FUNCTIONS SATISFYING LIPSCHITZ CONDITIONS

James D. Stein, Jr.

In memory of Professor A. Robert Brodsky (1940-1968)

Let (X, d) be a metric space, and let $\alpha > 0$. A real-valued function f on X is said to be of Lipschitz class α if

$$\sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)^{\alpha}} \;\middle|\; x, y \in X, \; x \neq y \right\}$$

is finite. The purpose of this paper is to investigate metric spaces that support non-constant functions of Lipschitz class α, with emphasis on the case $\alpha > 1$. In addition to investigating the metric spaces themselves, we shall also investigate the structure of various Banach algebras of functions satisfying Lipschitz conditions.

1. A PRELIMINARY PROPOSITION

Throughout the paper, we shall be concerned with real-valued functions on (X, d); if f is of Lipschitz class α, we denote by $\|f\|_{\alpha}$ the defining supremum. Let $\mathrm{Lip}_{\alpha}(X, d)$ denote the set of all bounded functions on X of Lipschitz class α; if $f \in \mathrm{Lip}_{\alpha}(X, d)$, let $\|f\|_{\infty} = \sup_{x \in X} |f(x)|$. The following proposition is of interest, since the proof differs from the argument in [3].

PROPOSITION 1.1. *For* $f \in \mathrm{Lip}_{\alpha}(X, d)$, *let* $\|f\| = \|f\|_{\alpha} + \|f\|_{\infty}$. *With this norm,* $\mathrm{Lip}_{\alpha}(X, d)$ *is a Banach algebra.*

Proof. The verification that $\mathrm{Lip}_{\alpha}(X, d)$ is a normed algebra parallels the argument in [4]; it remains to show $\mathrm{Lip}_{\alpha}(X, d)$ is complete. Let

$$f_n \in \mathrm{Lip}_{\alpha}(X, d) \qquad (n = 1, 2, \cdots),$$

and suppose $\|f_n - f_m\| \to 0$; then $\|f_n - f_m\|_{\infty} \to 0$, and therefore there exists a function $f \in C(X)$ such that $f_n \to f$ uniformly. Now

$$|f(x) - f(y)| \leq |f_n(x) - f_n(y)| + |(f - f_n)(x)| + |(f - f_n)(y)|,$$

and hence, given $\varepsilon > 0$ and $x \neq y$, we can choose N so that

$$\|f - f_N\|_{\infty} < \frac{\varepsilon}{2} d(x, y)^{\alpha}.$$

Then

$$|f(x) - f(y)| \leq \|f_N\|_{\alpha} d(x, y)^{\alpha} + \varepsilon\, d(x, y)^{\alpha} = (\|f_N\|_{\alpha} + \varepsilon) d(x, y)^{\alpha},$$

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and consequently $\|f\|_\alpha \leq \sup_n \|f_n\|_\alpha < \infty$ and $f \in \mathrm{Lip}_\alpha(X, d)$. We now show that $\|f_n - f\| \to 0$; it is clearly enough to show that if $\|f_n - f_m\|_\alpha \to 0$ and $f_n \to 0$ uniformly, then $\|f_n\|_\alpha \to 0$. Choose a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that $\|f_{n_k} - f_{n_{k+1}}\|_\alpha < 2^{-k}$. If we can show that $\|f_{n_k}\|_\alpha \to 0$, the proof will be complete, because a Cauchy sequence with a convergent subsequence is convergent. Assume there is an $\varepsilon > 0$ such that $\|f_{n_k}\|_\alpha \geq \varepsilon$. For each $x \neq y$, the inequality

$$\frac{|f_{n_k}(x) - f_{n_k}(y) - f_{n_{k+1}}(x) + f_{n_{k+1}}(y)|}{d(x, y)^\alpha} < 2^{-k}$$

implies that

$$\frac{|f_{n_k}(x) - f_{n_k}(y)|}{d(x, y)^\alpha} < 2^{-k} + \frac{|f_{n_{k+1}}(x) - f_{n_{k+1}}(y)|}{d(x, y)^\alpha} < \cdots$$

$$< 2^{-k} + \cdots + 2^{-k-j+1} + \frac{|f_{n_{k+j}}(x) - f_{n_{k+j}}(y)|}{d(x, y)^\alpha}$$

$$< 2^{-k+1} + \frac{|f_{n_{k+j}}(x) - f_{n_{k+j}}(y)|}{d(x, y)^\alpha}$$

for each $j > 0$. Choose x_0, y_0 so that $|f_{n_k}(x_0) - f_{n_k}(y_0)| \geq \varepsilon\, d(x_0, y_0)^\alpha$; then

$$\varepsilon \leq \frac{|f_{n_k}(x_0) - f_{n_k}(y_0)|}{d(x_0, y_0)^\alpha} < 2^{-k+1} + \frac{|f_{n_{k+j}}(x_0) - f_{n_{k+j}}(y_0)|}{d(x_0, y_0)^\alpha};$$

as $j \to \infty$, the last term goes to 0, since $f_n \to 0$ uniformly. Therefore $\varepsilon \leq 2^{-k+1}$ for all k, a contradiction. Hence $\mathrm{Lip}_\alpha(X, d)$ is a Banach algebra. ■

One might conceivably define $\mathrm{Lip}_\infty(X, d)$ as

$$\{f \in \mathrm{Lip}_\alpha(X, d) \mid \alpha > 0,\ \sup_\alpha \|f\|_\alpha < \infty\}.$$

As in the previous proposition, this can be shown to give rise to a Banach algebra; but the following is simpler. If $x, y \in X$, define $x \sim y$ if and only if there exist $x_1, \cdots, x_n \in X$ such that $x = x_1$, $y = x_n$, and $d(x_i, x_{i+1}) < 1$ for $i = 1, \cdots, n - 1$. It is easily seen that $\sim$ is an equivalence relation; call the equivalence classes under $\sim$ 1-components. If $f \in \mathrm{Lip}_\alpha(X, d)$ for all $\alpha > 0$ and $\sup_{\alpha > 0} \|f\|_\alpha$ is finite, then f is constant on 1-components. If $d(x, y) < 1$, then

$$|f(x) - f(y)| \leq (\sup_{\alpha > 0} \|f\|_\alpha)\, d(x, y)^\alpha$$

for all $\alpha > 0$; as $\alpha \to \infty$, the right-hand side goes to zero.

2. THE LIPSCHITZ INDEX OF (X, d)

Let $\beta > \alpha > 0$, and assume $f \in \mathrm{Lip}_\beta(X, d)$. Now

$$\|f\|_\alpha = \max\left(\sup\left\{\frac{|f(x) - f(y)|}{d(x, y)^\alpha} \,\middle|\, d(x, y) \le 1\right\}, \sup\left\{\frac{|f(x) - f(y)|}{d(x, y)^\alpha} \,\middle|\, d(x, y) > 1\right\}\right)$$

$$\le \max\left(\sup\left\{\frac{|f(x) - f(y)|}{d(x, y)^\beta} d(x, y)^{\beta-\alpha} \,\middle|\, d(x, y) \le 1\right\}, 2\|f\|_\infty\right)$$

$$\le \max(\|f\|_\beta, 2\|f\|_\infty);$$

therefore $f \in \mathrm{Lip}_\alpha(X, d)$. This prompts the following definition.

DEFINITION 2.1. *Let* (X, d) *be a metric space, and let* R *denote the space of real numbers. The Lipschitz index of* (X, d) *is* $L(X, d) = \inf\{\alpha \mid \mathrm{Lip}_\alpha(X, d) \cong R\}$; *if* $\mathrm{Lip}_\alpha(X, d)$ *is never isomorphic to the reals, we say* L(X, d) *is infinite.*

By [4, Proposition 1.4], it is clear that $L(X, d) \ge 1$ for each metric space (X, d). We first show that there exist metric spaces with arbitrary Lipschitz indices. In the following proposition, d^α $(0 < \alpha < 1)$ denotes the metric $d^\alpha(x, y) = d(x, y)^\alpha$.

PROPOSITION 2.1. (a) *If* $L(X, d) = \beta$ *and* $0 < \alpha < 1$, *then* $L(X, d^\alpha) = \beta/\alpha$ (β *is finite*).

(b) *Let* $\{(X_n, d_n)\}$ *denote a sequence of metric spaces such that*

$$\lim_{n\to\infty} L(X_n, d_n) = \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} \operatorname{diam} X_n < \infty,$$

and let $X = \prod_{n=1}^{\infty} X_n$ *and* $d((x_n), (y_n)) = \sum_{n=1}^{\infty} d_n(x_n, y_n)$. *Then* L(X, d) *is infinite.*

Proof. (a) Let $\gamma > \beta/\alpha$, and let

$$f \in \mathrm{Lip}_\gamma(X, d^\alpha) \Rightarrow \frac{|f(x) - f(y)|}{d(x, y)^{\alpha\gamma}} \le K;$$

since $\alpha\gamma > \beta$, f is constant. If $\alpha\gamma < \beta$, then there exists a nonconstant f on X such that

$$\frac{|f(x) - f(y)|}{d(x, y)^{\alpha\gamma}} \le K \Rightarrow f \in \mathrm{Lip}_\gamma(X, d^\alpha).$$

(b) The hypotheses imply that (X, d) is a metric space. Let $\alpha \ge 1$, and choose N so that $L(X_N, d_N) > \alpha$. Choose a nonconstant $f \in \mathrm{Lip}_\alpha(X_N, d_N)$, and let π denote the projection of X onto X_N. Let $g = f \circ \pi$; then g is clearly nonconstant. Also,

$$|g((x_n)) - g((y_n))| = |f(x_N) - f(y_N)| \le \|f\|_\alpha d_N(x_N, y_N)^\alpha \le \|f\|_\alpha d((x_n), (y_n))^\alpha,$$

and therefore $g \in \mathrm{Lip}_\alpha(X, d)$. ■

The simplest example of a metric space of infinite Lipschitz index is a two-point space, but Proposition 2.1 (b) assures us of the existence of arcwise connected metric spaces of infinite Lipschitz index. It is also clear that if we can write $X = A \cup B$, where $d(A, B) > 0$, then $L(X, d)$ is infinite. (The class $\mathrm{Lip}_\alpha(X, d)$ contains characteristic functions of A and B.) Pursuing this further, we define a relation $\approx$ among subsets of a space X by the rule that $A \approx B$ if and only if there exist subsets $A_1, \cdots, A_n$ of X such that $A = A_1$, $B = A_n$, and $d(A_i, A_{i+1}) = 0$ for $1 \leq i \leq n-1$, where $d(C, D)$ is defined by the formula

$$d(C, D) = \inf\{d(x, y) \mid x \in C,\ y \in D\}.$$

PROPOSITION 2.2. $A \approx B$ *is an equivalence relation. If* $X = \bigcup_\alpha A_\alpha$ *and* $A_\alpha \approx A_\beta$ *for each pair of indices* α *and* β, *and if* $L(A_\alpha, d) \leq c$ *for all* α, *then* $L(X, d) \leq c$.

Proof. If c is infinite, the conclusion is trivial. If f is of Lipschitz class $\gamma > c$ on X, then $f \mid A_\alpha$ is constant for each α. Given indices α and β, let $C_1, \cdots, C_n$ be subsets of X such that $A_\alpha = C_1$, $A_\beta = C_n$, and $d(C_i, C_{i+1}) = 0$ $(1 \leq i \leq n-1)$. If $f(C_i) \neq f(C_{i+1})$, let $M > 0$, and choose $x \in C_i$ and $y \in C_{i+1}$ so that

$$d(x, y)^\gamma < \frac{|f(C_i) - f(C_{i+1})|}{M};$$

then

$$\frac{|f(x) - f(y)|}{d(x, y)^\gamma} \geq M,$$

a contradiction. ∎

We now show that if $X = A \cup B$ and $d(A, B) = 0$, then $L(X, d)$ can be either the minimum or the maximum of $L(A, d)$ and $L(B, d)$. First, let $0 < \alpha < 1$, $A = [0, 1]$, $B = (0, 1]$; define

$$d(x, x') = |x - x'| \quad (x, x' \in A), \qquad d(y, y') = |y - y'|^\alpha \quad (y, y' \in B),$$

$$d(x, y) = x + y^\alpha \quad (x \in A,\ y \in B).$$

Let $X = A \cup B$. We note earlier that $L(A, d) = 1$ and $L(B, d) = 1/\alpha$; we now show that $L(X, d) = 1/\alpha$. By Proposition 2.2, $L(X, d) \leq 1/\alpha$. Define $f(x) = 0$ for $x \in A$ and $f(y) = y$ for $y \in B$, and let $\beta \in [1, 1/\alpha)$. Clearly,

$$x, x' \in A \Rightarrow |f(x) - f(x')| \leq d(x, x')^\beta,$$

and

$$y, y' \in B \Rightarrow |y - y'| \leq |y - y'|^{\alpha\beta} = d(y, y')^\beta.$$

Now let $x \in A$, $y \in B$; we show that $y \leq (x + y^\alpha)^\beta$. For fixed y, let $g(x) = (x + y^\alpha)^\beta - y$; then $g(0) = y^{\alpha\beta} - y \geq 0$ and $dg/dx = \beta(x + y^\alpha)^{\beta-1} \geq 0$, and therefore $g(x) \geq 0$. Hence $f \in \mathrm{Lip}_\beta(X, d)$ and $L(X, d) = 1/\alpha$.

Now let $A = (0, 1)$, $B = \{0, 1\}$, $X = A \cup B$ with $d(x, y) = |x - y|$. Clearly, $L(A, d) = 1$, $L(B, d) = \infty$, and $L(X, d) = 1$.

We now examine mappings of metric spaces, and their effect on the Lipschitz index. We note in advance that since (X, d) and (X, d^α) are uniformly homeomorphic but have different Lipschitz indices, we shall require stronger conditions.

PROPOSITION 2.3. *Let* (X, d), (Y, d') *be metric spaces, let* ϕ *map* Y *into* X, *and let* $L(Y, d') < \infty$.

(a) *Assume that, for every pair* $x, y \in X$, *there exists a map* $\psi: Y \to X$ *such that* $x, y \in \psi(Y)$ *and* $d(\psi(s), \psi(t)) \leq K d'(s, t)$ *for all* $s, t \in Y$. *Then* $L(X, d) \leq L(Y, d')$.

(b) *If* ϕ *is one-to-one and onto, and there exist* $M, K > 0$ *such that*

$$M \leq \frac{d(\phi(s), \phi(t))}{d'(s, t)} \leq K \quad \textit{for all } s, t \in Y,$$

then $L(X, d) = L(Y, d')$.

Proof. (a) Let $\alpha > L(Y, d')$, and assume $f \in \mathrm{Lip}_\alpha(X, d)$. If ψ maps Y into X so that $d(\psi(s), \psi(t)) \leq K d'(s, t)$ for all $s, t \in Y$, then

$$|(f \circ \psi)(s) - (f \circ \psi)(t)| \leq \|f\|_\alpha d(\psi(s), \psi(t))^\alpha \leq \|f\|_\alpha K^\alpha d'(s, t),$$

and this implies that $f \circ \psi$ is constant. Fix $x_0 \in X$, and choose ψ_x so that $x_0, x \in \psi_x(Y)$; then, if $x_0 = \psi_x(s)$ and $x = \psi_x(t)$, we have the relations

$$f(x_0) = f(\psi_x(s)) = f(\psi_x(t)) = f(x),$$

and therefore f is constant. Hence $L(X, d) \leq L(Y, d')$.

(b) Since $d(\phi(s), \phi(t)) \leq K d'(s, t)$ and ϕ is onto, it follows from (a) that $L(X, d) \leq L(Y, d')$. Since

$$d'(s, t) \leq \frac{1}{M} d(\phi(s), \phi(t))$$

and ϕ is one-to-one, this becomes

$$d'(\phi^{-1}(\phi(s)), \phi^{-1}(\phi(t))) \leq \frac{1}{M} d(\phi(s), \phi(t)) ;$$

therefore $L(Y, d') \leq L(X, d)$. ∎

If in the proposition above, $L(Y, d')$ is infinite, then (a) is trivial, and (b) follows immediately from (a) and the observation that if $L(X, d)$ is finite, then $L(Y, d')$ is also finite. The proposition gains interest if $Y = [0, 1]$ and d' is the absolute-value metric. In this instance, for $x, y \in X$, we define $x \sim y$ if there exists a $\gamma: [0, 1] \to X$ such that $\gamma(0) = x$, $\gamma(1) = y$, and

$$s, t \in [0, 1] \Rightarrow d(\gamma(s), \gamma(t)) \leq K |s - t| .$$

We show that this is an equivalence relation. Reflexivity and symmetry are trivial; now let $x \sim y$, with a map γ and a constant K_1, and let $y \sim z$ with a map ϕ and constant K_2. If $0 \leq s \leq 1/2$, let $\psi(s) = \gamma(2s)$, and if $1/2 < s \leq 1$, let $\psi(s) = \phi(2s - 1)$. Let $K = 2 \max(K_1, K_2)$. If $s, t \in [0, 1/2]$, then

$$d(\psi(s), \psi(t)) = d(\gamma(2s), \gamma(2t)) \leq 2K_1 |s - t| ,$$

and if $s, t \in (1/2, 1]$, then

$$d(\psi(s), \psi(t)) = d(\phi(2s - 1), \phi(2t - 1)) \leq 2K_2 |s - t| .$$

Finally, if $s \in [0, 1/2]$ and $t \in (1/2, 1]$, then

$$\begin{aligned} d(\psi(s), \psi(t)) &\leq d(\psi(s), y) + d(y, \psi(t)) = d(\gamma(2s), \gamma(1)) + d(\phi(1), \phi(2t - 1)) \\ &\leq K_1(1 - 2s) + K_2(2t - 1) \leq \max(K_1, K_2)(1 - 2s + 2t - 1) \\ &= K(t - s) = K|s - t| . \end{aligned}$$

Call the equivalence classes under $\sim$ L-components. Since every L-component has Lipschitz index 1, one might ask whether every arcwise connected set with Lipschitz index 1 is an L-component. This question appears to be fairly difficult.

3. EXTENSION THEOREMS

Suppose (X, d) is a metric space, and $Y \subset X$. Assume $f \in \mathrm{Lip}_\alpha(Y, d)$; the extension problem is to discover whether there exists an $F \in \mathrm{Lip}_\alpha(X, d)$ such that $F | Y = f$. If $\alpha \leq 1$, the extension problem is totally solved by [4, Proposition 1.4], which states that, for each $Y \subset X$ and each $f \in \mathrm{Lip}_\alpha(Y, d)$, there is an $F \in \mathrm{Lip}_\alpha(X, d)$ such that $F | Y = f$. If $\alpha > 1$, the problem is vastly more complex, and three possibilities occur. There are cases where f cannot be extended at all, cases where f can be extended but $\|F\|_\alpha \neq \|f\|_\alpha$, and cases where f can be extended so that $\|F\|_\alpha = \|f\|_\alpha$; this last situation is clearly the most desirable, and by [4] it occurs whenever $\alpha \leq 1$.

The simplest example of the first case is $X = [0, 1]$, with the absolute-value metric, and $Y = \{0, 1\}$. Let $f(0) = 0$, $f(1) = 1$; if $\alpha > 1$, we clearly cannot extend f. This is a rather trivial example; much more indicative of the complexity of the situation is the following proposition.

PROPOSITION 3.1. *Let* $X = \{1/n | n = 1, 2, \cdots\} \cup \{0\}$, *with the absolute-value metric. If* $\alpha > 1$, *then* $\mathrm{Lip}_\alpha(X, d)$ *is regular, and there exist a* $Y \subset X$ *and an* $f \in \mathrm{Lip}_\alpha(Y, d)$ *that cannot be extended to* $F \in \mathrm{Lip}_\alpha(X, d)$.

Proof. If $\alpha \leq 1$, then clearly $\mathrm{Lip}_\alpha(X, d)$ is regular, because the only closed sets in X are finite sets and the union of $\{0\}$ and sequences converging to 0. If $\alpha > 1$, choose an integer N such that $N^{\alpha-1} > 2$. For $k = 1, 2, \cdots$, let $x_k = N^{-k}$, and let $Y = \{x_k | k = 1, 2, \cdots\}$. Let $f(x_1) = 1$, and define f on Y recursively by

$$\frac{f(x_k) - f(x_{k-1})}{|x_k - x_{k-1}|^\alpha} = 2^{-k}.$$

Then, if $k > j$,

$$\frac{f(x_k) - f(x_j)}{|x_j - x_k|^\alpha} \leq \sum_{i=j+1}^{k} \frac{f(x_i) - f(x_{i-1})}{|x_i - x_{i-1}|^\alpha} \leq \sum_{n=2}^{\infty} 2^{-n} < 1 ,$$

and therefore $f \in \mathrm{Lip}_\alpha(Y, d)$.

To extend f, we must define it on all points between x_k and x_{k+1}, say $x_k = y_0 > \cdots > y_{n+1} = x_{k+1}$. Let

$$\Delta_k = \frac{f(x_{k+1}) - f(x_k)}{\sum_{i=1}^{n+1} |y_i - y_{i-1}|^\alpha} .$$

For $1 \leq j \leq n$, define

$$f(y_j) = f(x_k) + \Delta_k \sum_{i=1}^{j} |y_i - y_{i-1}|^\alpha .$$

We shall show that this choice of $f(y_j)$ minimizes

$$\max_{1\leq i\leq n+1} \frac{f(y_i) - f(y_{i-1})}{|y_i - y_{i-1}|^\alpha} .$$

Clearly, for $1 \leq j \leq n$,

$$\frac{f(y_j) - f(y_{j-1})}{|y_j - y_{j-1}|^\alpha} = \Delta_k ,$$

and

$$f(y_{n+1}) - f(y_n) = (f(x_{k+1}) - f(x_k)) - \Delta_k \sum_{i=1}^{n} |y_i - y_{i-1}|^\alpha$$

$$= \Delta_k \sum_{i=1}^{n+1} |y_i - y_{i-1}|^\alpha - \Delta_k \sum_{i=1}^{n} |y_i - y_{i-1}|^\alpha = \Delta_k |y_{n+1} - y_n|^\alpha .$$

Consequently, the definition of $f(y_1), \cdots, f(y_n)$ minimizes

$$\max_{1\leq i\leq n+1} \frac{f(y_i) - f(y_{i-1})}{|y_i - y_{i-1}|^\alpha} .$$

Now

$$\Delta_k = \frac{f(x_{k+1}) - f(x_k)}{|x_{k+1} - x_k|^\alpha} \frac{|x_{k+1} - x_k|^\alpha}{\sum_{i=1}^{n+1} |y_i - y_{i-1}|^\alpha} ,$$

and it remains to examine $\dfrac{|x_{k+1} - x_k|^\alpha}{\sum_{i=1}^{n+1} |y_i - y_{i-1}|^\alpha}$.

Let $n = N^k$, $m = N^{k+1}$. Then our quotient is

$$\frac{\left(\frac{1}{n} - \frac{1}{m}\right)^\alpha}{\sum_{j=0}^{m-n-1} \left(\frac{1}{n+j} - \frac{1}{n+j+1}\right)^\alpha} = \frac{(m - n)^\alpha}{\sum_{j=0}^{m-n-1} \frac{m^\alpha n^\alpha}{(n+j)^\alpha (n+j+1)^\alpha}} \geq \frac{(m - n)^\alpha}{(m - n)\left(\frac{m}{n}\right)^\alpha}$$

$$= \left(\frac{n}{m}\right)^\alpha (m - n)^{\alpha-1} = \left(\frac{1}{N}\right)^\alpha (N^k(N - 1))^{\alpha-1} = \frac{(N - 1)^{\alpha-1}}{N^\alpha} N^{(\alpha-1)k}.$$

Consequently,

$$\Delta_k \geq 2^{-k} \frac{(N - 1)^{\alpha-1}}{N^\alpha} N^{(\alpha-1)k} = \frac{(N - 1)^{\alpha-1}}{N^\alpha} \left(\frac{N^{\alpha-1}}{2}\right)^k ;$$

by the choice of N, $\lim_{k\to\infty} \Delta_k$ is infinite, and therefore f cannot be extended to $\mathrm{Lip}_\alpha(X, d)$. ■

It is clearly hopeless to seek an extension theorem for $\mathrm{Lip}_\alpha(X, d)$, if this algebra is not regular; for let F be closed, $x \notin F$; then, letting $f(x) = 1$ and $f(F) = 0$, we see that $f \in \mathrm{Lip}_\alpha(F \cup \{x\}, d)$. Even when we can extend functions, we may not be able to preserve the norm. Let $X = \{0, 1/2, 1\}$, let $f(0) = 0$, $f(1) = 1$; then $\|f\|_2 = 1$. If $f(1/2) = x$, then to preserve the norm in $\mathrm{Lip}_2(X, d)$ we must have $|x| \leq 1/4$ and $|1 - x| \leq 1/4$, which is clearly impossible.

The following proposition shows that we need concern ourselves only with the extension of functions defined on closed sets.

PROPOSITION 3.2. *Let* Y *be dense in* X, *and let* $f \in \mathrm{Lip}_\alpha(Y, d)$. *Then there exists* $F \in \mathrm{Lip}_\alpha(X, d)$ *such that* $F \mid Y = f$, $\|F\|_\alpha = \|f\|_\alpha$.

Proof. Since f is uniformly continuous on Y, there exists a continuous extension F of f to X. Let $x, y \in X$, and let $\varepsilon > 0$. Choose $\delta > 0$ so that

$$d(x, z) < \delta \Rightarrow |F(x) - F(z)| < \varepsilon/3, \qquad d(y, w) < \delta \Rightarrow |F(y) - F(w)| < \varepsilon/3.$$

Choose $x_0, y_0 \in Y$ so that $d(x, x_0) < \delta$, $d(y, y_0) < \delta$, and

$$d(x_0, y_0)^\alpha < d(x, y)^\alpha + \frac{\varepsilon}{3\|f\|_\alpha}.$$

Then

$$|F(x) - F(y)| \leq |F(x) - F(x_0)| + |F(x_0) - F(y_0)| + |F(y_0) - F(y)|$$
$$< \varepsilon/3 + \|f\|_\alpha d(x_0, y_0)^\alpha + \varepsilon/3 < \varepsilon + \|f\|_\alpha d(x, y)^\alpha,$$

and consequently $|F(x) - F(y)| \leq \|f\|_\alpha d(x, y)^\alpha \Rightarrow \|F\|_\alpha = \|f\|_\alpha$. ■

The next proposition relates the extension of a function in X to the extension of a function in $X \times X \sim \Delta$, where Δ is the diagonal in $X \times X$.

PROPOSITION 3.3. *Let* $f \in \mathrm{Lip}_\alpha(Y, d)$, *and define* g *on* $Y \times Y \sim \Delta$ *by*

$$g(x, y) = \frac{f(x) - f(y)}{d(x, y)^\alpha}.$$

Then f *admits an extension to* X *if and only if* g *admits a bounded extension* G *on* $X \times X \sim \Delta$ *satisfying, for any* $x, y, z \in X$, *the equation*

$$G(x, z)d(x, z)^\alpha = G(x, y)d(x, y)^\alpha + G(y, z)d(y, z)^\alpha.$$

The extension F *of* f *is norm-preserving if and only if*

$$\sup\{|g(x, y)| \mid (x, y) \in Y \times Y \sim \Delta\} = \sup\{|G(x, y)| \mid (x, y) \in X \times X \sim \Delta\}.$$

Proof. If f has an extension $F \in \mathrm{Lip}_\alpha(X, d)$, let

$$G(x, y) = \frac{F(x) - F(y)}{d(x, y)^\alpha} \quad \text{for } (x, y) \in X \times X \sim \Delta.$$

Clearly, G is an extension of g, and it is bounded. Since

$$F(x) - F(y) = G(x, y)\,d(x, y)^\alpha \quad \text{and} \quad F(y) - F(z) = G(y, z)\,d(y, z)^\alpha ,$$

we see (by adding these equations) that

$$F(x) - F(z) = G(x, z)\,d(x, z)^\alpha = G(x, y)\,d(x, y)^\alpha + G(y, z)\,d(y, z)^\alpha .$$

If g admits such a bounded extension G, fix $y_0 \in Y$, for $x \neq y_0$ define $F(x) = f(y_0) + G(x, y_0)\,d(x, y_0)^\alpha$, and let $F(y_0) = f(y_0)$. Now, if $x \in Y$, $x \neq y_0$, then

$$F(x) = f(y_0) + g(x, y_0)\,d(x, y_0)^\alpha = f(y_0) + f(x) - f(y_0) = f(x) .$$

If $(x, y) \in X \times X \sim \Delta$, then

$$F(x) - F(y) = G(x, y_0)\,d(x, y_0)^\alpha - G(y, y_0)\,d(y, y_0)^\alpha = G(x, y)\,d(x, y)^\alpha ,$$

and since G is bounded, $F \in \mathrm{Lip}_\alpha(X, d)$. Since

$$\|F\|_\alpha = \sup\{|G(x, y)| \mid (x, y) \in X \times X \sim \Delta\} ,$$

$$\|f\|_\alpha = \sup\{|g(x, y)| \mid (x, y) \in Y \times Y \sim \Delta\} .$$

the theorem is proved. ∎

We note in passing that we have been concerned only with extending a function so that the extension has a finite α-norm; we can maintain the bound on the function by truncating the extended function at the original bound, by [4, Proposition 1.3]. This will not increase the α-norm of the extended function.

As we have seen, we may not be able to obtain an extension theorem, even if $\mathrm{Lip}_\alpha(X, d)$ is regular. The standard proof of Tietze's extension theorem from Urysohn's lemma [2, p. 61] requires the existence of a uniformly norm-bounded family of functions f_{K_1}, f_{K_2} such that, for each pair of disjoint closed sets K_1 and K_2, $f_{K_1,K_2}(K_i) = i - 1$ $(i = 1, 2)$. Clearly, however, in $\mathrm{Lip}_\alpha(X, d)$ such a family of functions cannot be uniformly bounded, because

$$\|f_{K_1,K_2}\|_\alpha \geq \frac{1}{d(K_1, K_2)^\alpha} .$$

Under certain circumstances, we can extend a function by making it constant on certain sets. The next proposition concerns these trivial extensions for $\alpha > 1$.

PROPOSITION 3.4. (a) *Let* A *and* B *be subsets of* X *such that* $d(A, B) = \delta > 0$, *and let* $f \in \mathrm{Lip}_\alpha(A, d)$. *Then there exist* $F \in \mathrm{Lip}_\alpha(A \cup B, d)$ *such that* $F \mid A = f$, *and if* $\|f\|_\infty \leq \delta^\alpha \|f\|_\alpha$, *then* $\|F\|_\alpha = \|f\|_\alpha$.

(b) *Suppose* A *is a subset of* X, *and* $B \subset X \sim A$. *Let* $f \in \mathrm{Lip}_\alpha(A, d)$, *and assume there exists* $x \in A$ *such that* $d(z, y) \geq d(y, x)$ *for all* $y \in A$ *and* $z \in B$. *Then there exists* $F \in \mathrm{Lip}_\alpha(A \cup B, d)$ *such that* $F \mid A = f$, $\|F\|_\alpha = \|f\|_\alpha$.

Proof. (a) Define $F(B) = c$. If $x \in A$ and $y \in B$, then

$$\frac{|F(x) - F(y)|}{d(x, y)^\alpha} \leq \frac{|f(x) - c|}{\delta^\alpha} \leq \frac{\|f\|_\infty + |c|}{\delta^\alpha} .$$

If $\delta^\alpha \|f\|_\alpha \geq \|f\|_\infty$, choose $|c| \leq \delta^\alpha \|f\|_\alpha - \|f\|_\infty$.

(b) Define $F(B) = f(x)$; then, for $z \in B$ and $y \in A$, we have the inequalities

$$|F(z) - F(y)| \leq |F(z) - f(x)| + |f(x) - f(y)| \leq \|f\|_\alpha d(x, y)^\alpha \leq \|f\|_\alpha d(y, z)^\alpha .$$

Therefore $\|F\|_\alpha = \|f\|_\alpha$. ■

4. BANACH ALGEBRA STRUCTURE OF $\mathrm{Lip}_\alpha(X, d)$

The purpose of this section is to extend some of the results of [4] concerning the structure of $\mathrm{Lip}_\alpha(X, d)$ as a Banach algebra. Since some of the proofs are quite similar to those given in [4], we shall omit them and cite instead the appropriate passages in [4].

DEFINITION 4.1.

$$\mathrm{lip}_\alpha(X, d) = \left\{ f \in \mathrm{Lip}_\alpha(X, d) \,\middle|\, \lim_{x \to y} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} = 0 \right\}.$$

PROPOSITION 4.1. (a) $\mathrm{lip}_\alpha(X, d)$ *is a closed subalgebra of* $\mathrm{Lip}_\alpha(X, d)$.

(b) $\beta > \alpha \Rightarrow \mathrm{Lip}_\beta(X, d) \subset \mathrm{lip}_\alpha(X, d)$.

(c) $\mathrm{Lip}_\alpha(X, d)$ *and* $\mathrm{lip}_\alpha(X, d)$ *are inverse-closed.*

Proof. (a) The usual techniques show that $\mathrm{lip}_\alpha(X, d)$ is a subalgebra. Suppose $\{f_n \mid n = 1, 2, \cdots\} \subset \mathrm{lip}_\alpha(X, d)$ and $f_n \to f$. Let $\varepsilon > 0$, $y \in X$, and choose N and δ so that $\|f - f_N\|_\alpha < \varepsilon/2$ and

$$d(x, y) < \delta \Rightarrow \frac{|f_N(x) - f_N(y)|}{d(x, y)^\alpha} < \varepsilon/2 .$$

Then

$$d(x, y) < \delta \Rightarrow \frac{|f(x) - f(y)|}{d(x, y)^\alpha} \leq \frac{|f_N(x) - f_N(y)|}{d(x, y)^\alpha} + \frac{|(f - f_N)(x) - (f - f_N)(y)|}{d(x, y)^\alpha}$$

$$< \varepsilon/2 + \|f - f_N\|_\alpha < \varepsilon .$$

(b) $f \in \mathrm{Lip}_\beta(X, d)$ implies that

$$\frac{|f(x) - f(y)|}{d(x, y)^\alpha} = \frac{|f(x) - f(y)|}{d(x, y)^\beta} d(x, y)^{\beta-\alpha} ;$$

therefore

$$\lim_{x \to y} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} \leq \|f\|_\beta \lim_{x \to y} d(x, y)^{\beta - \alpha} = 0 ,$$

and $\|f\|_\alpha \leq \max(\|f\|_\beta, 2\|f\|_\infty)$.

(c) See [4, Proposition 1.7]. ■

If $\mathrm{Lip}_\beta(X, d)$ is regular and $\beta > \alpha$, then $\mathrm{Lip}_\beta(X, d) \subset \mathrm{lip}_\alpha(X, d)$, and therefore $\mathrm{lip}_\alpha(X, d)$ is regular.

PROPOSITION 4.2. *Let* $\beta > \alpha$, *and assume* $\mathrm{Lip}_\beta(X, d)$ *is regular. If* I *is a closed ideal of* $\mathrm{lip}_\alpha(X, d)$ *with hull* $K \subset X$, *then* I *consists of all functions in* $\mathrm{lip}_\alpha(X, d)$ *that vanish on* K. *If* (X, d) *is compact, every closed ideal is of this form.*

Proof. See [4, Theorem 4.2 and Corollary 4.3]. ■

Sherbert notes that for $0 < \alpha < 1$ it is much more difficult to obtain the ideal structure of $\mathrm{Lip}_1(X, d)$ than of $\mathrm{lip}_\alpha(X, d)$; it is even more difficult to obtain the ideal structure of $\mathrm{Lip}_\alpha(X, d)$ for $\alpha > 1$, because the lack of extension theorems is a definite handicap. As in [4], M(K) is the set of functions vanishing on K, and J(K) is the closure in $\mathrm{Lip}_\alpha(X, d)$ of all functions that vanish in a neighborhood of K.

PROPOSITION 4.3. *If* K *is a compact subset of* X, *then* $J(K) = \overline{M(K)^2}$ *in* $\mathrm{Lip}_\alpha(X, d)$.

Proof. See [4, Theorem 5.2]. ■

PROPOSITION 4.4. *Let* K *be a compact subset of* X, *and assume we can extend functions in* $\mathrm{Lip}_\alpha(Y, d)$ $(Y \subset X)$ *in a norm-preserving fashion. Then* $f \in \mathrm{Lip}_\alpha(X, d)$ *belongs to* J(K) *if and only if* $f(x) = 0$ *for all* $x \in K$ *and*

$$\frac{|f(x) - f(y)|}{d(x, y)^\alpha} \to 0 \qquad \text{as } (x, y) \to K \times K .$$

Proof. See [4, Theorem 5.1]. ■

For algebras such as we described in the previous proposition, it is still true, if (X, d) is compact, that J(K) is the intersection of the primary components containing it. If $\alpha > 1$, however, it is an open question whether every closed ideal is the intersection of every primary component containing it. Waelbroeck's proof in [5] is not extendable to this case.

The ideal structure of arbitrary $\mathrm{lip}_\alpha(X, d)$ or $\mathrm{Lip}_\alpha(X, d)$ is quite a difficult problem. By factoring out the common zero-set, if necessary, we can assume that these algebras are point-separating; but regularity is another matter. Similarly, the lack of extension theorems for $\mathrm{Lip}_\alpha(X, d)$ constitutes a major block to the discovery of its ideal structure.

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University of California
Los Angeles, California 90024