ON 3-MANIFOLDS THAT COVER THEMSELVES

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1. Let M be a compact, connected 3-manifold. We say that M *covers itself* if there is a nontrivial covering projection p: $M \to M$. We classify all nonprime 3-manifolds with this property, and we show that certain prime 3-manifolds fiber over the circle in the sense of Stallings [12].

This work was suggested by Kwun [6], who considered the class of closed, connected, orientable 3-manifolds (without boundary) that double-cover themselves. Kwun succeeded in classifying all nonprime manifolds in this class, and he showed that under certain technical restrictions the prime manifolds fiber over the circle. His results are special cases of our Theorems 1 and 3.

2. Recall that a closed 3-manifold is *prime* if it is not the connected sum of two 3-manifolds each of which is different from S^3. A compact 3-manifold with connected boundary is *prime* if it is not the disk sum of two 3-manifolds, each different from the closed 3-cell.

Milnor [9] has shown that every closed, orientable 3-manifold M is homeomorphic to a sum $P_1 \# P_2 \# \cdots \# P_n$ of prime manifolds, where the summands P_i are uniquely determined up to order and homeomorphism. (J. L. Gross [1], [2] has obtained a result analogous to Milnor's in the case where M is an orientable 3-manifold with nonvoid, connected boundary.) Raymond [11] observed that Kneser [5] actually "proved," modulo the validity of Dehn's lemma, a unique decomposition theorem for closed 3-manifolds, orientable or not. Kneser's theorem states that every closed 3-manifold can be written uniquely in normal form as the sum of prime manifolds ("in normal form" means that the number of nonorientable handles N in the sum is minimal).

Milnor proved that, with the exception of S^3 and $S^1 \times S^2$, an orientable, closed 3-manifold is prime if and only if it is irreducible. Raymond also observed that Milnor's proof extends to the nonorientable case if N is excluded.

In the next section, we classify closed, nonprime 3-manifolds that cover themselves.

3. Let P_3 denote real, projective 3-space, and consider the manifold $P_3 \# P_3$ (orientation need not be specified, because P_3 admits an orientation-reversing homeomorphism). For every integer $k > 0$, $P_3 \# P_3$ admits a k-sheeted covering by itself. This is exceptional behavior for a nonprime manifold, as the following theorem shows.

THEOREM 1. *A closed, connected, nonprime 3-manifold* M *covers itself if and only if* $M \approx P_3 \# P_3$.

First we indicate the unique manner in which $P_3 \# P_3$ covers itself, and then we devote the remainder of this section to three lemmas that complete the proof of

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Theorem 1. $P_3 \# P_3$ is homeomorphic to the sum $P(k) = P_3 \# S^3 \# \cdots \# S^3 \# P_3$, where S^3 occurs $k - 1$ times as a summand. Let $p: P(k) \to P_3 \# P_3$ be the k-sheeted covering projection by which the sphere-summands S^3 of P(k) alternately double-cover the P_3 summands of the base space, in which the first P_3 of P(k) covers the left half of $P_3 \# P_3$, and in which the last P_3 of P(k) covers the left (right) half of $P_3 \# P_3$ if k is even (odd). It will follow from the following arguments that this is essentially the only way $P_3 \# P_3$ can cover itself.

We adopt the following notational conventions. Let $H_0 = \{S^3\}$, and let H_1 denote the collection of nontrivial, prime, closed 3-manifolds. For $n \geq 2$, let H_n denote the collection of closed 3-manifolds that are homeomorphic to connected sums of exactly n elements of H_1.

LEMMA 1. *If* $M \in H_2$, *then* M *covers itself* k *times* $(k \geq 2)$ *if and only if* $M \approx P_3 \# P_3$.

Proof. Suppose $p: M \to M$ is a k-sheeted covering projection. Let $M \approx A \# B$ $(A, B \in H_1)$. Write $A \# B = A' \cup B'$, where A' (B') is obtained from A (from B) by deletion of a tame open 3-cell, so that $A' \cap B' = S$ is a 2-sphere. Then $p^{-1}(S)$ is a disjoint collection of 2-spheres $\{S_i\}_{i=1}^{k}$.

Case 1. Suppose each S_i separates $A \# B$ $(i = 1, \cdots, k)$. Then $M - p^{-1}(S)$ has $k + 1$ components, and the closure of each component covers either A' or B'. The closures of at least two components, say U_1 and U_2, have connected boundaries S_1 and S_2. Let C (D) be obtained from Cl U_1 (from Cl U_2) by sewing a 3-cell along S_1 (along S_2). Then C and D each cover either A or B exactly once, and therefore C and D belong to H_1. Since $M \in H_2$, we have the homeomorphism $A \# B \approx C \# D$, and each of the remaining components must be homeomorphic to S^3 minus two tame open 3-cells. An analysis of the situation reveals that either both A and B are double-covered by S^3, or $A \approx B$ and one of them is double-covered by S^3. Livesay [8] has shown that the orbit space of the action on S^3 by a free involution must be P_3. This proves that $A \approx B \approx P_3$.

Case 2. Suppose $(A \# B) - S_i$ is connected for some i, say $i = 1$. Since $(A \# B) - S_1$ is connected, either A or B must be a handle, that is, it must be homeomorphic to either $S^1 \times S^2$ or N. Suppose B is a handle. Again, the closure of each component of $M - p^{-1}(S)$ covers either A' or B'. If the closure of a component U covers B', then the space consisting of Cl U plus some 3-cells sewn along its boundary components is homeomorphic to a handle. This is so because $\pi_1(B') \cong Z$ has exactly one subgroup of a given finite index, and hence B' has a unique, connected, m-sheeted covering, for each m.

It follows that the closure of only one component of $M - p^{-1}(S)$ covers B'. Otherwise we should have at least three handles upstairs, in violation of M being in H_2. Therefore only Cl U covers B', and hence this covering is k-to-1. The boundary of Cl U must be $p^{-1}(S)$. Since $M - S_1$ is connected, there must be another handle upstairs, in addition to Cl U with k 3-cells attached. Thus both A and B must be handles. The closure of some component, say Cl V, covers A' k times and is such that Cl U $\cap$ Cl V $= p^{-1}(S)$. But this implies that there are $k - 2$ handles upstairs, in addition to Cl U and Cl V (with appropriate 3-cells attached). Since $M \in H_2$, it follows that $k = 2$; but we shall now show that this is impossible (our argument is essentially due to Kwun [6, Case 2 of Proposition 3.1]).

Because neither S_1 nor S_2 separates $A \# B$, the space $A \# B - (S_1 \cup S_2)$ has two components P and Q. Since $p(P \cup Q) = (A \# B) - S$ is disconnected, $p(P) = A' - B'$ and $p(Q) = B' - A'$ (by proper choice of labeling). Let C and D be closed manifolds obtained from Cl P and Cl Q, respectively, by attaching 3-cells. Then

$$A \# B \approx C \# (S^1 \times S^2) \# D \quad \text{or} \quad A \# B \approx C \# N \# D.$$

By the uniqueness of the decomposition, either A or B (say A) is a handle. But this implies that C is a handle, which in turn implies that both C and D are handles. Hence the connected sum of three handles would be homeomorphic to the connected sum of two handles. This contradiction rules out Case 2, completing the proof of the lemma.

LEMMA 2. *Suppose that* $M_1 \in H_m$ *and* $M_2 \in H_n$, *and that* M_1 *contains no handles. If* $m \geq 1$ *and there exists a nontrivial* k-*to*-1 *covering projection* $p: M_1 \to M_2$, *then* $m \geq n$.

Proof. We use induction on m. Let $m = 1$. Write $M_2 \approx A \# B$ ($A \in H_1$ and $B \in H_{n-1}$). As usual, we write $A \# B = A' \cup B'$, where $A' \cap B' = S$ is a 2-sphere. Since M_1 has no handles, each component S_i of $p^{-1}(S) = S_1 \cup \cdots \cup S_k$ must separate M_1. There are at least two components of $M_1 - p^{-1}(S)$ such that the closure of each covers either A' or B' exactly once. But unless $n = 1$, this is impossible, since M_1 is prime.

We now show that if the lemma is true for all $m \leq q$, it also holds for $m = q + 1$. Suppose $M_1 \in H_{q+1}$ and $M_2 \in H_n$. We assume that $n > q + 1$, and we show this leads to a contradiction. Write $M_2 \approx A \# B = A' \cup B'$, where $A \in H_1$, $B \in H_{n-1}$, and $A' \cap B' = S$ is a 2-sphere. Again, every component S_i of $p^{-1}(S)$ separates M_1. Hence there must be at least two components of $M_1 - p^{-1}(S)$ such that the closure of each gives a one-to-one covering either of A' or of B'. Let C' be a component of $p^{-1}(B')$, and let C be the manifold obtained by capping the 2-sphere boundary components of C' with 3-cells. It is easy to see that $C \in H_j$ ($1 \leq j \leq q$). However, p induces a covering of B by C, which violates our induction hypothesis, since $j < n$. This completes the proof.

LEMMA 3. *If* $M \in H_n$ ($n > 2$), *then* M *does not cover itself* k *times for any* $k \geq 2$.

Proof. Let $M \approx A_1 \# A_2 \# \cdots \# A_n$ ($A_i \in H_1$ for $1 \leq i \leq n$).

Case 1. Suppose at least one A_i is either $S^1 \times S^2$ or N. Suppose the indices are chosen so that the A_i are handles for $1 \leq i \leq m$, but not handles for $i > m$. Write $M = A_1' \cup A_2' \cup \cdots \cup A_n'$, where A_1' and A_n' are obtained from A_1 and A_n, respectively, by deletion of a tame open 3-cell, and where A_i' ($1 < i < n$) is obtained from A_i by deletion of tame open 3-cells. Since M is connected, $A_i' \cap A_{i+1}'$ is a 2-sphere S_i. A connected, t-sheeted covering of A_i' ($1 < i < m$ or $i = m$ if $m \neq 1, n$) must be a handle minus 2t open 3-cells (or t 3-cells if $i = 1$ or $i = m = n$). Hence each component of $M - p^{-1}(S)$ that covers A_i' ($i \leq m$) is of this form. To cover $A_1 \# A_2 \# \cdots \# A_m$ k times, we require at least $k(m - 1)$ handles. But M has exactly m handles, and the inequality $k(m - 1) \leq m$ holds only when $k = m = 2$. An analysis of this special situation reveals that it takes more than two handles upstairs for double-covering M. Since M has only two handles, such an M cannot double-cover itself. This rules out Case 1.

Case 2. Suppose no A_i is a handle. Write $M \approx A \# B$, where $A = A_1$ and $B = A_2 \# \cdots \# A_n$. As before, let $A \# B = A' \cup B'$, where $A' \cap B'$ is a 2-sphere S. Since M contains no handles, each component of $p^{-1}(S)$ must separate M. Hence the closures of at least two components of $M - p^{-1}(S)$ must have connected 2-sphere boundaries and provide one-to-one coverings. Call two of these components U and V. Then $\mathrm{Cl}\, U \approx \mathrm{Cl}\, V \approx A'$, for otherwise we would have too many summands upstairs (at least $2(n - 1) > n$). Let W' be a component of $\mathrm{Cl}(p^{-1}(B'))$. If we cap the 2-sphere boundary components of W' and B' with 3-cells, we get the closed

3-manifolds W and B ($W \in H_q$ and $B \in H_{n-1}$, where $q < n - 1$). But $p \mid W'$ induces a covering of B by W, which is impossible, in view of Lemma 2. This rules out Case 2. Thus we have completed the proof of this lemma and consequently the proof of Theorem 1.

Remark. In the proof of Lemma 2, we showed that a nonprime 3-manifold cannot be covered by an irreducible, closed 3-manifold. However, a covering by a prime 3-manifold is possible (for example, $S^1 \times S^2$ double-covers $P_3 \# P_3$). It remains an open question whether a closed covering space of a prime, closed 3-manifold must be prime.

4. THEOREM 2. *If a compact, connected 3-manifold M with connected boundary covers itself, then M is a prime, irreducible 3-manifold, and Bd M is either a torus or a Klein bottle.*

Proof. Suppose $p: M \to M$ is a k-to-1 covering projection ($k \geq 2$). Then $p \mid \text{Bd } M: \text{Bd } M \to \text{Bd } M$ is also a k-to-1 covering projection. Let $\chi(\text{Bd } M)$ denote the Euler characteristic of Bd M. The relation $\chi(\text{Bd } M) = k\chi(\text{Bd } M)$ [1, p. 277] implies that $\chi(\text{Bd } M) = 0$. Hence $\text{Bd } M \approx S^1 \times S^1$ or $\text{Bd } M \approx K$ (the Klein bottle).

Now suppose that M is not prime. Then M is homeomorphic to the disk sum of two 3-manifolds A and B, neither of which is a 3-cell. With proper choice of notation, we may suppose that $\text{Bd } A \approx \text{Bd } M$ and $\text{Bd } B \approx S^2$. Consider 2M, the *double* of M, obtained by sewing two copies of M together along their boundaries by the identity map. It is clear that $2M \approx 2A \# 2B$, where 2A and 2B are nontrivial. The projection p induces a k-to-1 covering of 2M by itself. By Theorem 1, $2M \approx P_3 \# P_3$. By the uniqueness of the connected-sum decomposition, $2B \approx P_3$. But this implies that $Z_2 \cong \pi_1(B) * \pi_1(B)$, where $*$ denotes the free product. This is a contradiction, and therefore M must be prime.

We want to show that M is also irreducible. Since 2M covers itself, there are only three cases to consider, namely $2M \approx P_3 \# P_3$, 2M irreducible, and 2M homeomorphic to a handle. For clarity of notation, we suppose that $2M = M \cup M'$, where $M = M'$ and $M \cap M' = \text{Bd } M = \text{Bd } M'$.

First suppose that $2M \approx P_3 \# P_3$. Let $S \subset \text{Int } M \subset 2M$ be a tamely embedded 2-sphere. If S does not bound a 3-cell in M, then S must bound P_3 less a tame, open 3-cell. By the symmetry of 2M, a corresponding tame 2-sphere S' in Int M' also bounds a copy of P_3 less a tame, open 3-cell. If we let $\overline{M}$ and $\overline{M}'$ denote M and M', respectively, with the P_3's removed and replaced by 3-cells, we find that $2\overline{M} \approx S^3$. Since there is a retraction of $2\overline{M}$ onto $\overline{M}$, $\pi_1(\overline{M}) \cong 0$. It follows that $\pi_1(M) \cong Z_2$. But this is impossible ($\pi_1(M)$ contains subgroups of index k^n for all $n \geq 1$, since M covers itself k times). Hence, if this case occurs, every 2-sphere tamely embedded in M bounds a 3-cell.

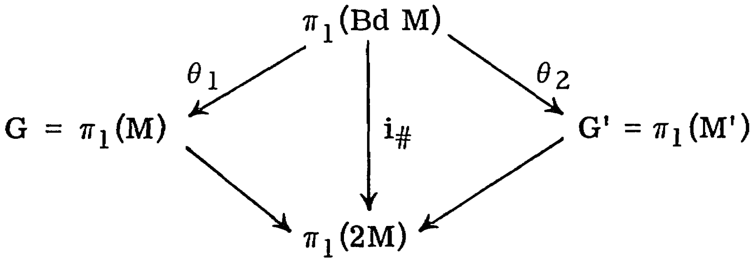
Now suppose 2M is irreducible. Let $S \subset \text{Int } M \subset 2M$ be a tamely embedded 2-sphere. Let $2M - S = A \cup B$, and suppose that $A \subset \text{Int } M$. Then Cl A must be a 3-cell. For if A were not a 3-cell, Cl B would have to be. But there is a 2-sphere S' in Int M' corresponding to S. We have the inclusions $S' \subset \text{Int } M' \subset \text{Cl } B$, and S' bounds a homeomorphic copy of A in Cl B. This is a contradiction, since Cl B would be a 2-cell and would therefore be irreducible. Hence, if 2M is irreducible, then M is irreducible.

The next lemma takes care of the last case, where 2M is a handle.

LEMMA 4. *Let M be a compact 3-manifold with connected boundary. If $2M \approx S^1 \times S^2$ or $2M \approx N$, then M is irreducible.*

Proof. Let $S \subset \text{Int } M$ be a tamely embedded 2-sphere. If S separates 2M, then S must bound a 3-cell in 2M, and this 3-cell lies in Int M. Hence it is sufficient to show that every 2-sphere tamely embedded in $\text{Int } M \subset 2M$ must separate 2M.

Let i: Bd M $\to$ 2M be the inclusion map. Suppose that $S \subset \text{Int } M \subset 2M - \text{Bd } M$ is a nonseparating, tame 2-sphere. Then $i(\text{Bd } M) \subset 2M - S$, and the induced map $i_\#: \pi_1(\text{Bd } M) \to \pi_1(2M)$ is trivial, since $2M - S \approx S^2 \times (0, 1)$. Consider the commutative diagram



obtained from the Van Kampen theorem. If we let

$$\pi_1(\text{Bd } M) \cong (\bar{z} : \bar{t}), \qquad G \cong (\bar{x} : \bar{r}), \qquad G' \cong (\bar{y} : \bar{s}),$$

we can write $\pi_1(2M) \cong (\bar{x}, \bar{y} : \bar{r}, \bar{s}, \{\theta_1(Z_k)\,\theta_2(Z_k)^{-1} : Z_k \in \overline{Z}\})$. Since $i_\# = 0$ and M is a retract of 2M, θ_1 and θ_2 are trivial. Therefore $\pi_1(2M) \cong G * G'$, where $G \cong G'$. But this is a contradiction, since $\pi_1(2M) \cong Z$ and Z is not the free product of two isomorphic groups. Therefore every tame 2-sphere S in Int M must separate 2M and hence bound a 3-cell in M.

Remark. Using Lemma 4, Kwun [7] observed that if $2M \approx S^1 \times S^2$ and Bd M is connected, then M is a solid torus. It also follows that if $2M \approx N$ and Bd M is connected, then M is the product of the Möbius band with the unit interval.

5. In this section, we show that certain prime 3-manifolds that cover themselves fiber over the circle. A complete classification of such manifolds seems difficult. Let p: M $\to$ M be a regular k-sheeted covering projection ($k \geq 2$ and prime). If some covering transformation of this covering space is homotopic to the identity homeomorphism of M, we say that M *properly* covers itself. This is equivalent to saying that the action on M by the group of covering transformations is *proper* [4].

THEOREM 3. *Let* $k \geq 2$ *be a prime integer. Suppose that* M *is a compact, connected, orientable* 3-*manifold such that* $H_1(M; Z)$ *has no element of order* k *and* Bd M *is either empty or connected. If* M *properly covers itself* k *times, then* M *can be fibered over the circle.*

Proof. M admits a proper free action by the group Z_k of covering transformations. If M is closed, then M is a prime, closed 3-manifold (by Theorem 1), since each covering of $P_3 \# P_3$ by itself is not proper. $S^1 \times S^2$ fibers over the circle; therefore, if M is closed, we may assume for this argument that M is irreducible. On the other hand, Theorem 5 implies that if Bd M $\neq \emptyset$, then M is prime and irreducible. In either case, we can now apply the main result of [14] to show that M fibers over the circle.

Remark. Fiberings over the circle such as those given by Theorem 3 do not necessarily have unique connected fibers. For example, in [13] we show that for every integer $n \geq 2$ there is a fibering of $T_2 \times S^1$ over S^1 with fiber T_n, where T_n denotes a closed surface of genus n. Results about nonunique fibers have also been obtained by W. Jaco [4].

6. In this section we describe a class of nontrivial manifolds that properly cover themselves. But first we consider a class of trivial examples, to motivate a question. If T is a compact 2-manifold and k is an integer $(k \geq 2)$, then the map $1_T \times p'\colon T \times S^1 \to T \times S^1$ (where p' is the standard k-sheeted covering projection of the circle) is a proper k-sheeted covering projection. It is natural to seek conditions of this type that characterize products of the form $T \times S^1$. Kwun has posed the following question.

Question. If a closed, orientable 3-manifold M covers itself properly k times, for every prime k, is M a product of a 2-manifold and S^1?

We show in Theorem 3 that such a manifold fibers over the circle, and thus is at least "almost" a product. If we alter the question by requiring merely that M cover itself properly k times for every *odd* prime k, we can answer in the negative with an example.

Let T_2 be a closed surface of genus 2. Consider the homeomorphism $h\colon T_2 \to T_2$ obtained by interchanging the holes of T_2 in such a way that h has exactly two fixed points, and such that h^2 is the identity. Let M be the manifold obtained from the product $T_2 \times I$ by identifying $(T \times 0)$ and $(T \times 1)$ by h, that is, $M = T \times I/\{(t, 0) \sim (h(t), 1)\}$. Then M covers itself properly k times for every positive odd integer k.

This example is contained in the following class of manifolds that cover themselves in such a manner that the action of the group of covering transformations can be extended to an effective SO(2) action (see [10] for notation). Let M be the closed, orientable, irreducible 3-manifold

$$\{-1; (o, g, 0, 0); (\lambda + 1, 1), (\lambda + 1, \lambda)\} \qquad (\lambda, g > 0).$$

M is a proper k-sheeted covering of itself for every $k \equiv 1 \pmod{\lambda + 1}$. Moreover, $H_1(M; Z)$ is a free abelian group of rank $2g + 1$, but M is not a product of the form $T \times S^1$. (The proofs of these facts are essentially computational, and they depend on the classification theorems in [10].)

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