ON ADDITIVE FUNCTIONS

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Introduction. A number-theoretic function f is said to be *additive* if $f(mn) = f(m) + f(n)$ whenever $(m, n) = 1$; we denote the class of such functions by $\mathscr{A}$. Because of the special nature of the subclass $\mathscr{B}$ of functions of the form $f(n) = c \log n$, it is of interest to find conditions on functions f in $\mathscr{A}$ under which f is also in $\mathscr{B}$.

The first investigation in this direction was made by P. Erdös [1], who proved that if $f \in \mathscr{A}$ and $f(n+1) - f(n) \geq 0$ for each natural number n, then $f \in \mathscr{B}$. Erdös conjectured that the same conclusion holds if the monotonicity condition is relaxed to the requirement that $f(n+1) - f(n) \geq 0$ for almost all n, and this conjecture was subsequently proved by I. Kátai [2]. Erdös also proved that if $f \in \mathscr{A}$ and $\lim_{n \to \infty} [f(n+1) - f(n)] = 0$, then $f \in \mathscr{B}$, and he conjectured that the condition on $f(n+1) - f(n)$ can be replaced by the condition

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} |f(n+1) - f(n)| = 0 .$$

The last conjecture was recently established by Kátai (proof to appear). E. Wirsing subsequently found an elegant proof of this result, and since the proof of Theorem 1 is based on some of the ideas in his proof, we shall give an outline of his method, at the end of Lemma 3.

Finally, we mention a long-standing conjecture of Erdös, recently proved by Wirsing [3]:

THEOREM (Wirsing). *Suppose that* $f \in \mathscr{A}$ *and that the set of differences* $f(n+1) - f(n)$ *is bounded. Then* $f(n) = c \log n + g(n)$, *where* g *is a bounded, additive function.*

Some time ago, I conjectured that the following is true:

CONJECTURE. *If* $f \in \mathscr{A}$ *and*

$$\liminf_{x \to \infty} \frac{1}{x} \sum_{n \leq x} |f(n+1) - f(n)| = 0, \tag{1}$$

then $f \in \mathscr{B}$.

The conjecture is still open; but in this paper I prove the following weaker version of it.

THEOREM 1. *Let* $f \in \mathscr{A}$, *and let* f *satisfy condition* (1) *and*

$$f(n) = O(\log n). \tag{2}$$

Then $f \in \mathscr{B}$.

I shall also prove the following result.

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THEOREM 2. *Let* $f \in \mathscr{A}$, *and let* f *satisfy the conditions*

$$\liminf_{x\to\infty} \frac{1}{\log x} \sum_{n \leq x} \left| \frac{f(n+1)}{n+1} - \frac{f(n)}{n} \right| = 0 \tag{3}$$

and

$$f(n) \geq 0 ; \tag{4}$$

then $f \in \mathscr{B}$.

In the course of the proof of Theorem 1, it will emerge (see Lemma 3) that if f is additive and satisfies (1), then f is completely additive, that is, $f(mn) = f(m) + f(n)$ for *all* natural numbers m and n.

Notation. The symbols k, l, m, n, n_1, n_2 will always denote natural numbers, and p and q will denote prime numbers. The symbols ε and x will denote a small, positive number and a positive parameter that tends to $+\infty$. The statement $f(n) = O(\log n)$ will mean that $|f(n)| \leq B \log n$ for all natural numbers n, where B is a nonnegative constant. For a positive function g(x), the expression $f(x) = \omega(g(x))$ will mean that there is an infinite set of values of x that tend to $+\infty$ and that the ratio f/g tends to zero on this set.

LEMMA 1. *Let* f *satisfy condition* (1). *Then, for each* ε $(0 < \varepsilon < 1)$, *there exist infinitely many numbers* $x_i = x_i(\varepsilon)$ $(x_{i+1} \geq 1 + x_i)$ *such that, for each* i, *there are more than* $(1 - \varepsilon)x_i$ *integers* $n \leq x_i$ *for which* $|f(n+1) - f(n)| < \varepsilon$.

Proof. Let f satisfy condition (1). Then, for each ε $(0 < \varepsilon < 1)$, there exist infinitely many positive numbers $x_i = x_i(\varepsilon)$ with $x_{i+1} > 1 + x_i$ such that

$$\frac{1}{x_i} \sum_{n \leq x_i} |f(n+1) - f(n)| < \varepsilon^2. \tag{5}$$

For each such x_i, if there were at least εx_i integers $n \leq x_i$ for which $|f(n+1) - f(n)| \geq \varepsilon$, then the expression in (5) would be at least ε^2, a contradiction. Thus, there are more than $(1 - \varepsilon)x_i$ integers $n \leq x_i$ for which $|f(n+1) - f(n)| < \varepsilon$, and this proves the lemma.

LEMMA 2. *Let* f *satisfy condition* (1), *and let* p^m *denote a prime power. Then for each* ε *for which*

$$\varepsilon(m+1)p^{m+1} < 1, \tag{6}$$

there exist natural numbers $\lambda_0, \lambda_1, \cdots, \lambda_m$ *such that*

$$(\lambda_0, p) = 1, \tag{7}$$

$$\lambda_j = p\lambda_{j-1} + 1 \quad (1 \leq j \leq m), \tag{8}$$

$$|f(\lambda_j) - f(p\lambda_{j-1})| < \varepsilon \quad (1 \leq j \leq m), \tag{9}$$

$$|f(\lambda_m) - f(p^m \lambda_0)| < \varepsilon. \tag{10}$$

Proof. Let p^m be a prime power, and let ε satisfy condition (6). For $x > p^m$, define the sets $S_k = S_k(\varepsilon, x)$ $(0 \leq k \leq m)$ by

$$S_0 = \{n \le xp^{-m}\colon (n, p) = 1,\ |f(np^m + p^{m-1} + \cdots + p + 1) - f(np^m)| < \varepsilon\},$$

$$S_1 = \{n \le xp^{-m}\colon |f(np + 1) - f(np)| < \varepsilon\},$$

$$S_k = \{n \le xp^{-m}\colon |f(np^k + p^{k-1} + \cdots + p + 1) - f(np^k + p^{k-1} + \cdots + p)| < \varepsilon\} \quad (2 \le k \le m).$$

The idea of the proof is to show that we can choose x so that some natural number n lies in the intersection of the S_k $(0 \le k \le m)$. Then, if we put $\lambda_0 = n$ and define a sequence $\{\lambda_j\}_1^m$ of natural numbers inductively by (8), it is clear from the definition of the sets S_k that the λ_j satisfy conditions (7) to (10). Therefore, the proof will be complete if we can show that, for a suitable x, there is an n in $\bigcap_{k=0}^m S_k$.

From Lemma 1, we easily deduce that there are infinitely many numbers $x_i > p^m$ $(x_{i+1} > 1 + x_i)$ for which

$$|S_k| > p^{-m}x_i - \varepsilon x_i \quad (1 \le k \le m), \tag{11}$$

$$|S_0| > (1 - p^{-1})p^{-m}x_i - \varepsilon x_i. \tag{12}$$

Using a simple counting argument, we obtain from (11) and (12) for each such x_i the inequality

$$\left|\bigcap_{k=0}^{m} S_k\right| > p^{-m}x_i - m\varepsilon x_i - (p^{-m-1} + \varepsilon)x_i = \beta x_i,$$

where $\beta > 0$ by the choice of ε in (5). This proves the lemma.

LEMMA 3. *Let* $f \in \mathscr{A}$, *and let* f *satisfy* (1). *Then* f *is completely additive.*

Proof. To show that f is completely additive, it is sufficient to prove that

$$f(p^m) = mf(p) \tag{13}$$

for each prime power p^m. Let ε satisfy (6). By Lemma 2, there exist natural numbers $\lambda_0, \lambda_1, \cdots, \lambda_m$ satisfying (7) to (10). Therefore, $f(\lambda_0)$ can be expressed as follows:

$$\begin{aligned}
f(\lambda_0) &= f(p\lambda_0) - f(p) \\
&= f(\lambda_1) - f(p) + \varepsilon_1 && (|\varepsilon_1| < \varepsilon) \\
&= f(p\lambda_1) - 2f(p) + \varepsilon_1 \\
&= f(\lambda_2) - 2f(p) + \varepsilon_2 && (|\varepsilon_2| < 2\varepsilon) \\
&\cdots \\
&= f(p\lambda_{m-1}) - mf(p) + \varepsilon_{m-1} && (|\varepsilon_{m-1}| < (m-1)\varepsilon) \\
&= f(\lambda_m) - mf(p) + \varepsilon_m && (|\varepsilon_m| < m\varepsilon) \\
&= f(p^m\lambda_0) - mf(p) + \varepsilon_{m+1} && (|\varepsilon_{m+1}| < (m+1)\varepsilon),
\end{aligned}$$

and since ε can be chosen arbitrarily small, (13) holds. This proves the lemma.

Now suppose that $f \in \mathscr{A}$ and that $\lim_{x\to\infty} \sum_{n\leq x} |f(n+1) - f(n)| = 0$. Then, by Lemma 3, f is completely additive, and we can show (following Wirsing) that $f(n) = c \log n$. To do this, let $S(x) = \sum_{n\leq x} f(n)$. For each natural number m,

$$S(x) = xf(m) + mS(x/m) + o(x)$$

(the proof of the last equation is almost identical to the proof of (22)). Iterating the above expression $K = [\log x/\log m]$ times, we obtain the estimates

$$\begin{aligned} S(x) &= xf(m) + mS(x/m) + o(x) \\ &= 2xf(m) + m^2 S(x/m^2) + o(2x) \\ &\cdots \\ &= Kxf(m) + m^K S(x/m^K) + o(Kx) \\ &= (f(m)/\log m)\, x \log x + o(x \log x), \end{aligned}$$

and it follows that

$$f(m)/\log m = \lim_{x\to\infty} S(x)/x \log x = \text{constant}.$$

Now, if instead of $\lim_{x\to\infty} \sum_{n\leq x} |f(n+1) - f(n)| = 0$, we assume condition (1), then it is clear that the above procedure is too weak to show that $f(n) = c \log n$, because the iteration procedure is no longer valid. We can overcome this difficulty, however, if we also assume that condition (2) holds.

Proof of Theorem 1. Let $f \in \mathscr{A}$, and let f satisfy (1) and (2). Then f is completely additive, by Lemma 3. The proof consists in showing that for natural numbers n_1 and n_2, we can make $|f(n_1) - f(n_2)|$ arbitrarily small by choosing the ratio n_2/n_1 sufficiently close to 1; or, equivalently, by choosing $\log n_2 - \log n_1$ sufficiently close to 0. The result will follow from a theorem of Erdös (which says that if $f \in \mathscr{A}$ and $f(n+1) - f(n) \to 0$, then $f \in \mathscr{B}$), or directly, as follows:

Let p be a prime for which $f(p) \neq 0$ (if there is no such prime, Theorem 1 is true automatically), and let q be any other prime. Now suppose it can be shown that there exists a constant C such that if ζ is a small, positive number, then

$$|af(p) - bf(q)| < C\zeta\, |\log \zeta| \tag{14}$$

whenever the natural numbers a and b satisfy the condition

$$|a \log p - b \log q| < \zeta. \tag{15}$$

It follows from (14) and (15) that

$$\left|\frac{f(q)}{f(p)} - \frac{\log q}{\log p}\right| \leq \left|\frac{a}{b} - \frac{\log q}{\log p}\right| + \left|\frac{f(q)}{f(p)} - \frac{a}{b}\right| < \frac{C\zeta\,|\log \zeta|}{bf(p)} + \frac{\zeta}{b \log p},$$

and since ζ can be taken arbitrarily small, we obtain the relation

$$\frac{f(q)}{\log q} = \text{constant}$$

for all primes q. The proof of Theorem 1 will be complete, then, if it is shown that (14) is true whenever (15) holds. (Note that the left-hand side of (15) can be made arbitrarily small, by Dirichlet's theorem on the approximation of real numbers). This will be deduced from (23), with the assumption that $|f(n)| \leq B \log n$ for all natural numbers n, where B is a nonnegative constant. For $x \geq 1$, we define

$$S(x) = \sum_{n \leq x} f(n), \tag{16}$$

and we choose two natural numbers n_1 and n_2 such that

$$n_2/n_1 = 1 + \zeta, \tag{17}$$

where ζ is a small, positive number. For n_i $(i = 1, 2)$, we have the equation

$$\begin{aligned} S(x) &= \sum_{k=1}^{n_i} \sum_{\substack{n \leq x \\ n \equiv k \pmod{n_i}}} f(n) \\ &= n_i \sum_{\substack{n \leq x \\ n \equiv n_i \pmod{n_i}}} f(n) + \sum_{k=1}^{n_i - 1} \left\{ \sum_{\substack{n \leq x \\ n \equiv k \pmod{n_i}}} f(n) - \sum_{\substack{n \leq x \\ n \equiv n_i \pmod{n_i}}} f(n) \right\} \\ &= [x/n_i]\, n_i f(n_i) + n_i S(x/n_i) + E(x, n_i), \end{aligned} \tag{18}$$

where

$$\begin{aligned} E(x, n_i) &= \sum_{k=1}^{n_i - 1} \left\{ \sum_{\substack{n \leq x \\ n \equiv k \pmod{n_i}}} f(n) - \sum_{\substack{n \leq x \\ n \equiv n_i \pmod{n_i}}} f(n) \right\} \\ &= \sum_{k=1}^{n_i - 1} \left\{ \sum_{j=k}^{n_i - 1} \left(\sum_{\substack{n \leq x \\ n \equiv j \pmod{n_i}}} f(n) - \sum_{\substack{n \leq x \\ n \equiv j+1 \pmod{n_i}}} f(n) \right) \right\} \\ &= \sum_{k=1}^{n_i - 1} \left\{ \sum_{j=k}^{n_i - 1} \left(\sum_{\substack{n \leq x-1 \\ n \equiv j \pmod{n_i}}} (f(n) - f(n+1)) + \sum_{\substack{x-1 < n \leq x \\ n \equiv j \pmod{n_i}}} f(n) \right) \right\}. \end{aligned}$$

Therefore

$$|E(x, n_i)| \leq n_i \left(\sum_{n \leq x-1} |f(n+1) - f(n)| + |f([x])| \right),$$

and since $f(1) = 0$, this does not exceed

$$2n_i \sum_{n \le x-1} |f(n+1) - f(n)| .$$

Given $\varepsilon > 0$, choose $x = x(\varepsilon, n_1, n_2)$ so that

$$\frac{1}{x} \sum_{n \le x} |f(n+1) - f(n)| < \varepsilon/4n_1 n_2 \tag{19}$$

and

$$|n_i f(n_i)| < \varepsilon x/2 \qquad (i = 1, 2). \tag{20}$$

With this choice of x, we have the inequality

$$|E(x, n_i)| < \varepsilon x/2, \tag{21}$$

and from (18) and (21) we deduce that

$$S(x) = xf(n_i) + n_i S(x/n_i) + \omega(x) \qquad (i = 1, 2). \tag{22}$$

Subtracting equation (22) with $i = 2$ from equation (22) with $i = 1$, we obtain the relation

$$n_1 S(x/n_1) - n_2 S(x/n_2) = x(f(n_2) - f(n_1)) + \omega(x). \tag{23}$$

Since f is completely additive, we can write (for $y \ge 0$)

$$\begin{aligned} S(y) &= \sum_{n \le y} f(n) = \sum_{n \le y} \sum_{p^e \| n} f(p^e) = \sum_{p^e \le y} f(p^e) \sum_{\substack{n \le y \\ p^e \| n}} 1 \\ &= \sum_{p^e \le y} e f(p) \{[y/p^e] - [y/p^{e+1}]\} = \sum_{p^e \le y} f(p)[y/p^e], \end{aligned} \tag{24}$$

where the sums are taken over prime powers p^e. By virtue of (24), we can rewrite the left member of (23) in the form

$$\begin{aligned} n_1 S(x/n_1) - n_2 S(x/n_2) &= \sum_{p^e \le \frac{x}{n_1}} f(p) \{n_1 [x/p^e n_1] - n_2 [x/p^e n_2]\} \\ &= \sum_{p \le \frac{x}{n_1}} f(p) \{n_1 [x/pn_1] - n_2 [x/pn_2]\} + O(\sqrt{x} \log^3 x) \\ &= T(x, n_1, n_2) + \omega(x); \end{aligned}$$

here we have used, at the penultimate step, the facts that the number of prime powers $p^e \le x$ $(e \ge 2)$ is $O(\sqrt{x} \log x)$, that $f(p) = O(\log x)$, and that

$$|n_1[y/n_1] - n_2[y/n_2]| \leq \max(n_1, n_2) = n_2 = O(\log x).$$

We now obtain an upper bound for the sum $T(x, n_1, n_2)$. First, we divide the sum into two parts:

$$T(x, n_1, n_2) = \sum_{p \leq \frac{\zeta x}{n_1}} + \sum_{\frac{\zeta x}{n_1} < p \leq \frac{x}{n_1}} = T_1(x, n_1, n_2) + T_2(x, n_1, n_2),$$

where ζ is defined in (17). For T_1, we have the estimate

$$T_1 \leq n_2 B \sum_{p \leq \frac{\zeta x}{n_1}} \log p \leq 2B\zeta x \tag{25}$$

for all sufficiently large x. We now estimate T_2. For each prime p $(\zeta x/n_1 < p \leq x/n_1)$, we distinguish two cases depending on whether there is an integer between x/pn_2 and x/pn_1. If there is no such integer for the prime p, then, by (17),

$$|n_1[x/pn_1] - n_2[x/pn_2]| = (n_2 - n_1)[x/pn_1] \leq n_1 \zeta x/pn_1 = \zeta x/p,$$

and therefore the contribution of these primes to the sum T_2 does not exceed

$$B\zeta x \sum_{\frac{\zeta x}{n_1} < p \leq \frac{x}{n_1}} \frac{\log p}{p} \leq 2Bx\zeta\, |\log \zeta|, \tag{26}$$

for all sufficiently large x. We now count the number $N = N(x, \zeta, n_1)$ of primes p $(\zeta x/n_1 < p \leq x/n_1)$ for which there is an integer j between x/pn_2 and x/pn_1. Let the integer j satisfy the condition

$$x/pn_2 < j < x/pn_1, \tag{27}$$

in which case we have the relations

$$x/jn_2 < p \leq x/jn_1. \tag{28}$$

Thus the number of primes p that satisfy (27) for a particular integer j is equal to

$$\pi(x/jn_1) - \pi(x/jn_2). \tag{29}$$

Moreover, from the range of summation for p it is clear that $1 \leq j \leq \zeta^{-1}$. Consequently, from (28) and (29) we obtain the inequality

$$N \leq \sum_{1 \leq j \leq \zeta^{-1}} \{\pi(x/jn_1) - \pi(x/jn_2)\}.$$

Having chosen n_1 and n_2, choose x so that, in addition to (19) and (20), it also satisfies the condition

$$\log^{-1}(x/jn_2) \leq 2 \log^{-1} x. \tag{30}$$

Then, by the prime number theorem, N does not exceed the quantity

$$\frac{4x\zeta}{n_2 \log x} \sum_{1 \leq j \leq \zeta^{-1}} \frac{1}{j} \leq \frac{8x\zeta\,|\log \zeta|}{n_2 \log x},$$

so that the contribution of these primes to the sum T_2 does not exceed the quantity

$$\frac{8x\xi\,|\log \zeta|}{n_2 \log x}(B \log x)\cdot \max(|n_1[x/pn_1] - n_2[x/pn_2]|) \leq 8Bx\zeta\,|\log \zeta|. \tag{31}$$

From (25), (26), and (31) we see that

$$T \leq Cx\zeta\,|\log \zeta|$$

when x is chosen sufficiently large; here C is a constant independent of x, ε, ζ, n_1, and n_2. Therefore, taking absolute values in (23), we obtain the estimate

$$x\,|f(n_2) - f(n_1)| \leq C\zeta\,|\log \zeta|\, x + \omega(x), \tag{32}$$

provided x is sufficiently large and satisfies the conditions (19), (20), and (30). It is then clear from (32) with $n_2 = q^b$ and $n_1 = p^a$ that (14) must be true whenever (15) holds, for some constant C. This proves Theorem 1.

Proof of Theorem 2. From condition (3) we can deduce that f is completely additive, by a sequence of lemmas similar to Lemmas 1, 2, and 3. The proofs of these lemmas will not be given; we simply quote the following result.

LEMMA 4. *Let* $f \in \mathscr{A}$, *and let* f *satisfy* (3). *Then* f *is completely additive.*

Now for $x \geq 1$, define

$$U(x) = \sum_{n \leq x} \frac{f(n)}{n}.$$

Choose any two natural numbers n_1 and n_2. By an argument similar to the one used to deduce equation (23), we can show that

$$U(x) = f(n_i)\log x + U(x/n_i) + \omega(\log x) \qquad (i = 1, 2), \tag{33}$$

provided x is chosen large compared with n_1 and n_2 and x is a number for which

$$\frac{1}{\log x} \sum_{n \leq x} \left| \frac{f(n+1)}{n+1} - \frac{f(n)}{n} \right| < \varepsilon$$

and

$$|f(n_i) \log n_i| < \varepsilon \log x.$$

Subtracting equation (33) with $i = 2$ from equation (33) with $i = 1$, we find that

$$(f(n_1) - f(n_2))\log x = U(x/n_2) - U(x/n_1) + \omega(\log x). \tag{34}$$

Since $f(n) \geq 0$ by condition (4), we see from (34) that $f(n_1) - f(n_2)$ is nonnegative if and only if $U(x/n_2) - U(x/n_1)$ is nonnegative, which is true if and only if $n_1 - n_2$ is nonnegative. Thus, f is a nondecreasing, additive function, and so $f(n) = c \log n$, by a result of Erdös [1]. This proves Theorem 2.

It is clear from the proof of Theorem 2 that the same result will hold if condition (4) is replaced by the condition $f(n) \leq 0$. In fact, the result holds if we assume either (i) $f(n) \geq -K \log n$, or (ii) $f(n) \leq K \log n$, where K is a constant. This can be deduced immediately from the fact that if $f(n)$ satisfies the hypotheses of Theorem 2, then the same is true of the additive functions $f(n) \pm K \log n$.

REFERENCES

1. P. Erdös, *On the distribution function of additive functions.* Ann. of Math. (2) 47 (1946), 1-20.

2. I. Kátai, *A remark on number theoretical functions.* Acta Arith. 14 (1968), 409-415.

3. E. Wirsing, *A characterization of the logarithm as an additive function.* Proceedings of the Rome Conference of Number Theory (1968) (to appear).

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