A CHARACTERIZATION OF ZERO SETS FOR A^{∞}

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1. INTRODUCTION

Let U and T denote the open unit disk $\{z \mid |z| < 1\}$ and the unit circle $\{z \mid |z| = 1\}$. The space A^{∞} consists of all nonconstant analytic functions f in U such that for each positive integer n, the n^{th} derivative $f^{(n)}$ is bounded in U.

The following characterization of the possible zero sets for functions in this class is due to B. A. Taylor and D. L. Williams [3].

THEOREM 1.1 (Taylor and Williams). *In order that a closed subset* Z *of* $\overline{U} = U \cup T$ *be the zero set of a function in* A^{∞}, *it is necessary and sufficient that* (a) *the set* $Z \cap U = \{r_k e^{i\theta_k}\}_{k=1}^{\infty}$ *satisfy the condition*

$$\sum_{k=1}^{\infty} (1 - r_k) < \infty, \tag{1.1}$$

and (b)

$$\int_{-\pi}^{\pi} \log \operatorname{dist}(e^{i\theta}, Z)\, d\theta > -\infty. \tag{1.2}$$

The main result of this paper provides an intrinsic characterization of such zero sets; at the same time, it yields a shorter proof of Theorem 1.1.

THEOREM 1.2. *Let* Z *be a closed subset of* $\overline{U}$, *and put* $E_1 = Z \cap T$. *Then* Z *is the zero set of a function in* A^{∞} *if and only if* (a) *holds and*

(c) $E_2 = E_1 \cup \{e^{i\theta_k}\}_{k=1}^{\infty}$ *is a Carleson set.*

This result clearly depends only on the radial projection of Z on T, modulo the Blaschke condition (1.1). Our next theorem gives a condition under which (c) is always satisfied.

THEOREM 1.3. *Suppose* Z *is a closed subset of* $\overline{U}$ *satisfying* (a) *and that* $E = Z \cap T$ *is a Carleson set. If*

$$\sum_{k=1}^{\infty} [\operatorname{dist}(e^{i\theta_k}, E)]^{\alpha} < \infty \tag{1.3}$$

for some $\alpha \geq 1$, *then* Z *is the zero set of a function in* A^{∞}.

The hypothesis of Theorem 1.3 appears in [4], where it is shown only that Z is the zero set of a function whose derivative belongs to H^1.

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2. PROOFS OF THE MAIN THEOREMS

A Carleson set is a closed subset of T of measure zero whose complementary open arcs $\{I_n\}_{n=1}^{\infty}$ satisfy the condition

$$\sum_{n=1}^{\infty} \varepsilon_n \log(1/\varepsilon_n) < \infty ,$$

where ε_n denotes the length of I_n $(n = 1, 2, \cdots)$. For a closed subset E of T, this definition is equivalent to the condition

$$\int_{-\pi}^{\pi} \log d(e^{i\theta}, E)\, d\theta > -\infty , \tag{2.1}$$

where $d(e^{i\theta}, E) = \text{dist}(e^{i\theta}, E)$ denotes the distance from $e^{i\theta}$ to E. From (2.1) it follows immediately that a closed subset of a Carleson set is itself a Carleson set.

Before proceeding with the proofs, we state two elementary lemmas. The first is a direct consequence of the monotonicity of the function $g(x) = x \log(1/x)$ on $(0, 1/e)$. The second follows from the convexity of the function g.

LEMMA 2.1. *Let* $\{\varepsilon_k\}_{k=1}^{\infty}$ *and* $\{\delta_k\}_{k=1}^{\infty}$ *be sequences of positive numbers such that* $\delta_k \leq \varepsilon_k$ *for each index* k. *If* $\sum_{k=1}^{\infty} \varepsilon_k \log(1/\varepsilon_k) < \infty$, *then* $\sum_{k=1}^{\infty} \delta_k \log(1/\delta_k) < \infty$.

LEMMA 2.2. *Let* n *be a positive integer. For each set* $\{x_0, x_1, \cdots, x_{n+1}\}$ *of* $n+2$ *points in the interval* $[a, b]$ *that satisfy the condition*

$$a = x_0 \leq x_1 \leq \cdots \leq x_n \leq x_{n+1} = b , \tag{2.2}$$

put

$$c_i = \begin{cases} (x_i - x_{i-1}) \log(x_i - x_{i-1})^{-1} & (x_i \neq x_{i-1}), \\ 0 & (x_i = x_{i-1}) \end{cases}$$

for $i = 1, 2, \cdots, n+1$. *The maximum value taken by* $\sum_{i=1}^{n+1} c_i$ *over all sets of* $n+2$ *points satisfying* (2.2) *occurs when* $x_1 - x_0 = x_2 - x_1 = \cdots = x_n - x_{n-1} = x_{n+1} - x_n$.

In addition to Lemmas 2.1 and 2.2, we shall also use the inequality

$$|1 - re^{i\theta}|^2 \leq (1-r)^2 + \theta^2 \qquad (0 \leq r \leq 1,\ -\pi \leq \theta \leq \pi).$$

Proof of Theorem 1.2. Suppose Z is a closed subset of $\overline{U}$ satisfying (a) and (c). Since $\{e^{i\theta_k}\}_{k=1}^{\infty}$ is a set of points on T whose closure is a Carleson set, it follows by a result of Caughran [1, Theorem 2] that there exists a function f_1 in A^{∞} whose zero set is the closure of $\{r_k e^{i\theta_k}\}_{k=1}^{\infty}$. Thus the zeros of f_1 are the points of Z except those on T that are not in the closure of $\{r_k e^{i\theta_k}\}_{k=1}^{\infty}$. Since E_1 is a closed subset of E_2, E_1 is a Carleson set, and, by a theorem of P. Novinger [2], there exists $f_2 \in A^{\infty}$ such that E_1 is the zero set of f_2. The function $f = f_1 f_2$ is in A^{∞}, and its zero set is Z.

Now suppose f is in A^∞, and Z is its zero set in $\overline{U}$. Then Z satisfies (a), and (1.2) follows from the integrability of $\log|f|$ on T and the fact that f satisfies a Lipschitz condition on $\overline{U}$, that is,

$$|f(z_1) - f(z_2)| \leq M|z_1 - z_2| \qquad (z_1, z_2 \in \overline{U}).$$

Because (1.2) holds, $E_1 = Z \cap T$ is a Carleson set, and it remains to show that $E_2 = E_1 \cup \{e^{i\theta_k}\}_{k=1}^\infty$ is a Carleson set. We show that this is implied by (1.1) and (1.2).

Without loss of generality, assume that 1 is in E_1. We shall first prove the theorem under the assumptions that $e^{i\theta_k} \notin E_1$ for any k and that $\theta_k \neq \theta_j$ if $k \neq j$. In the remainder of the proof, E_1 will be considered as a subset of $[-\pi, \pi]$ in the usual manner.

Since E_1 is a Carleson set, it follows that $[-\pi, \pi] \setminus E_1$ is the countable union of disjoint open intervals,

$$[-\pi, \pi] \setminus E_1 = \bigcup_{n=1}^\infty (a_n, b_n)$$

and

$$\sum_{n=1}^\infty (b_n - a_n)\log(b_n - a_n)^{-1} < \infty. \tag{2.3}$$

For each positive integer n, let $\{\theta_k^n\}_k$ denote the terms of $\{\theta_k\}_{k=1}^\infty$ that lie in (a_n, b_n). For positive integers k and n, let $I_k^n = (w_k^n, \theta_k^n)$, where w_k^n is the greatest element of E_2 less than θ_k^n. If b_n is not a limit point of $\{\theta_k^n\}_k$, let J_n denote the interval (w_n, b_n), where w_n is the greatest element of E_2 less than b_n. It now follows that

$$[-\pi, \pi] \setminus E_2 = \left(\bigcup_{n=1}^\infty \bigcup_{k=1}^\infty I_k^n\right) \cup \left(\bigcup_{s=1}^\infty J_s\right),$$

the intervals in this union being pairwise disjoint.

Relabel the collection of intervals $\{I_k^n\}_{n,k=1}^\infty$, and denote it by $\{I_n\}_{n=1}^\infty$; denote the right-hand endpoint of I_n by θ_n, the length of I_n by ε_n, and the length of J_n by ρ_n.

For each integer s, $J_s \subset (a_s, b_s)$, hence $\rho_s \leq b_s - a_s$, and from (2.3) and Lemma 2.1 it follows that $\sum_{s=1}^\infty \rho_s \log(1/\rho_s) < \infty$. Thus it will suffice to show that $\sum_{n=1}^\infty \varepsilon_n \log(1/\varepsilon_n) < \infty$.

To this end, denote by $\{m_k\}_{k=1}^\infty$ the subsequence of the positive integers for which

$$1 - r_{m_k} \leq \varepsilon_{m_k} \log(1/\varepsilon_{m_k}),$$

and by $\{n_k\}_{k=1}^\infty$ the subsequence for which

$$1 - r_{n_k} > \varepsilon_{n_k} \log(1/\varepsilon_{n_k}) .$$

Clearly, $\sum_{k=1}^{\infty} \varepsilon_{n_k} \log (1/\varepsilon_{n_k}) < \infty$.

If N is a positive integer, then

$$\infty > \int_{-\pi}^{\pi} \log \frac{4}{d(e^{i\theta}, Z)^2} d\theta \geq \sum_{k=1}^{N} \int_{I_{m_k}} \log \frac{1}{d(e^{i\theta}, Z)^2} d\theta$$

$$\geq \sum_{k=1}^{N} \int_{I_{m_k}} \log |e^{i\theta} - z_{m_k}|^{-2} d\theta \geq \sum_{k=1}^{N} \int_{I_{m_k}} \log[(1 - r_{m_k})^2 + (\theta - \theta_{m_k})^2]^{-1} d\theta$$

$$\geq \sum_{k=1}^{N} \int_{I_{m_k}} \log[\varepsilon_{m_k}^2 \log^2(1/\varepsilon_{m_k}) + \varepsilon_{m_k}^2]^{-1} d\theta$$

$$= \sum_{k=1}^{N} \varepsilon_{m_k} \log[\varepsilon_{m_k}^2 \log^2(1/\varepsilon_{m_k}) + \varepsilon_{m_k}^2]^{-1}$$

$$= 2 \sum_{k=1}^{N} \varepsilon_{m_k} \log(1/\varepsilon_{m_k}) - \sum_{k=1}^{N} \varepsilon_{m_k} \log(1 + \log^2(1/\varepsilon_{m_k}))$$

$$\geq 2 \sum_{k=1}^{N} \varepsilon_{m_k} \log(1/\varepsilon_{m_k}) - 2 \text{ const.} \sum_{k=1}^{N} \varepsilon_{m_k} \log\log(1/\varepsilon_{m_k})$$

$$= 2 \sum_{k=1}^{N} (\varepsilon_{m_k} \log(1/\varepsilon_{m_k})) \left(1 - \frac{\text{const.} \log\log(1/\varepsilon_{m_k})}{\log(1/\varepsilon_{m_k})}\right).$$

Since

$$\lim_{k \to \infty} \left(1 - \frac{\text{const.} \log\log(1/\varepsilon_{m_k})}{\log(1/\varepsilon_{m_k})}\right) = 1 ,$$

it follows that $\sum_{k=1}^{\infty} \varepsilon_{m_k} \log(1/\varepsilon_{m_k}) < \infty$. Therefore, $\sum_{n=1}^{\infty} \varepsilon_n \log(1/\varepsilon_n) < \infty$, and E_2 is a Carleson set.

The assumptions on the sequence $\{r_k e^{i\theta_k}\}_{k=1}^{\infty}$ in the proof cause no problem. If $\{z_k\}_{k=1}^{\infty}$ does not satisfy these conditions, choose a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ such that the conditions are satisfied and $E_2 = E_1 \cup \{e^{i\theta_{n_k}}\}_{k=1}^{\infty}$. The integral in (1.2) will still be finite if we replace Z by $Z' = E_1 \cup \{z_{n_k}\}_{k=1}^{\infty}$. This completes the proof of Theorem 1.2.

We have shown that (1.1) and (1.2) imply that E_2 is a Carleson set. An application of the result of Caughran now proves the sufficiency in Theorem 1.1.

Proof of Theorem 1.3. We show that $E \cup \{e^{i\theta_k}\}_{k=1}^{\infty}$ is a Carleson set, and we again apply the result of Caughran [1], as in the proof of Theorem 1.2.

Assume that $E \subset [-\pi, \pi]$ and $\{\theta_k\}_{k=1}^{\infty}$ is a sequence in $[-\pi, \pi] \setminus E$. The condition (1.3) is equivalent to the condition

$$\sum_{k=1}^{\infty} [d(\theta_k, E)]^{\alpha} < \infty . \tag{2.4}$$

Furthermore, it is no loss of generality to assume that $\alpha > 1$.

We now define a new sequence $\{\phi_k\}_{k=1}^{\infty}$ in $[-\pi, \pi] \setminus E$ such that $\{\theta_k\}_{k=1}^{\infty} \subset \{\phi_k\}_{k=1}^{\infty}$, and we show that $E \cup \{e^{i\phi_k}\}_{k=1}^{\infty}$ is a Carleson set. Since every closed subset of a Carleson set is a Carleson set, it will follow that $E \cup \{e^{i\theta_k}\}_{k=1}^{\infty}$ is a Carleson set.

Let $\{(a_n, b_n)\}_{n=1}^{\infty}$ be the set of complementary intervals of E, and let $\varepsilon_n = b_n - a_n$ $(n = 1, 2, \cdots)$. Set $\beta = 1/(\alpha - 1)$, and for each positive integer n, let s_n be the least positive integer for which $s_n^{-\beta} < \varepsilon_n/2$. Denote by ρ and ω the functions $\rho(x) = 1/x$ and $\omega(x) = x \log(1/x)$.

Now consider the set consisting of the union of the sequence $\{\theta_k\}_{k=1}^{\infty}$, the sequence of midpoints $\{(a_n + b_n)/2\}_{n=1}^{\infty}$, and the countable collection of sequences

$$\{a_n + k^{-\beta}\}_{k=s_n, n=1}^{\infty} \quad \text{and} \quad \{b_n - k^{-\beta}\}_{k=s_n, n=1}^{\infty} .$$

For each positive integer n, let $\{\phi_{nm}\}_{m=1}^{\infty}$ be a decreasing enumeration of the elements of this set that belong to $(a_n, a_n + \varepsilon_n/2]$, and let $\{\phi_{nm}^*\}_{m=1}^{\infty}$ be an increasing enumeration of those in $[b_n - \varepsilon_n/2, b_n)$.

Note that for each positive integer n,

$$\sum_{k=s_n}^{\infty} [d(a_n + k^{-\beta}, E)]^{\alpha} = \sum_{k=s_n}^{\infty} k^{-\beta\alpha} \leq \text{const.} \int_{s_n}^{\infty} \rho(x)^{\beta+1}\, dx \leq \text{const.}\ \beta^{-1} \varepsilon_n . \tag{2.5}$$

Thus it follows from (2.5) that

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} [d(\phi_{nk}, E)]^{\alpha} \leq \sum_{k=1}^{\infty} [d(\theta_k, E)]^{\alpha} + \text{const.}\ \beta^{-1} \sum_{n=1}^{\infty} \varepsilon_n + 2^{-\alpha} \sum_{n=1}^{\infty} \varepsilon_n^{\alpha} < \infty . \tag{2.6}$$

Fix an integer n. Without loss of generality, assume that $a_n = 0$, so that $d(\phi_{nk}, E) = \phi_{nk}$. We now obtain a bound on

$$\sum_{k=1}^{\infty} \omega(\phi_{nk} - \phi_{n,k+1}) . \tag{2.7}$$

All constants obtained will be independent of n. Let M_0^n denote the number of terms of $\{\phi_{nk}\}_{k=1}^{\infty}$ that lie strictly between $s_n^{-\beta}$ and $\varepsilon_n/2$, and M_k^n the number between

$(s_n+k)^{-\beta}$ and $(s_n+k-1)^{-\beta}$ $(k = 1, 2, \cdots)$. Since we seek a bound on (2.7), we may by Lemma 2.2 assume that the points of $\{\phi_{nk}\}_{k=1}^{\infty}$ between $(s_n+k-1)^{-\beta}$ and $(s_n+k)^{-\beta}$ $(k = 1, 2, \cdots)$ are distributed evenly, likewise those between $s_n^{-\beta}$ and $\varepsilon_n/2$. Redistributing the points in this manner will not destroy condition (2.6).

Now,

$$\begin{aligned}\sum_{k=1}^{\infty} \omega(\phi_{nk} - \phi_{n,k+1}) &= (M_0^n+1)\,\omega[(M_0^n+1)^{-1}(\varepsilon_n/2 - s_n^{-\beta})] \\ &\quad + \sum_{k=1}^{\infty} (M_k^n+1)\,\omega[(M_k^n+1)^{-1}((s_n+k-1)^{-\beta} - (s_n+k)^{-\beta})] \\ &= (\log(M_0^n+1))(\varepsilon_n/2 - s_n^{-\beta}) + \omega(\varepsilon_n/2 - s_n^{-\beta}) \\ &\quad + \sum_{k=1}^{\infty} [\log(M_k^n+1)][(s_n+k-1)^{-\beta} - (s_n+k)^{-\beta}] \\ &\quad + \sum_{k=1}^{\infty} \omega[(s_n+k-1)^{-\beta} - (s_n+k)^{-\beta}]\,. \end{aligned} \tag{2.8}$$

We estimate the terms of the right-hand side of (2.8). First,

$$(\log(M_0^n+1))(\varepsilon_n/2 - s_n^{-\beta}) \le (M_0^n+1)((s_n-1)^{-\beta} - s_n^{-\beta}) \le \text{const.}\,(M_0^n+1)\,s_n^{-\beta\alpha}\,. \tag{2.9}$$

Likewise,

$$\sum_{k=1}^{\infty} [\log(M_k^n+1)][(s_n+k-1)^{-\beta} - (s_n+k)^{-\beta}] \le \text{const.} \sum_{k=1}^{\infty} (M_k^n+1)(s_n+k)^{-\beta\alpha}. \tag{2.10}$$

Furthermore,

$$(M_0^n+1)\,s_n^{-\beta\alpha} + \sum_{k=1}^{\infty} (M_k^n+1)(s_n+k)^{-\beta\alpha} \le \sum_{k=1}^{\infty} \phi_{nk}^{\alpha}\,. \tag{2.11}$$

Finally,

$$\begin{aligned}&\sum_{k=1}^{\infty} \omega[(s_n+k-1)^{-\beta} - (s_n+k)^{-\beta}] \\ &\quad \le \text{const.} \sum_{k=1}^{\infty} \omega[(s_n+k)^{-1-\beta}] + \text{const.} \sum_{k=1}^{\infty} (s_n+k)^{-1-\beta} \\ &\quad \le \text{const.} \int_{s_n+1}^{\infty} \omega(x^{-\beta-1})\,dx + \text{const.}\ \beta^{-1}\varepsilon_n \\ &\quad = \text{const.}\ \beta^{-2}(\omega[(s_n+1)^{-\beta}] + (s_n+1)^{-\beta}) + \text{const.}\ \beta^{-1}\varepsilon_n\,. \end{aligned} \tag{2.12}$$

From expressions (2.9) to (2.12) we see that

$$\sum_{k=1}^{\infty} \omega(\phi_{nk} - \phi_{n,k+1})$$

$$\leq \text{const.} \sum_{k=1}^{\infty} \phi_{nk}^{\alpha} + \text{const.}\ \varepsilon_n + \text{const.}\ \omega[(s_n + 1)^{\beta}] + \omega(\varepsilon_n/2 - s_n^{-\beta}) .$$

Our assumption that $a_n = 0$ implies that $\phi_{nk}^{\alpha} = [d(\phi_{nk}, E)]^{\alpha}$. Similar estimates on each interval $(a_n, a_n + \varepsilon_n/2]$ $(n = 1, 2, \cdots)$ yield the bound

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \omega(\phi_{nk} - \phi_{n,k+1}) \leq \text{const.} \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} [d(\phi_{nk}, E)]^{\alpha} + \text{const.} \sum_{n=1}^{\infty} \varepsilon_n$$
$$+ \text{const.} \sum_{n=1}^{\infty} \omega[(s_n + 1)^{-\beta}] + \text{const.} \sum_{n=1}^{\infty} \omega(\varepsilon_n/2 - s_n^{-\beta}), \tag{2.13}$$

and each series on the right side of (2.13) is convergent. The last two series converge, by Lemma 2.1, and convergence of the first two is clear.

Similar estimates on the points $\{\phi_{nm}^{*}\}_{n,m=1}^{\infty}$ are possible, and it follows that $E \cup \{e^{i\theta_k}\}_{k=1}^{\infty}$ is a Carleson set. This completes the proof of Theorem 1.3.

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