ON THE SOLUTION OF THE RIEMANN PROBLEM WITH GENERAL STEP DATA FOR AN EXTENDED CLASS OF HYPERBOLIC SYSTEMS

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1. The Riemann problem for a quasi-linear hyperbolic system of equations is a specific initial-value problem, namely

$$U_t + F(U)_x = 0,$$

$$U(0, x) = \begin{cases} U_\ell & (x < 0), \\ U_r & (x > 0). \end{cases}$$

Here F is a vector-valued function of $U = U(t, x) \in E^n$ ($n \geq 2$, $-\infty < x < \infty$, $t \geq 0$), and U_ℓ and U_r are constant vectors. The assumption that the system is hyperbolic means that the matrix $dF(U)$ has real and distinct eigenvalues for all values of the argument U in question. By a solution of this problem we mean a function $U = U(x/t)$ consisting of $n + 1$ constant states separated by shock and rarefaction waves, satisfying the Rankine-Hugoniot condition across shocks, and satisfying the equation in the usual sense at all other points (see [5] for the definitions of these concepts). Such a solution is necessarily a weak solution in the sense of the theory of distributions.

In 1957, P. Lax [5] solved the Riemann problem for the case where U_ℓ and U_r are sufficiently close. In this paper, we shall allow U_ℓ and U_r to be arbitrary constant vectors, but we shall restrict ourselves to the case $n = 2$ for the same systems that were studied in [2]. For these systems we shall show that at each point U_ℓ in the plane there originate four smooth curves that divide the plane into four unbounded regions, with the property that if U_r lies in three of these regions, then the Riemann problem can be solved without any additional assumptions. We shall then show by an example (due essentially to J. L. Johnson [1], who also considered certain special cases of our results) that our assumptions are not strong enough to solve all Riemann problems if U_r lies in the fourth region. However, by putting additional assumptions on F, we can guarantee that all Riemann problems are solvable. Also, by generalizing the notion of solution, we are able to solve all Riemann problems without any additional restriction on F.

2. In the case $n = 2$, we can write our system in the form

(1) $$u_t + f(u, v)_x = 0, \qquad v_t + g(u, v)_x = 0,$$

with initial data

(2) $$(u(0, x), v(0, x)) = \begin{cases} U_\ell = (u_\ell, v_\ell) & (x < 0), \\ U_r = (u_r, v_r) & (x > 0). \end{cases}$$

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We denote by F the smooth mapping (f, g) from E^2 into E^2, and we let ℓ_i and r_i $(i = 1, 2)$, be the left and right eigenvectors of dF, for the eigenvalues λ_i $(\lambda_1 < \lambda_2)$. We shall require that $f_v g_u > 0$ in E^2, and we can thus assume that $f_v < 0$ and $g_u < 0$. (This implies that the system (1) is hyperbolic.) Throughout the paper, we shall assume that the system (1) is genuinely nonlinear and satisfies the shock interaction conditions (see [2]) in E^2. As was shown in [2], these two conditions can be written as $\ell_i d^2 F(r_j, r_j) > 0$ $(i, j = 1, 2)$ under the normalizations $d\lambda_i(r_i) > 0$, $\ell_i r_i > 0$ $(i = 1, 2)$.

Let $U_\ell = (u_\ell, v_\ell)$ be any point in E^2. Under our assumptions, it can be shown, by the methods of [2], that there exist four smooth curves originating at U_ℓ and representing the states that can be connected to U_ℓ by shock waves and rarefaction waves of both characteristic families. These can be written as

$$v = w_1(u; U_\ell), \qquad v = s_2(u; U_\ell) \qquad (u \geq u_\ell),$$

$$v = w_2(u; U_\ell), \qquad v = s_1(u; U_\ell) \qquad (u \leq u_\ell),$$

where the equations $v = s_i(u; U_\ell)$ represent the shock-wave curves and the equations $v = w_i(u; U_\ell)$ represent the rarefaction-wave curves. It was shown in [2] that the curve $v = w_1(u; U_\ell)$ is a smooth, increasing, concave curve (that is, $dw_1/du > 0$ and $d^2w_1/du^2 < 0$), so that the curve is defined for all $u \geq u_\ell$. Similarly, it was also shown in [2] that the curve $v = s_2(u; U_\ell)$ is a decreasing convex curve and is also defined for all $u \geq u_\ell$. Analogously, the curve $u = w_2(u; U_\ell)$ is decreasing and convex, and the curve $v = s_1(u; U_\ell)$ is increasing and concave. We define

$$S_1 = \{(u, v): v = s_1(u; U_\ell),\ u \leq u_\ell\},$$

$$S_2 = \{(u, v): v = s_2(u; U_\ell),\ u \geq u_\ell\},$$

$$W_1 = \{(u, v): v = w_1(u; U_\ell),\ u \geq u_\ell\},$$

$$W_2 = \{(u, v): v = w_2(u; U_\ell),\ u \leq u_\ell\}.$$

These sets divide the uv-plane into four regions illustrated in Figure 1. To solve the Riemann problem in the large means, given $U_\ell = (u_\ell, v_\ell)$, to solve (1) for each point $U_r = (u_r, v_r)$ in E^2. It is somewhat surprising that without any additional assumptions on F, we can already solve the Riemann problem if U_r is in regions I, II, or III.

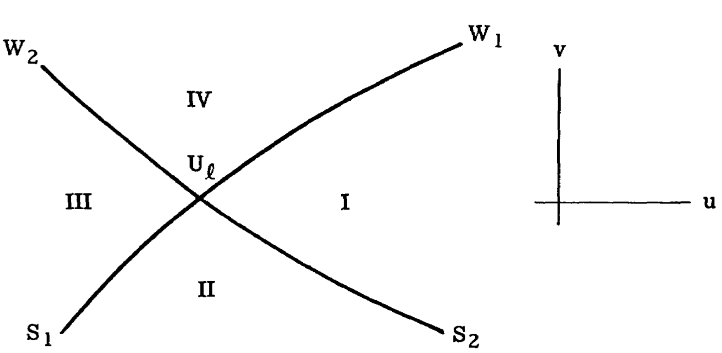

Figure 1.

THEOREM 1. *Let* $U_\ell = (u_\ell, v_\ell)$ *be any point in* E^2. *If* $U_r = (u_r, v_r)$ *is any point in regions* I, II, *or* III, *then the Riemann problem* (1)-(2) *is solvable.*

Proof. It was shown in [2] that if U_r is in region I, then the problem (1)-(2) is solvable. Suppose U_r is in region II. We consider two cases: $u_r > u_\ell$ and $u_r \leq u_\ell$. If $u_r > u_\ell$, let Λ denote the vertical line $u = u_r$, and consider the mapping ϕ from S_1 to Λ given by

$$\phi\colon (u,\ s_1(u;\ U_\ell)) \to (u_r,\ s_2(u_r;\ u,\ s_1(u;\ U_\ell))) .$$

Since the curves $v = s_2(u; U)$ $(U \in S_1)$ have negative slopes, they cut Λ transversally, and thus ϕ is continuous. Now, $\phi(U_\ell)$ is not below U_r on Λ, and furthermore, there exists a $\bar{u}$ such that $s_1(\bar{u}; U_\ell) < v_r$, so that $\phi(\bar{u}, s_1(\bar{u}; U_\ell))$ is below U_r on Λ. Since the portion of S_1 between U_ℓ and $s(\bar{u}; U_\ell)$ is connected, it follows that there exists a point U on S_1 for which $\phi(U) = U_r$. The solution to the Riemann problem is now obvious; namely, we go from the constant state U_ℓ to the constant state U by a 1-shock, and then from U to the constant state U_r by a 2-shock.

If $u_r \leq u_\ell$, we again let Λ denote the line $u = u_r$, which now cuts S_1. Choose $\bar{u}$ so that $s_1(\bar{u}; U_\ell) < v_r$. Then the 2-shock curve originating at $s_1(\bar{u}; U_\ell)$ meets Λ at a point (u, v) $(v < v_r)$. Let K denote the compact region bounded by this curve, by Λ, and by S_1. Since the slopes of the 2-shock curves are continuous, they must be bounded in K. Hence there exists a u' for which the 2-shock curve originating at $(u', s_1(u'; U_r))$ cuts Λ above U_r, and as before, there exists a point U on S_1 for which the 2-shock curve originating at U passes through U_r. The solution of the problem (1)-(2) now follows as in the previous case.

Finally, we consider the case where U_r lies in region III. In this case we solve the initial-value problem

$$\frac{dv}{du} = a_2(u, v), \qquad v(u_r) = v_r,$$

and we consider the solution in $u \geq u_r$. Since $a_2 < 0$ and $(a_2)_u + a_2(a_2)_v > 0$ (see [2]), we see that this solution curve is decreasing and convex, for all $u \geq u_r$. Now the family of curves W_2 is governed by the same differential equation as above, so that by standard theorems in ordinary differential equations, the curve $v = w_2(u; U_r)$ $(u \geq u_r)$ must cut S_1 in a point U. Hence, the problem (1)-(2) is also solvable in this case. The solution consists of going from U_ℓ to U by a 1-shock and from U to U_r by a 2-rarefaction wave. This completes the proof of the theorem.

In view of this theorem, it remains only to consider the case where U_r is in region IV. We shall now show, by means of an example, that our problem (1)-(2) is not always solvable if U_r is in region IV. Consider the system

$$u_t - v_x = 0, \qquad v_t + (e^{-2u}/2)_x = 0. \tag{3}$$

It is easy to verify that this system is hyperbolic and genuinely nonlinear and that it satisfies the shock-interaction conditions in E^2. Let $U_\ell = (0, 0)$, and let $U_r = (0, 2k)$. We assert that for $k \geq 1$, the system (3) with this initial-datum is not solvable. Indeed, the Riemann invariants of the system (3) are given by

$$v \pm \int^u e^{-x}dx,$$

so that the equation for W_1 is

$$v = \int_0^u e^{-x}dx = 1 - e^{-u};$$

similarly, the equation for the 2-rarefaction wave curve through U is given by $v = 2k - 1 + e^{-u}$. Therefore these curves never meet if $k \geq 1$. Thus the Riemann problem is not solvable for all data. Note that if $k < 1$, then these curves do meet and we can therefore solve the Riemann problem.

This example gives the clue to the general case. Namely, we must find conditions on f and g under which none of the curves $v = w_1(u; U_\ell)$ $(u \geq u_\ell)$ has a horizontal asymptote. Of course, it is easy to see that we could equally well solve the Riemann problem if none of the curves $v = w_2(u; U_\ell)$ $(u \geq u_\ell)$ has a horizontal asymptote. Note that in a situation where both families of curves have horizontal asymptotes (as in the above example), we can solve certain Riemann problems where U_r is in region IV. Namely, it is sufficient that the points U_ℓ and U_r be sufficiently close so that the curve $v = w_1(u; U_\ell)$ $(u \geq u_\ell)$ meets $v = w_2(u; U_r)$ $(u \geq u_r)$.

THEOREM 2. *The Riemann problem* (1)-(2) *is solvable in the large for all* U_ℓ *and* U_r *if and only if for each triple of numbers* u_0, v_1, v_2 $(v_2 \geq v_1)$ *the curves* $v = w_1(u; u_0, v_1)$ *and* $v = w_2(u; u_0, v_2)$ *intersect.*

Proof. Suppose that the condition is satisfied, and let U_ℓ and U_r be any two points in E^2. In view of Theorem 1, it suffices to consider the case where U_r is in region IV. Suppose first that $u_r > u_\ell$, and consider the curves

$$v = w_2(u; U_r) \quad (u \geq u_r) \qquad \text{and} \qquad v = w_1(u; u_r, w_1(u_r; U_\ell)) \quad (u \geq u_r).$$

By hypothesis, these curves meet at a point $U = (\bar{u}, \bar{v})$ with $\bar{u} \geq u_r$, so that we can solve the Riemann problem by going from the constant state U_ℓ to the constant state U by a 1-rarefaction wave and then going from U to the constant state U_ℓ by a 2-rarefaction wave. If $u_r \leq u_\ell$, then the curve $v = w_2(u; u_\ell, v_r)$ $(u \geq u_\ell)$ meets W_1; hence the curve $v = W_2(u; U_r)$ $(u \geq u_r)$ also meets W_1. Therefore the Riemann problem is solvable also in this case, so that the condition is sufficient. The necessity is obvious.

The next theorem gives a sufficient condition for the solvability of the Riemann problem.

THEOREM 3. (a) *If* $U_\ell = (u_\ell, v_\ell)$ *is a point in* E^2 *and the curve* $v = w_1(u; U_\ell)$ $(u \geq u_\ell)$ *is not asymptotic to any line* $v = \text{const.}$, *then the problem* (1)-(2) *is solvable for each* $U_r \in E^2$.

(b) *Suppose that* $U_\ell = (u_\ell, v_\ell)$ *is a point in* E^2, *that* $U_r = (u_r, u_r)$ *is a point in* E^2, *and that the curve* $v = w_2(u; U_r)$ $(u \geq u_\ell)$ *is not asymptotic to any line* $v = \text{const.}$ *Then the problem* (1)-(2) *is solvable.*

Proof. We shall first prove (a). Let $U_r = (u_r, v_r)$ be any point in E^2. Again using Theorem 1, we need only consider the case where U_r is in region IV. Let Λ be the line $v = v_r$, which by hypothesis intersects W_1. It is clear that Λ also intersects W_2. Since the region enclosed by Λ, W_1, and W_2 is compact, and since the curves $v = w_2(u; U')$ $(u \leq u')$ with $U' = (u', v')$ on W_1 have negative slopes, an argument similar to that in the proof of Theorem 1 shows that there exists a point U on W_1 for which $v_r = w_2(u_r; U)$. The solution of the Riemann problem now follows as before. To prove (b), it again suffices to consider the case when U_r is in region IV. Then, by hypothesis, the curve $v = w_2(u; U_r)$ $(u \geq u_r)$ meets W_1, and hence the Riemann problem is again solvable. This completes the proof of the theorem.

Let R and S be the classical Riemann-invariants of the system (1), (see [2]). It is well-known that since

$$\begin{pmatrix} R_u & R_v \\ S_u & S_v \end{pmatrix} = \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix},$$

the mapping $(u, v) \to (R, S) = (R(u, v), S(u, v))$ is locally one-to-one. However, in view of the monotonicity of the curves W_1 and W_2 (where W_1 and W_2 are represented by $R = \text{const.}$ and by $S = \text{const.}$), we can easily see that this mapping is also globally one-to-one. It is interesting to note that a necessary and sufficient condition for the solvability of all Riemann problems (1)-(2) is that this mapping is also onto.

THEOREM 4. *The Riemann problem* (1)-(2) *is solvable for every pair* U_ℓ, U_r *if and only if the mapping* $(u, v) \to (R, S)$ *is onto.*

Proof. Suppose the mapping is onto. As before, it suffices to consider the case where U_r is in region IV. Then U_r lies on a curve $S = S_0$, U_ℓ lies on a curve $R = R_0$, and there is a point (u_0, v_0) such that $R_0 = R(u_0, v_0)$ and $S_0 = S(u_0, v_0)$. Thus the curves $R = R_0$ and $S = S_0$ meet at (u_0, v_0), and the Riemann problem is solvable. Conversely, if all Riemann problems are solvable, let (R_0, S_0) be an arbitrary point. The curves $R = R_0$ and $S = S_0$ meet at a point (u_0, v_0) (otherwise not all Riemann problems would be solvable), so that $R_0 = R(u_0, v_0)$ and $S_0 = S(u_0, v_0)$, and the mapping is onto.

We shall now give some precise and easily verified conditions under which the Riemann problem (1)-(2) is always solvable. We recall that the differential equations of W_1 and W_2 are

$$\frac{dv}{du} = a_1(u, v) \qquad \text{and} \qquad \frac{dv}{du} = a_2(u, v),$$

where $a_1 > 0$, $a_2 < 0$, $(a_1)_u + a_1(a_1)_v < 0$, and $(a_2)_u + a_2(a_2)_v > 0$.

THEOREM 5. *The Riemann problem* (1)-(2) *is solvable for all* U_ℓ *and* U_r *if one of the following six conditions is satisfied.*

(a) *There exists a* u_0 *such that either*

$$\int_{u_0}^{\infty} a_1(x, \phi(x))\, dx = +\infty \tag{4}$$

for every smooth $\phi(x)$ *with* $\phi' > 0$ *and* $\phi'' < 0$, *or else*

$$\int_{u_0}^{\infty} a_2(x, \psi(x))\, dx = -\infty \tag{4'}$$

for every smooth $\psi(x)$ *with* $\psi' < 0$ *and* $\psi'' > 0$.

(b) *There exists a* u_0 *such that either*

$$\int_{u_0}^{\infty} a_1(x, c)\, dx = +\infty \qquad \textit{for every } c \tag{5}$$

and $(a_1)_v \le 0$, *or else*

$$\int_{u_0}^{\infty} a_2(x, c)\, dx = -\infty \qquad \textit{for every } c$$

and $(a_2)_v \le 0$.

(c) *Either* $(a_1)_u \geq 0$, *or* $(a_2)_u \leq 0$.

(d) *Either condition* (4) *holds, and* $\left|\iint_D (a_1)_v \, du\, dv\right| < \infty$ *for every region* D *bounded by any line* v = const. *and any increasing concave curve asymptotic to that line, or else* (4') *holds, and* $\left|\iint_{D'} (a_2)_v \, du\, dv\right| < \infty$ *for every region* D' *bounded by any line* v = const. *and any decreasing convex curve asymptotic to that line.*

(e) *Either* $a_1(u, v) \geq \alpha(|u|)\beta(v)$, *where* $\beta(v)$ *is continuous and nonnegative and* $\alpha(r) > 0$ *for* $r > 0$, *and there exists a* u_0 *such that*

$$\int_{u_0}^{\infty} \alpha(r)\, dr = +\infty,$$

or else $-a_2(u, v) \geq \alpha(|u|)\beta(v)$, *where* α *and* β *satisfy the same hypotheses as above.*

(f) *Suppose that* R *and* S *are the Riemann invariants of the system* (1), *where* R = const. *represents the curves* W_1 *and* S = const. *represents the curves* W_2, *and that* $u = \phi(R, S)$, $v = \psi(R, S)$; *then for all* R_0 *and* S_0 *either*

$$\int_{S_0}^{\infty} a_1(\phi(R_0, S), \psi(R_0, S))\, dS = +\infty, \quad \textit{or else} \quad \int_{R_0}^{\infty} a_2(\phi(R, S_0), \psi(R, S_0))\, dR = -\infty .$$

Proof. We use Theorem 3, and therefore we need only show that we can rule out horizontal asymptotes. We shall only prove the first half of each condition. It will follow that the curves W_1 have no horizontal asymptotes. The other conditions imply that the curves W_2 have no horizontal asymptotes, and the proofs are similar.

First note that we can write the equation for W_1 as

$$v = v_\ell + \int_{u_\ell}^{u} a_1(x, w_1(x; U_\ell))\, dx \qquad (u \geq u_\ell),$$

so that W_1 has no horizontal asymptote if

$$\int_{u_\ell}^{\infty} a_1(x, w_1(x; U_\ell))\, dx = +\infty .$$

In terms of the Riemann invariants R and S,

$$\int_{S_0}^{\infty} a_1(\phi(R_0, S), \psi(R_0, S))\, dS = +\infty, \tag{6}$$

where $u = \phi(R, S)$, $v = \psi(R, S)$, and this proves (f).

Since W_1 is increasing and concave, we see that (6) holds if (4) holds. Now, if (5) holds and $(a_1)_v \leq 0$, then W_1 has no horizontal asymptote; for if $v = w_1(u; U_\ell)$ is asymptotic to $v = c$, say, then $c \geq w_1(u; U_\ell)$, so that

$$+\infty = \int_{u_\ell}^{\infty} a_1(x, c)\,dx \leq \int_{u_\ell}^{\infty} a_1(x, w_1(x; U_\ell))\,dx,$$

and this is a contradiction. This proves (b). Note that (b) is unnecessarily strong. Given any c, we need only the condition $(a_1)_v \leq 0$ for each sequence of disjoint intervals $\{I_n\}$ tending to $+\infty$ for which $\sum_n \int_{I_n} a_1(x, c)\,dx = +\infty$.

To see that (c) is sufficient, note that $(a_1)_v \leq 0$, since $(a_1)_u + a_1(a_1)_v < 0$ and $a_1 > 0$. Furthermore, if c is any real number and $h(u) = a_1(u, c)$, then $h > 0$ and $h' \geq 0$, so that also (5) holds. (The condition (c) does not hold for the test system $u_t - v_x = 0$, $v_t + g(u)_x = 0$, where $g' < 0$ and $g'' > 0$; nor does it hold for the systems $u_t + f(v)_x = 0$, $v_t + g(u)_x = 0$ considered in [3].)

Next, using line integrals and Green's theorem, we see that we can replace the condition $(a_1)_v \leq 0$ by the double-integral condition in (d). Finally, the sufficiency of (e) follows from well-known results in ordinary differential equations; see for example [6, pp. 5-6]. The proof of the theorem is complete.

3. In this section we shall first make some additional remarks on the solution of Riemann problems in E^2 and then sketch some results for some other open sets.

Suppose that we are in the situation of the example of Section 2; that is, suppose the curves

$$v = w_1(u; U_\ell) \quad (u \geq u_\ell) \qquad \text{and} \qquad v = w_2(u; U_r) \quad (u \geq u_r)$$

do not intersect. We can write these curves in terms of the Riemann invariants R and S as $R = R_\ell$ and $S = S_r$. The fact that these curves do not meet implies that along $R = R_\ell$, S has a limiting value S_∞ and along $S = S_r$, R has a limiting value $S = S_\infty$. This is illustrated in Figure 2. We shall show that $\lambda_1(+\infty, v_\ell + \alpha)$ and

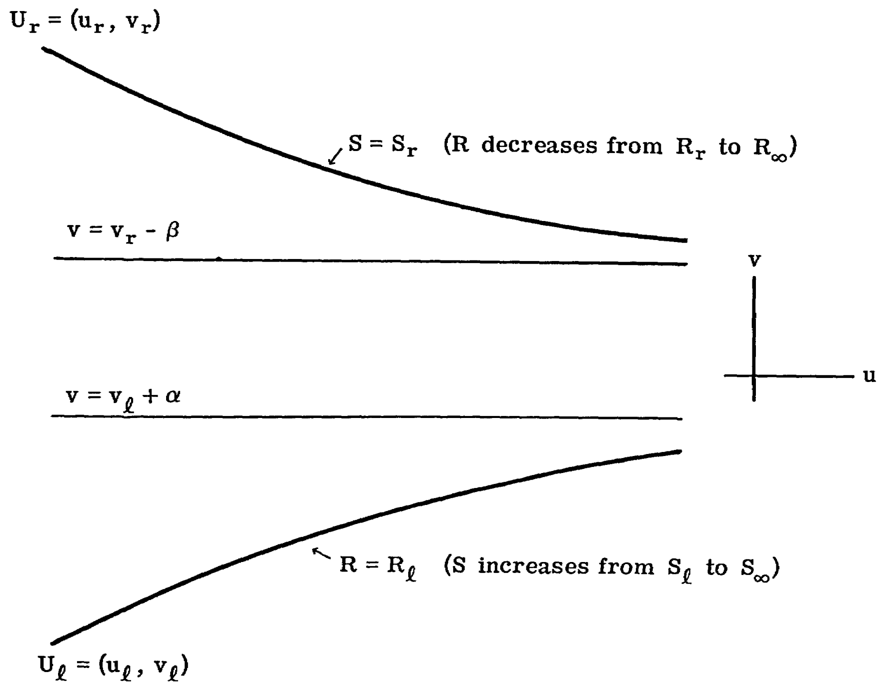

Figure 2.

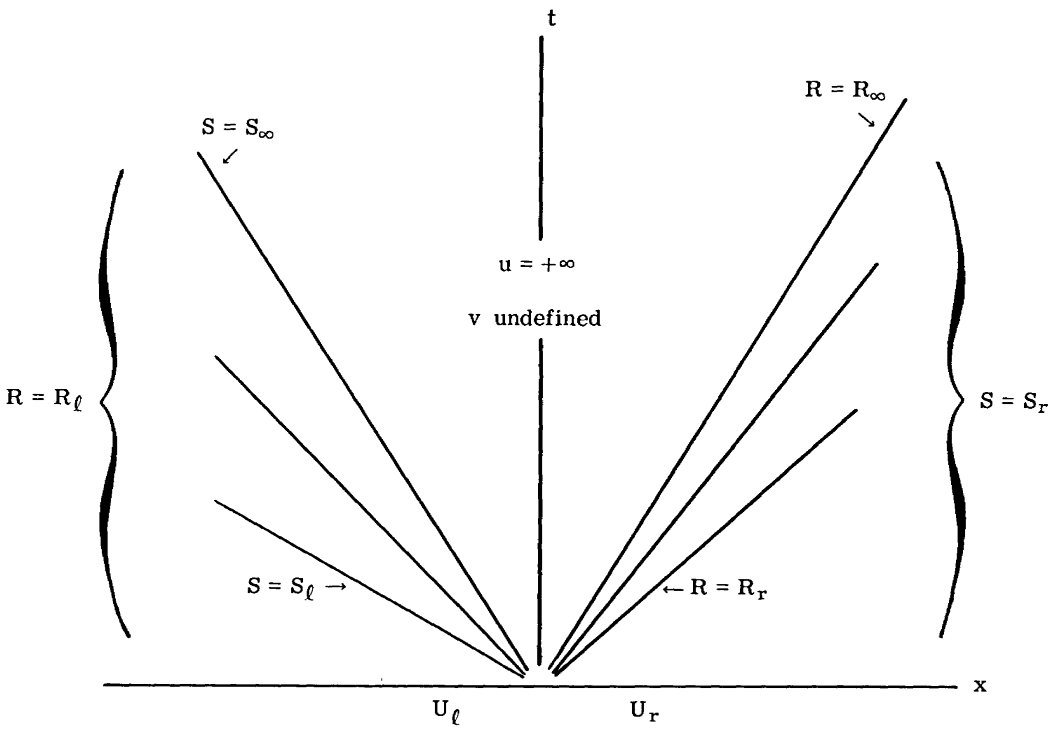

Figure 3.

$\lambda_2(+\infty, v_r - \beta)$ are both finite. First, to show that $\lambda_1(+\infty, v_\ell + \alpha)$ is finite, choose points P_n on $R = R_\ell$ such that $P_n \to \infty$. Then $v = w_1(u; P_n)$ meets $S = S_r$ at points Q_n $(Q_n \to \infty)$. At each Q_n, construct the curve $v = s_1(u; Q_n)$; it meets $R = R_\ell$ at points $\overline{P}_n$ $(\overline{P}_n \to \infty)$. Since λ_2 decreases along $S = S_r$ (as u increases), we have the inequalities

$$\lambda_2(U_r) > \lambda_2(Q_n) > \lambda_1(Q_n) > \lambda_1(\overline{P}_n).$$

Since λ_1 increases along $R = R_\ell$, this shows that λ_1 is bounded along this curve. The proof that λ_2 is bounded along $S = S_r$ is somewhat easier. For any point P on $S = S_r$, construct the curve $v = s_2(u; P)$; it meets the curve $R = R_\ell$ at the point Q. Since

$$\lambda_2(P) > \lambda_2(Q) > \lambda_1(Q) > \lambda_1(U_\ell),$$

we see that λ_2 is bounded from below along $S = S_r$. But λ_2 decreases along $S = S_r$, as u increases, so that λ_2 is bounded along this curve.

Now we can "solve" the Riemann problem as follows. We connect U_ℓ to a complete 1-rarefaction wave $R = R_\ell$, S varying from S_ℓ to S_∞, and we connect U_r (on the left) to a complete 2-rarefaction wave $S = S_r$, R varying from R_∞ to R_r. These two rarefaction waves cannot have any points in common, for otherwise there would be a point on $S = S_r$ and $R = R_\ell$, contrary to our assumption. Thus there is a "void" region where we make $u = +\infty$ and leave v undefined. Physically speaking, if we are dealing with the equations of gas dynamics with one space-variable and with constant entropy, and if u represents the specific volume (the reciprocal of the density) and v is the velocity, this region could be regarded as a cavity, that is, as a region where there is no gas (see [4, p. 364], for example). In the particular case of a shock tube where both gases are moving in the same direction, say to the left, this situation is realized if the right boundary of the gas on the left moves at a greater velocity than the left boundary of the gas on the right. The situation is illustrated in

Figure 3. It appears plausible that one can change variables so as to avoid the situation where $u = +\infty$ and v is undefined.

Next we consider similar questions on the existence of the solutions to the Riemann problem in other open sets. We shall here only sketch the results for the half-space $u > 0$. Suppose that all of our hypotheses are satisfied on the half-plane $u > 0$, in other words, that

$$f_v g_u > 0 \quad \text{and} \quad \ell_i d^2 F(r_j, r_j) > 0 \quad (i, j = 1, 2;\ u > 0).$$

To eliminate possible difficulties in region IV, we must put an additional restriction on the system. For example, we can require that one of the conditions of Theorem 5 holds.

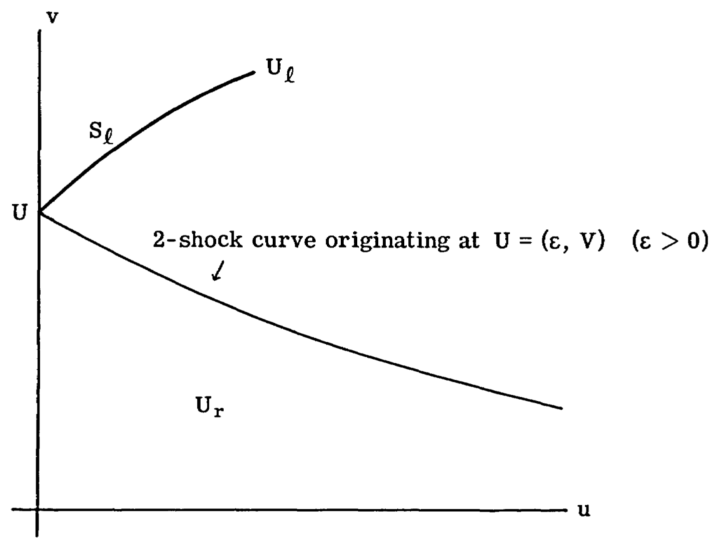

Figure 4.

Figure 4 illustrates still another difficulty that can arise. Here U_r is in region II, and the 2-shock curves originating on S_ℓ (in $u > 0$) do not pass through U_r if $|U_\ell - U_r|$ is sufficiently large. We can rule this out by requiring that $v \to -\infty$ as $u \to 0$ along the S_ℓ-curves. Since the shock curves satisfy the equation

$$(u - u_\ell)(g(u, v) - g(u_\ell, v_\ell)) = (v - v_\ell)(f(u, v) - f(u_\ell, v_\ell)),$$

it is easy to see that a sufficient condition for this is that $\lim_{u \to 0} g(u, v) = \pm\infty$ for every v.

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