AN EXTENSION OF THE HAUSDORFF-YOUNG THEOREM

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1. INTRODUCTION

The classical Hausdorff-Young theorem consists of the following pair of mutually dual assertions. For $1 \le p \le 2$, define p' by $1/p + 1/p' = 1$. If $f \in L^p$ (on the unit circle), then the Fourier transform $\hat{f}$ of f is a member of $\ell^{p'}$, and $\|\hat{f}\|_{p'} \le \|f\|_p$. If $\lambda \in \ell^p$, then there exists an $f \in L^{p'}$ such that $\hat{f} = \lambda$ and $\|f\|_{p'} \le \|\lambda\|_p$. A discussion of the Hausdorff-Young theorem and some of its consequences can be found in [2(II), Chapter 13] and [8(II), Chapter XII].

In this paper, we prove an extension of the Hausdorff-Young theorem to the setting of mixed-norm spaces (see [1] and [5]). We then apply the extended version to obtain sufficient conditions for membership in the multiplier spaces (L^p, L^q) and (H^p, H^q). The sufficient conditions significantly extend the results found in [5] and [2(II), p. 268].

Our method is to characterize first the multipliers of the mixed-norm spaces and then, by means of Hedlund's results [5], to prove the extended Hausdorff-Young theorem. The theorem's ultimate dependence on interpolation and on a theorem of Hardy and Littlewood [4, p. 167] is somewhat obscured by this approach.

2. MULTIPLIERS OF MIXED-NORM SPACES

In this section, we give definitions and preliminary results. We begin by defining the mixed-norm spaces $L^{p,q}$ and $H^{p,q}$. Corresponding to a bounded sequence $\lambda = \{\lambda(n)\}_{n=-\infty}^{\infty}$ and real numbers p and q in the interval $[1, \infty]$, we let

$$\|\lambda\|_{p,q} = \left(\sum_{m=-\infty}^{\infty} \left[\sum_{n \in I(m)} |\lambda(n)|^p \right]^{q/p} \right)^{1/q},$$

where

$$I(m) = \begin{cases} \{n \in Z: 2^{m-1} \le n < 2^m\} & \text{if } m > 0, \\ \{0\} & \text{if } m = 0, \\ \{n \in Z: -2^{-m} < n \le -2^{-m-1}\} & \text{if } m < 0. \end{cases}$$

In the case where p or q is infinite, replace the corresponding sum by a supremum. We define $L^{p,q}$ to be the set of all bounded sequences λ such that $\|\lambda\|_{p,q} < \infty$. The symbol $H^{p,q}$ denotes the set of all $\lambda \in L^{p,q}$ such that $\lambda(n) = 0$ for $n < 0$.

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With suitable modifications, the following remarks made about $L^{p,q}$ apply to $H^{p,q}$. It is clear that $L^{p,q}$ (with the norm $\|\cdot\|_{p,q}$) is a Banach space and that $L^{p,p}$ is precisely the space ℓ^p of doubly-infinite p-summable sequences. It is easy to see that $L^{p,q}$ has many of the properties of ℓ^p, such as

$$L^{p,q} \subset L^{r,q} \quad \text{if } 1 \leq p \leq r \leq \infty,$$

$$L^{p,q} \subset L^{p,s} \quad \text{if } 1 \leq q \leq s \leq \infty,$$

and

$$(L^{p,q})^* = L^{p',q'} \quad \text{if } 1 \leq p,\, q < \infty \text{ and } 1/p + 1/p' = 1/q + 1/q' = 1\,.$$

For any two subsets A and B of ℓ^∞, we define the set of multipliers from A to B (denoted by (A, B)) to be the set of all $\lambda \in \ell^\infty$ such that $\lambda a = \{\lambda(n)\, a(n)\}_{n=-\infty}^{\infty}$ is an element of B for all $a \in A$.

The following theorem characterizes the multiplier spaces $(L^{r,s}, L^{u,v})$. In addition to being interesting in its own right, it is useful in some of our later calculations.

THEOREM 1. *Let* r, s, u, *and* v *be real numbers in* $[1, \infty]$, *and define* p *and* q *by*

$$1/p = 1/u - 1/r \quad \textit{if } r > u, \qquad p = \infty \quad \textit{if } r \leq u,$$

$$1/q = 1/v - 1/s \quad \textit{if } s > v, \qquad q = \infty \quad \textit{if } s \leq v.$$

Then $(L^{r,s}, L^{u,v}) = L^{p,q}$.

Proof. If $1 \leq r \leq u \leq \infty$ and $1 \leq s \leq v \leq \infty$, then $L^{r,s} \subset L^{u,v}$,

$$\ell^\infty \subset (L^{r,s}, L^{u,v}) \subset \ell^\infty,$$

and the result follows.

We next suppose that $1 \leq u < r \leq \infty$ and that $1 \leq v < s \leq \infty$. We shall show that if $\lambda \in L^{p,q}$ and $x \in L^{r,s}$, then $\lambda x \in L^{u,v}$. By applying Hölder's inequality, first to the inside sum with $\alpha = r/(r-u)$ and then to the outside sum with $\beta = s/(s-v)$, we see that

$$\left(\sum_{m=-N}^{N}\left[\sum_{n\in I(m)} |\lambda(n)\,x(n)|^u\right]^{v/u}\right)^{1/v} \leq \|\lambda\|_{p,q}\ \|x\|_{r,s}$$

for each positive integer N. It now follows that $\lambda x \in L^{u,v}$, $\lambda \in (L^{r,s}, L^{u,v})$, and $L^{p,q} \subset (L^{r,s}, L^{u,v})$.

In order to show the reverse inclusion relation, choose $\lambda \in (L^{r,s}, L^{u,v})$. It follows from the closed-graph theorem that T_λ, defined by $T_\lambda(x) = \lambda x$ $(x \in L^{r,s})$, is a bounded linear operator from $L^{r,s}$ to $L^{u,v}$; we denote its operator norm by $\|T_\lambda\|_0$. For each positive integer N, define the bounded linear operator T_N from $L^{r,s}$ to $L^{u,v}$ by

$$T_N(y)(n) = \begin{cases} \lambda(n)\,y(n) & \text{if } -2^N < n < 2^N, \\ 0 & \text{otherwise}. \end{cases}$$

The computations given above show that

$$\|T_N\|_0 \leq \left(\sum_{m=-N}^{N} \left[\sum_{n\in I(m)} |\lambda(n)|^p \right]^{q/p} \right)^{1/q}.$$

We shall show that equality holds. First, choose N large enough to guarantee that

$$\sum_{n=-2^N+1}^{2^N-1} |\lambda(n)| > 0 .$$

Then, for $-N \leq m \leq N$, let

$$c_m = \begin{cases} \left(\sum_{n\in I(m)} |\lambda(n)|^p \right)^{(qu-pv)/puv} & \text{if } \sum_{n\in I(m)} |\lambda(n)| \neq 0, \\ 0 & \text{otherwise}. \end{cases}$$

Finally, define the sequence x by

$$x(n) = \begin{cases} d\, c_m |\lambda(n)|^{p/r} & \text{if } -N \leq m \leq N \text{ and } n \in I(m), \\ 0 & \text{otherwise}, \end{cases}$$

where

$$d = \left(\sum_{m=-N}^{N} \left[\sum_{n\in I(m)} |\lambda(n)|^p \right]^{q/p} \right)^{(v-q)/qv}.$$

A rather long but elementary calculation then shows that $\|x\|_{r,s} = 1$ and that

$$\|T_N(x)\|_{u,v} = \left(\sum_{m=-N}^{N} \left[\sum_{n\in I(m)} |\lambda(n)|^p \right]^{q/p} \right)^{1/q}.$$

Hence

$$\|T_N\|_0 = \left(\sum_{m=-N}^{N} \left[\sum_{n\in I(m)} |\lambda(n)|^p \right]^{q/p} \right)^{1/q}.$$

Since $\|T_N\|_0 \leq \|T_\lambda\|_0$, the partial sums of $\|\lambda\|_{p,q}$ are bounded. Therefore $\lambda \in L^{p,q}$ and $(L^{r,s}, L^{u,v}) \subset L^{p,q}$.

Since the remaining two cases are quite similar, we give an argument only for $1 \leq u < r \leq \infty$ and $1 \leq s \leq v \leq \infty$. The proof that

$$L^{p,q} \subset (L^{r,s}, L^{u,v}) \quad \text{and} \quad \|T_N\|_0 \leq \max_{-N\leq m\leq N} \left(\sum_{n\in I(m)} |\lambda(n)|^p \right)^{1/p}$$

proceeds just as above, if we note that $\|\cdot\|_{u,v} \leq \|\cdot\|_{u,s}$. Choose N as before, and then choose $m_0 \in [-N, N]$ so that

$$\left(\sum_{n \in I(m_0)} |\lambda(n)|^p \right)^{1/p} = \max_{-N \leq m \leq N} \left(\sum_{n \in I(m)} |\lambda(n)|^p \right)^{1/p} .$$

Let $c = \left(\sum_{n \in I(m_0)} |\lambda(n)|^p \right)^{-1/r}$, and define the sequence x by

$$x(n) = \begin{cases} c\,|\lambda(n)|^{p/r} & \text{if } n \in I(m_0) , \\ 0 & \text{otherwise.} \end{cases}$$

This choice of x shows that

$$\|T_N\|_0 = \max_{-N \leq m \leq N} \left(\sum_{n \in I(m)} |\lambda(n)|^p \right)^{1/p} ,$$

and the conclusion $(L^{r,s}, L^{u,v}) \subset L^{p,q}$ follows as before.

3. THE HAUSDORFF-YOUNG THEOREM

In this section, we present an extension of the classical Hausdorff-Young theorem. For $1 \leq p < \infty$, we denote by L^p the usual space of equivalence classes of functions on $[0, 2\pi]$ normed by

$$\|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p dx \right)^{1/p} .$$

For each $f \in L^1$, the Fourier transform $\hat{f}$ of f is defined by

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) \exp(-inx)\, dx \qquad (n \in Z) .$$

The Hardy space H^p is the closed subspace of L^p consisting of those functions $f \in L^p$ for which $\hat{f}(n) = 0$ $(n < 0)$.

THEOREM 2. *Suppose that* $1 \leq p \leq 2$, *and let* $1/p + 1/p' = 1$. *Then there exists a constant* A_p *such that for each* $f \in H^p$,

$$\hat{f} \in H^{p',2} \qquad \textit{and} \qquad \|\hat{f}\|_{p',2} \leq A_p \|f\|_p .$$

Proof. The proof of this theorem uses the result of J. H. Hedlund [5, p. 1068] that for $1 \leq p \leq 2$,

$$H^{2p/(2-p),\infty} \subset (H^p, H^2) .$$

Note that we regard H^p as a subset of ℓ^∞ by identifying it with the corresponding space of Fourier transforms. Hedlund's result and Theorem 1 imply that

$$H^p \subset (H^{2p/(2-p),\infty}, H^2) = (H^{2p/(2-p),\infty}, H^{2,2}) = H^{p',2} .$$

Thus, $\hat{f} \in H^{p',2}$ for every $f \in H^p$. The inclusion operator mapping H^p into $H^{p',2}$ is closed and is thus bounded by the closed-graph theorem. If A_p denotes its norm, it follows that

$$\|\hat{f}\|_{p',2} \leq A_p \|f\|_p .$$

An application of the Riesz projection theorem [7, p. 217] now yields the following result.

THEOREM 3. *If* $1 < p \leq 2$ *and* $1/p + 1/p' = 1$, *then there exists a constant* C_p *such that if* $f \in L^p$, *then*

$$\hat{f} \in L^{p',2} \quad \textit{and} \quad \|\hat{f}\|_{p',2} \leq C_p \|f\|_p .$$

It should be noted that the restriction $1 < p$ in Theorem 3 is necessary. To see this, choose any nonnegative $\lambda \in \ell^\infty$ such that $\lim_{|n|\to\infty} \lambda(n) = 0$ and $\lambda \notin L^{\infty,2}$. By a theorem in Edwards [2(I), p. 117], there exists an $f \in L^1$ such that $|\hat{f}(n)| \geq \lambda(n)$ for all $n \in Z$. Hence $\hat{f} \notin L^{\infty,2}$.

We proceed to the second half of our extension of the Hausdorff-Young theorem.

THEOREM 4. *Suppose that* $1 < p \leq 2$, *and let* $1/p + 1/p' = 1$. *Then there exists a constant* B_p *such that for each* $\lambda \in L^{p,2}$, *there exists an* $f \in L^{p'}$ *such that*

$$\hat{f} = \lambda \quad \textit{and} \quad \|f\|_{p'} \leq B_p \|\lambda\|_{p,2} .$$

Proof. The convolution $L^1 * L^p$ is contained in L^p. Therefore, by Theorem 3,

$$\hat{f}\hat{g} = (f * g)\hat{} \in (L^p)\hat{} \subset L^{p',2}$$

for all $f \in L^1$ and $g \in L^p$. It now follows that $(L^1)\hat{} \subset (L^p, L^{p',2})$. Since $(L^p, L^{p',2}) = (L^{p,2}, L^{p'})$, we conclude that

$$L^{p,2} \subset (L^1, L^{p'}) .$$

However, $(L^1, L^{p'}) = (L^{p'})\hat{}$ (see [2(II), p. 255]), and hence $L^{p,2} \subset (L^{p'})\hat{}$. As in the proof of Theorem 2, the inclusion operator mapping $L^{p,2}$ into $L^{p'}$ is closed and is thus bounded by the closed-graph theorem. If B_p denotes its norm, the conclusion follows.

The following example shows that the restriction $1 < p$ in Theorem 4 is necessary. For any real number t $(1/2 < t \leq 1)$, let

$$f(x) = \sum_{m=0}^{\infty} (m+1)^{-t} \exp(i\, 2^m x) .$$

It is clear that $f \notin H^\infty$ and that $\hat{f} \in H^{1,2}$. The author thanks the referee for noting the following general example. Let $\{n_j\}$ be any lacunary sequence of integers, take $c \in \ell^2 \setminus \ell^1$, and set

$$\lambda(n) = \begin{cases} c(j) & \text{if } n = n_j, \\ 0 & \text{otherwise} . \end{cases}$$

Then $\lambda = \hat{f}$, where $f(x) = \sum_{n=-\infty}^{\infty} \lambda(n) \exp(inx)$. Clearly, $\lambda \in L^{1,2}$ and $f \in L^2$. But $f \notin L^{\infty}$, for each lacunary series in L^{∞} satisfies the condition $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$ [8(I), p. 247].

In a positive direction, we can assert that

$$H^{1,2} \subset (H^1, \ell^1) = (L_+^{\infty})^{\wedge},$$

where $(L_+^{\infty})^{\wedge}$ is the restriction of $(L^{\infty})^{\wedge}$ to the nonnegative integers. The inclusion is due to Hedlund [6], and the equality to G. I. Gaudry [3].

For $1 < p < 2$, ℓ^p and $L^{p',2}$ are proper subsets of $L^{p,2}$ and $\ell^{p'}$, respectively. Thus, our theorems are proper extensions of the classical Hausdorff-Young theorem.

4. SUFFICIENT CONDITIONS FOR MULTIPLIERS

We now use the results of the preceding sections to obtain sufficient conditions for a bounded sequence to belong to (H^p, H^q) or (L^p, L^q). The theorems will significantly extend Hedlund's Theorem 1 [5] and a theorem in Edwards [2(II), p. 268].

THEOREM 5. *If* $1 \leq p \leq 2 \leq q < \infty$ *and* $1/s = 1/p - 1/q$, *then* $H^{s,\infty} \subset (H^p, H^q)$.

Proof. Choose $\lambda \in H^{s,\infty}$ and $f \in H^p$. Since $f \in H^p$, Theorem 2 guarantees that $\hat{f} \in H^{p',2}$. By Hölder's inequality,

$$\left(\sum_{n \in I(m)} |\lambda(n) \hat{f}(n)|^{q'} \right)^{2/q'} \leq \left(\sum_{n \in I(m)} |\lambda(n)|^s \right)^{2/s} \left(\sum_{n \in I(m)} |\hat{f}(n)|^{p'} \right)^{2/p'}.$$

It follows that

$$\|\lambda \hat{f}\|_{q',2} \leq \|\lambda\|_{s,\infty} \|\hat{f}\|_{p',2} < \infty.$$

Theorem 4 now implies that $\lambda \hat{f} \in (H^q)^{\wedge}$, and hence $H^{s,\infty} \subset (H^p, H^q)$.

If in the previous argument we use Theorem 3 in place of Theorem 2, we obtain the following result.

THEOREM 6. *If* $1 < p \leq 2 \leq q < \infty$ *and* $1/s = 1/p - 1/q$, *then* $L^{s,\infty} \subset (L^p, L^q)$.

We note that, except for $q = 2$ and $p = 1$ or $p = 2$ in Theorem 5, and for $p = q = 2$ in Theorem 6, the sufficient conditions obtained are far from necessary.

REFERENCES

1. A. Benedek and R. Panzone, *The space* L^p, *with mixed norm.* Duke Math. J. 28 (1961), 301-324.

2. R. E. Edwards, *Fourier series: A modern introduction.* Vols. I and II. Holt, Rinehart and Winston, New York, 1967.

3. G. I. Gaudry, H^p *multipliers and inequalities of Hardy and Littlewood.* J. Austral. Math. Soc. 10 (1969), 23-32.

4. G. H. Hardy and J. E. Littlewood, *Notes on the theory of series. XX: Generalizations of a theorem of Paley.* Quart. J. Math. Oxford Ser. 8 (1937), 161-171.

5. J. H. Hedlund, *Multipliers of* H^p *spaces*. J. Math. Mech. 18 (1968/69), 1067-1074.

6. ———, *Multipliers of* H^1 *and Hankel matrices*. Proc. Amer. Math. Soc. 22 (1969), 20-23.

7. W. Rudin, *Fourier analysis on groups*. Interscience Tracts in Pure and Applied Mathematics, No. 12. Interscience Publ., New York-London, 1962.

8. A. Zygmund, *Trigonometric series*. 2nd ed. Vols. I and II. Cambridge University Press, New York, 1959.

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