to prove Theorem 2.5, since $|K| = |F(B)| = |F[B]|$ and since $|\{1/d_s\}_{s \in S-B}| \leq |S - B| \leq |S|$.

We let $B = \{Y_\alpha\}_{\alpha \in A}$. Then $|F[B]| = |F|\,|A|\,\aleph_0$. We prove that if $\mathscr{F}$ is a complete set of nonassociate prime elements of F[B], then $|\mathscr{F}| = |F[B]|$. It is well known that if F and A are finite, then $F[Y_\alpha]$ (and hence F[B]) contains an irreducible polynomial in Y_α of any positive degree k. Hence $|\mathscr{F}| = \aleph_0 = |F[B]|$ if F and A are finite. If F or A is infinite, then $|F[B]| = |F|\,|A|$, and $\{Y_\alpha - t \mid \alpha \in A,\ t \in F\}$ is a set of nonassociate prime elements of F[B] of cardinality $|A|\,|F|$. Hence $|\mathscr{F}| \geq |F[B]|$, and equality holds: $|\mathscr{F}| = |F[B]|$. We let $\mathscr{F} = \{f_\mu\}_{\mu \in M}$. Fix μ in M. Since $F(B) = F[B][\{1/d_s\}_{s\in S-B}]$, there exists a finite subset S_μ of $\{1/d_s\}$ such that $1/f_\mu \in F[B][S_\mu]$, and this implies that f_μ divides d_s in F[B] for some d_s such that $1/d_s$ is in S_μ. For each $1/d_s$, we let H_s denote the finite set of elements of $\mathscr{F}$ that divide d_s. We have just proved that $\mathscr{F} = \bigcup H_s$. Hence

$$|\mathscr{F}| = |F[B]| = \left|\bigcup H_s\right| \leq \aleph_0 \, |\{d_s\}_{s\in S-B}| \, .$$

Finally, $|\{d_s\}_{s\in S-B}|$ is infinite, by [9, Theorem 21], and hence

$$\aleph_0 \, |\{d_s\}_{s\in S-B}| = |\{d_s\}_{s\in S-B}| \, .$$

This completes the proof of Theorem 2.5.

2.6. PROPOSITION. *If* R *is a field, then* S/M *is algebraic over* R *for each maximal ideal* M *of* S *if and only if* $|\Lambda| < |R|\,\aleph_0$.

Proof. We assume first that $|\Lambda| < |R|\,\aleph_0$. If R is finite, this implies that Λ is finite, and (HN5) implies that S/M is algebraic over R for each maximal ideal M of S. If R is infinite, we consider a maximal ideal M of S. If $x_\lambda = X_\lambda + M$ for each λ in M, then $S/M = R[\{x_\lambda\}]$ is a field; moreover, $|\{x_\lambda\}| \leq |\Lambda| < |R| \leq |S/M|$. Theorem 2.5 then implies that S/M is algebraic over R.

If $|\Lambda| \geq |R| \aleph_0$, then there exists an R-homomorphism ϕ from S onto R(Y), since $|R(Y)| = |R[Y]| = |R| \aleph_0$. Hence the kernel M of S has the property that S/M is isomorphic to R(Y), a field not algebraic over R.

2.7. COROLLARY. *If* R *is an algebraically closed field, then each maximal ideal of* S *is of the form* $(\{X_\lambda - t_\lambda\})$ *for some elements* t_λ *in* R *if and only if* $|\Lambda| < |R|$.

2.8. COROLLARY. *Suppose that* F *is a subfield of* K, *that* K/F *is algebraic, and that* $\{X_\lambda\}_{\lambda \in \Lambda}$ *is a set of indeterminates over* K. *If* $D_F = F[\{X_\lambda\}]$ *and* $D_K = K[\{X_\lambda\}]$, *then* D_F/M *is algebraic over* F *for each maximal ideal* M *of* D_F *if and only if* K *and* D_K *satisfy the analogous condition.*

Proof. If F is finite, then $|K| \leq \aleph_0$ and $|F| \aleph_0 = |K| \aleph_0$; if F is infinite, then $|K| = |F|$ and again $|F| \aleph_0 = |K| \aleph_0$. Hence Corollary 2.8 follows from Proposition 2.6.

Using Propositions 2.1 and 2.2, we can prove the following generalization of Corollary 2.8.

Suppose that R *is a commutative ring with identity, that* T *is a commutative unitary overring of* R *such that* T *is integral over* R, *and that* $\{X_\lambda\}_{\lambda \in \Lambda}$ *is a set of indeterminates over* T. *Let* M_2 *be a prime ideal of* $T[\{X_\lambda\}]$, *and define* M_1, P_2, *and* P_1 *by*

$$M_1 = M_2 \cap R[\{X_\lambda\}], \qquad P_2 = M_2 \cap T, \qquad P_1 = M_2 \cap R = M_1 \cap R.$$

Then $T[\{X_\lambda\}]/M_2$ *is algebraic (integral) over* T/P_2 *if and only if* $R[\{X_\lambda\}]/M_1$ *is algebraic (integral) over* R/P.

The proof is straightforward, and we omit it.

2.9. THEOREM. *If* R *is a field, then* S *is a Hilbert ring if and only if* $|\Lambda| < |R| \aleph_0$.

Proof. If $|\Lambda| > |R| \aleph_0$, then there exists a homomorphism from S onto $R[Y]_{(Y)}$, for $|R[Y]_{(Y)}| = |R[Y]| = |R| \aleph_0$. Since $R[Y]_{(Y)}$ is not a Hilbert ring, it follows that S is not a Hilbert ring.

We assume that $|\Lambda| < |R| \aleph_0$. If Λ is finite, then S is a Hilbert ring; if Λ is infinite, we prove that S is a Hilbert ring by showing that each maximal ideal M of S[Y] meets S in a maximal ideal of S. Thus, if $P = M \cap S$, then $R \subseteq S/P \subseteq S[Y]/M$. Proposition 2.6 shows that S[Y]/M (and hence S/P) is algebraic over R. Therefore S/P is a field, and P is a maximal ideal of S.

In summary, we assert the following. For $1 \leq i \leq 7$, let (HNi)' be obtained from (HNi) by the following modifications. In (HN1), (HN2), and (HN3), we take an arbitrary set $\{f_\alpha\}$ of elements of $K[\{X_\lambda\}]$, and we take $A = (\{f_\alpha\})$. (Since each ideal of $K[\{X_\lambda\}]$ has a basis of at most $|\Lambda| \aleph_0$ elements, we could restrict to consideration of sets of cardinality at most $|\Lambda| \aleph_0$.) (HN4)' is the statement that whenever $K[\{a_\lambda\}_{\lambda \in \Lambda}]$ is a field, then each a_λ is algebraic over K. For $5 \leq i \leq 7$, we obtain (HNi)' by replacing $\{X_1, \cdots, X_n\}$ throughout (HNi) by $\{X_\lambda\}$. We define (HN8)' to be the statement $|\Lambda| < |K| \aleph_0$. Results 2.4, 2.6, 2.7, 2.8, and 2.9 of this section show that conditions (HN5)' to (HN8)' are equivalent. It is clear that (HN4)' is equivalent to (HN5)'; the implication (HN2)' $\rightarrow$ (HN3)' is likewise clear. Since (HN7)' implies (HN5)', (HN7)' implies (HN2)'. Finally, we observe that (HN3)' implies (HN 5)'. Thus, take a maximal ideal M of $K[\{X_\lambda\}]$ and take a basis $\{f_\alpha\}$ for M. Since $(\{f_\alpha\}) = M \subset K[\{X_\lambda\}]$, (HN 3)' implies that the polynomials f_α have a common algebraic zero $\{t_\lambda\}_{\lambda \in \Lambda}$ over $\overline{K}$. Consequently,

$$M\overline{K}[\{X_\lambda\}] \subseteq (\{X_\lambda - t_\lambda\}) = \overline{M},$$

a maximal ideal of $\overline{K}[\{X_\lambda\}]$. Therefore $M = \overline{M} \cap K[\{X_\lambda\}]$, and

$$K \subseteq K[\{X_\lambda\}]/M \subseteq \overline{K}[\{X_\lambda\}]/\overline{M} = \overline{K}.$$

It follows that $K[\{X_\lambda\}]/M$ is algebraic over K. We have proved the following result (S. Lang [11] establishes the equivalence of (HN 2)', (HN 3)', (HN 4)', and (HN 8)' under the assumption that K is algebraically closed).

2.10. THEOREM. *If* K *is a field with algebraic closure* $\overline{K}$, *and if* $\{X_\lambda\}_{\lambda\in\Lambda}$ *is a set of indeterminates over* K, *then the conditions* (HN 2)' *to* (HN 8)' *are equivalent.*

We remark that condition (HN 1)' is satisfied in $K[\{X_\lambda\}]$ for any set $\{X_\lambda\}$ of indeterminates over K; hence (HN 1)' does not imply the conditions (HN 2)' to (HN 8)'.

3. A MORE GENERAL THEOREM

We turn to the problem of determining when $R[\{X_\lambda\}_{\lambda\in\Lambda}]$ (where R denotes a commutative ring with identity) is a Hilbert ring. We need consider only the case where Λ is infinite, and since a homomorphic image of a Hilbert ring is again a Hilbert ring, R is a Hilbert ring if $R[\{X_\lambda\}]$ is a Hilbert ring. A further necessary condition for $R[\{X_\lambda\}]$ to be a Hilbert ring is implied by our next results.

3.1. PROPOSITION. *If* $|\Lambda| \geq |R/M|\aleph_0$ *for some maximal ideal* M *of* R, *then* $R[\{X_\lambda\}]$ *is not a Hilbert ring.*

Proof. By Theorem 2.9, the ring $R[\{X_\lambda\}]/M[\{X_\lambda\}]$ is isomorphic to $(R/M)[\{X_\lambda\}]$, which is not a Hilbert ring. Hence $R[\{X_\lambda\}]$ is not a Hilbert ring.

Before proving that $R[\{X_\lambda\}]$ is a Hilbert ring whenever R is a Hilbert ring and $|\Lambda| < |R/M|\aleph_0$, we need a basic lemma.

3.2. LEMMA. *Suppose that* R_1 *is a commutative ring with identity, that* R_2 *is a commutative unitary overring of* R_1 *such that* R_2/R_1 *is integral, that* P_2 *is a prime ideal of* R_2, *and* $P_1 = P_2 \cap R_1$. *Then* P_2 *is the intersection of the set of prime ideals of* R_2 *that properly contain* P_2 *if and only if* P_1 *is the intersection of the set of prime ideals of* R_1 *that properly contain* P_1.

Proof. R_2/P_2 is an integral domain that is integral over the domain R_1/P_1, and hence Lemma 2 of [3] and the observation that each nonzero ideal of R_2/P_2 meets R_1/P_1 in a nonzero ideal imply that the pseudoradical of R_2/P_2 is zero if and only if the pseudoradical of R_1/P_1 is zero. This assertion, however, is equivalent to the conclusion of the lemma.

3.3. THEOREM. *The ring* $S = R[\{X_\lambda\}]$ *is a Hilbert ring if and only if* R *is a Hilbert ring and* $|\Lambda| < |R/M|\aleph_0$ *for each maximal ideal* M *of* R.

Proof. By (HR 3), it suffices to prove that each nonmaximal proper prime ideal P of S is the intersection of a set of prime ideals of S that properly contain P. By passage to (R/P') and to $(R/P')[\{X_\lambda\}] \simeq S/P'[\{X_\lambda\}]$, where $P' = P \cap R$, we can assume that R is an integral domain and that $P \cap R = (0)$. If R is a field, then Theorem 2.9 shows that S is a Hilbert ring, and hence P is the intersection of a set of prime ideals of S properly containing P. If R is not a field, then we denote by $\overline{K}$ an algebraic closure of the quotient field K of R, and by $\overline{R}$ the integral closure of R in $\overline{K}$. Since $\overline{R}[\{X_\lambda\}]$ is integral over $R[\{X_\lambda\}]$ (see [1, Proposition 12, Section 1, No. 3] or [4, Theorem 8.7]), there exists a prime ideal P' of $R[\{X_\lambda\}]$ lying over P. Moreover, $|\overline{R}/\overline{M}|\aleph_0 \geq |R/(\overline{M} \cap R)|\aleph_0 > |\Lambda|$ for each maximal ideal $\overline{M}$ of $\overline{R}$. By Lemma 3.2, it suffices to consider the case where $R = \overline{R}$ and $P = P'$. (We note that if $(P' \cap \overline{R}) \cap R = P \cap R = (0)$ where the domain $\overline{R}$ is integral over R, then $P' \cap R = (0)$.) Thus we assume, without loss of generality, that R is integrally closed, K is algebraically closed, $R \neq K$, and $P \cap R = (0)$.

We set $S = R[\{X_\lambda\}]$ and $N = R - \{0\}$. Then $S_N = K[\{X_\lambda\}]$ and PS_N is a proper ideal of S_N. Now $|\Lambda| < |R| = |K| = |K|\aleph_0$, so that S_N is a Hilbert ring, by Theorem 2.9. If PS_N is not maximal in S_N, it follows that there exists a family $\{P_a\}$ of prime ideals of S properly containing P such that $PS_N = \bigcap_a P_a S_N$, and hence $P = \bigcap_a P_a$. Corollary 2.7 shows that if PS_N is maximal in S_N, then PS_N is the set of elements of S_N that vanish at $\xi = \{\xi_\lambda\}$ for some elements ξ_λ in K. We denote by ϕ_ξ the K-homomorphism from $K[\{X_\lambda\}]$ onto K determined by the condition $X_\lambda \to \xi_\lambda$ for each λ. Then

$$P = PS_N \cap S = \{f \in S \mid \phi_\xi(f) = 0\}.$$

We denote by $\{V_\sigma\}_{\sigma \in \Sigma}$ the set of nontrivial valuation rings on K that contain R, and by M_σ the maximal ideal of V_σ (for each σ in Σ). Then $\{M_\sigma \cap R\}_{\sigma \in \Sigma}$ is the set of nonzero prime ideals of R. Thus $\bigcap_{\sigma \in \Sigma} (M_\sigma \cap R) = (0)$, since R is a Hilbert ring and (0) is not maximal in R. Further, $\bigcap_{\sigma \in \Sigma} M_\sigma \subseteq \bigcap V_\sigma = R$, so that

$$\bigcap M_\sigma = \left(\bigcap M_\sigma\right) \cap R = \bigcap (M_\sigma \cap R) = (0).$$

For $\sigma \in \Sigma$, we let $P_\sigma = \{f \in S \mid \phi_\xi(f) \in M_\sigma\}$. We prove that P_σ is a prime ideal of S properly containing P and that $\bigcap_{\sigma \in \Sigma} P_\sigma = P$. It is easy to verify that P_σ is an ideal of S, and clearly $P \subseteq P_\sigma$. The inclusion is proper, since $(0) \subset M \cap R \subseteq P_\sigma$, while $P \cap R = (0)$. Also, P_σ is prime in S, since the condition $\phi_\xi(fg) = \phi_\xi(f)\phi_\xi(g)$ in M_σ implies that $\phi_\xi(f)$ or $\phi_\xi(g)$ is in M_σ, the maximal ideal of a valuation ring on K. Finally

$$\bigcap_{\sigma \in \Sigma} P_\sigma = \left\{f \in S \mid \phi_\xi(f) \in \bigcap_{\sigma \in \Sigma} M_\sigma\right\} = \{f \in S \mid \phi_\xi(f) = 0\} = P.$$

This completes the proof of Theorem 3.3.

3.4. COROLLARY. *If* R *is a Hilbert ring, then* $R[\{X_\lambda\}]$ *is a Hilbert ring if and only if* $(R/M)[\{X_\lambda\}]$ *is a Hilbert ring for each maximal ideal* M *of* R.

Proof. Apply Theorems 2.9 and 3.3.

In connection with (HR 4) and (HR 5), we remark that if each maximal ideal of $R[\{X_\lambda\}]$ meets R in a maximal ideal, then R is a Hilbert ring, but $R[\{X_\lambda\}]$ need not be a Hilbert ring. Such an example occurs if R is a field and $|R| < |\Lambda|$. More precisely, we have the following result.

3.5. THEOREM. *If* R *is a commutative ring with identity and if* $\{X_\lambda\}_{\lambda \in \Lambda}$ *is a set of indeterminates over* R, *then conditions* (1) *and* (2) *are equivalent; these conditions imply that* $R[\{X_\lambda\}]$ *is a Hilbert ring, but not conversely.*

(1) *For each maximal ideal* M *of* $R[\{X_\lambda\}]$, *the ideal* $M \cap R$ *is maximal in* R, *and* $R[\{X_\lambda\}]/M$ *is algebraic over* $R/(M \cap R)$.

(2) *For each maximal ideal* M *of* $R[\{X_\lambda\}]$, *the residue field* $R[\{X_\lambda\}]/M$ *is integral over* $R/(M \cap R)$.

Proof. (1) and (2) are equivalent because if a field is integral over a subring, then the subring must be a subfield.

We prove that (1) implies that $R[\{X_\lambda\}]$ is a Hilbert ring. First we show that R is a Hilbert ring, by proving that for a fixed element σ of Λ each maximal ideal M of $R[X_\sigma]$ meets R in a maximal ideal of R. Since

$$MR[\{X_\lambda\}] = M[\{X_\lambda\}_{\lambda \neq \sigma}] \subset R[\{X_\lambda\}],$$

M is contained in a maximal ideal M_1 of $R[\{X_\lambda\}]$, and consequently $M_1 \cap R[X_\sigma] = M$, so that $M_1 \cap R = M \cap R$. By (1), it follows that R is a Hilbert ring. To prove that $R[\{X_\lambda\}]$ is a Hilbert ring, it suffices, by Corollary 3.4, to prove that $(R/P)[\{X_\lambda\}]$ is a Hilbert ring for each maximal ideal P of R. The maximal ideals $\bar{N}$ of $(R/P)[\{X_\lambda\}]$ are in one-to-one correspondence with the maximal ideals N of $R[\{X_\lambda\}]$ that contain $P[\{X_\lambda\}]$, and $N \cap R = P$ for any such N. Therefore (1) shows that $\{(R/P)[\{X_\lambda\}]\}/\bar{N}$ (which is isomorphic to $R[\{X_\lambda\}]/N$) is algebraic over R/P for each $\bar{N}$. By (HN5)' of Theorem 2.10, $(R/P)[\{X_\lambda\}]$ is a Hilbert ring.

As we shall show, the hypothesis that $R[\{X_\lambda\}]$ is a Hilbert ring does not generally imply that $M \cap R$ is maximal in R for each maximal ideal M of $R[\{X_\lambda\}]$. (It is true, however, that if $R[\{X_\lambda\}]$ is a Hilbert ring and M is a maximal ideal of $R[\{X_\lambda\}]$ such that $M \cap R$ is maximal in R, then $R[\{X_\lambda\}]/M$ is algebraic over $R/(M \cap R)$; this statement follows from Corollary 3.4 and (HN5)' of Theorem 2.10.) Thus, we take C to be the field of complex numbers, we take S to be the multiplicative system in C[X] generated by $\{X - \alpha \mid \alpha \in C - Z\}$, and we take $R = C[X]_S$. Then R is a one-dimensional Hilbert ring with the following properties. The set $\{M_i\}_{i=-\infty}^{\infty}$ of maximal ideals of R is countable (we can take $M_i = (X - i)R$ for each i), and $R/M_i \simeq C$ for each i. By Theorem 3.3, $R[\{X_i\}_1^\infty]$ is a Hilbert ring, and since

$$C(X) = R[\{1/(X - i)\}_{-\infty}^{\infty}],$$

there exists an R-homomorphism ϕ from $R[\{X_i\}_1^\infty]$ onto C(X). The kernel of ϕ is a maximal ideal M of $R[\{X_i\}_1^\infty]$ such that $M \cap R = (0)$, where (0) is not maximal in R.

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