ON THE THEORY OF SIMPLE Γ-RINGS

Jiang Luh

1. INTRODUCTION

Let M and Γ be two additive abelian groups. If for all x, y, z ∈ M and all $\alpha, \beta \in \Gamma$, the conditions

(1) $$x\alpha y \in M,$$

(2) $$(x + y)\alpha z = x\alpha z + y\alpha z, \quad x(\alpha + \beta)z = x\alpha z + x\beta z, \quad x\alpha(y + z) = x\alpha y + x\alpha z,$$

(3) $$(x\alpha y)\beta z = x\alpha(y\beta z)$$

are satisfied, then, following Barnes [1], we call M a *Γ-ring*. If these conditions are strengthened to

(1') $$x\alpha y \in M, \quad \alpha x\beta \in \Gamma,$$

(2') the same as (2),

(3') $$(x\alpha y)\beta z = x(\alpha y\beta)z = x\alpha(y\beta z),$$

(4') $$x\alpha y = 0 \text{ for all } x, y \in M \text{ implies } \alpha = 0,$$

then M is called a *Γ-ring in the sense of Nobusawa*. Clearly, every associative ring A is a Γ-ring, but it need not be a Γ-ring in the sense of Nobusawa if Γ = A. In [4], Nobusawa obtained an analogue of Wedderburn's theorem, for simple Γ-rings with minimal condition on one-sided ideals. In an earlier paper, the author developed the concept of primitivity for Γ-rings, and he characterized the primitive Γ-rings in the sense of Nobusawa having minimal one-sided ideals, by means of certain Γ-rings of continuous semilinear transformations. This characterization generalized a result of Jacobson in ordinary ring theory.

In this paper, we extend the notions of simplicity and complete primeness to Γ-rings. Our definition of simple Γ-rings differs slightly from Nobusawa's original definition, and the simple Γ-rings defined by Nobusawa are now called *completely prime Γ-rings*. However, the two concepts are identical for a Γ-ring in the sense of Nobusawa with minimum condition on one-sided ideals. We study the relations among simplicity, primeness, primitivity, and complete primeness for Γ-rings. Much of the development is analogous to the corresponding part of ring theory. We also define socles for Γ-rings, and we discuss their basic properties. One of our main results is the generalized Litoff theorem for simple Γ-rings having minimal left ideals. Finally, we determine completely the one-sided ideals of a simple Γ-ring having minimal one-sided ideals.

We refer to [2] for all notions relevant to ring theory.

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2. PRELIMINARIES

Let M be a Γ-ring. If $S, T \subseteq M$ and $\Gamma_0 \subseteq \Gamma$, we shall write $S\Gamma_0 T$ for the set of finite sums $\sum_i s_i \alpha_i t_i$, where $s_i \in S$, $t_i \in T$, $\alpha_i \in \Gamma_0$. A subgroup I of M is a *left (right) ideal* of M if $M\Gamma I \subseteq I$ ($I\Gamma M \subseteq I$). If I is both a left and a right ideal of M, then I is a *two-sided ideal* or simply an *ideal* of M. A one-sided ideal I is *strongly nilpotent* if $I^n = I\Gamma I\Gamma \cdots \Gamma I = 0$ for some positive integer n. A nonzero left (right) ideal I of M is *minimal* if the only left (right) ideals of M contained in I are 0 and I itself. We note that, for a minimal left ideal I of M, either $I\Gamma I = 0$, or $I = M\gamma e$, where $\gamma \in \Gamma$, $e \in M$, and $e\gamma e = e$.

Let F be the free abelian group generated by the set of all ordered pairs (x, α) with $x \in M$, $\alpha \in \Gamma$. Let G be the subgroup of elements $\sum_i m_i(x_i, \alpha_i) \in F$, where the m_i are integers such that $\sum_i m_i(x_i \alpha_i x) = 0$ for all $x \in M$. Denote by L the factor group F/G and by $[x, \alpha]$ the coset $G + (x, \alpha)$. Clearly, every element in L can be expressed as a finite sum $\sum_i [x_i, \alpha_i]$. Also, for all $x, y \in M$ and $\alpha, \beta \in \Gamma$,

$$[x, \alpha] + [x, \beta] = [x, \alpha + \beta] \quad \text{and} \quad [x, \alpha] + [y, \alpha] = [x + y, \alpha].$$

We define multiplication in L by

$$\sum_i [x_i, \alpha_i] \cdot \sum_j [y_j, \beta_j] = \sum_{i,j} [x_i \alpha_i y_j, \beta_j].$$

Then L forms a ring. Furthermore, M is a left L-module, with the definition

$$\sum_i [x_i, \alpha_i] x = \sum_i x_i \alpha_i x \qquad \text{for } x \in M.$$

We call the ring L the *left operator ring* of M. Similarly, we can define the right operator ring R of M. For $S \subseteq M$ and $\Gamma_0 \subseteq \Gamma$, we denote by $[S, \Gamma_0]$ the set of all finite sums $\sum_i [x_i, \alpha_i]$ in L with $x_i \in S$ and $\alpha_i \in \Gamma_0$.

A Γ-ring M is *left (right) primitive* if (i) the left (right) operator ring of M is a left (right) primitive ring, and (ii) $x\Gamma M = 0$ ($M\Gamma x = 0$) implies $x = 0$. M is a *two-sided primitive* Γ*-ring* (or simply a primitive Γ-ring) if it is both left and right primitive. It is known [3] that every one-sided primitive Γ-ring having minimal one-sided ideals is a two-sided primitive Γ-ring. Since no left primitive Γ-ring has nonzero strongly nilpotent one-sided ideals, every minimal left ideal of a primitive Γ-ring M is of the form $M\gamma e$, where $e\gamma e = e$. We note that any primitive ring A (having minimal left ideals) is a primitive Γ-ring in the sense of Nobusawa (having minimal left ideals), if $\Gamma = A$.

Let (V, W) and (V', W') be two pairs of dual vector spaces over division rings D and D', respectively, and let σ be an isomorphism of D onto D'. We denote by $\mathscr{L}(V, V')$ the additive group of all continuous semilinear transformations of V (topologized by the W-topology) into V' (topologized by the W'-topology), and by $\mathscr{F}(V, V')$ the subgroup of $\mathscr{L}(V, V')$ consisting of all continuous semilinear transformations of V into V' of finite rank. We shall need the following result from [3].

THEOREM 2.1. *Let* M *be a* Γ*-ring. Then* M *is a left primitive* Γ*-ring in the sense of Nobusawa having minimal one-sided ideals if and only if there exist two* *pairs of dual vector spaces* (V, W) *and* (V', W'), *over isomorphic division rings* D *and* D', *respectively, such that* M *is isomorphic to the* Γ'*-ring* M', *where*

$$\mathscr{F}(V, V') \subseteq M' \subseteq \mathscr{L}(V, V') \quad \text{and} \quad \mathscr{F}(V', V) \subseteq \Gamma' \subseteq \mathscr{L}(V', V),$$

and where the composition $x\alpha y$ *for* $x, y \in M'$ *and* $\alpha \in \Gamma'$ *is the composition of mappings. Moreover,* $\mathscr{F}(V, V')$ *is the unique minimal two-sided ideal of* M'.

3. SIMPLICITY, PRIMENESS, PRIMITIVITY, AND COMPLETE PRIMENESS OF Γ-RINGS

A Γ-ring M is said to be *simple* if (i) $M\Gamma M \neq 0$ and (ii) M has no ideals other than 0 and M itself. A Γ-ring M is said to be *completely prime* if $a\Gamma b = 0$ implies $a = 0$ or $b = 0$. We recall Barnes' definition: Let M be a Γ-ring. An ideal P of M is prime if, for all pairs of ideals S and T of M, $S\Gamma T \subseteq P$ implies $S \subseteq P$ or $T \subseteq P$. A Γ-ring M is *prime* if the zero ideal is prime.

The following theorem characterizes primeness for ideals in Γ-rings. The proof is a minor modification of the proof of the corresponding theorem in ring theory, and we omit it.

THEOREM 3.1. *If* P *is an ideal in a* Γ*-ring* M, *the following four conditions are equivalent:*

(i) P *is a prime ideal.*

(ii) *If* $a, b \in M$ *and* $a\Gamma M\Gamma b \subseteq P$, *then* $a \in P$ *or* $b \in P$.

(iii) *If* I *and* J *are right ideals in* M *and* $I\Gamma J \subseteq P$, *then* $I \subseteq P$ *or* $J \subseteq P$.

(iv) *If* I *and* J *are left ideals in* M *and* $I\Gamma J \subseteq P$, *then* $I \subseteq P$ *or* $J \subseteq P$.

COROLLARY 3.1. *A* Γ*-ring* M *is prime if and only if* $a\Gamma M\Gamma b = 0$ *implies* $a = 0$ *or* $b = 0$.

From this we can see that every completely prime Γ-ring is prime. Let A be an associative ring. If A is simple (prime, completely prime), then A, regarded as a Γ-ring where $\Gamma = A$, is simple (prime, completely prime). We also note that, for a Γ-ring in the sense of Nobusawa, primeness and complete primeness are equivalent.

THEOREM 3.2. *If* M *is a simple* Γ*-ring, then* M *is prime.*

Proof. Suppose that M is not prime and that $S\Gamma T = 0$, where S and T are nonzero ideals of M. Then, by the simplicity of M, $S = T = M$, and hence $M\Gamma M = 0$, which contradicts the simplicity of M.

THEOREM 3.3. *If* M *is a left primitive* Γ*-ring, then* M *is prime.*

Proof. Let N be a faithful irreducible left L-module, where L is the left operator ring of M. Suppose, contrary to the theorem, that M is not prime and that $S\Gamma T = 0$, where S and T are nonzero ideals of M. We claim first that $[T, \Gamma]N = N$. For otherwise, since $[T, \Gamma]N = 0$, it would follow that $[T, \Gamma] = 0$, so that $T\Gamma M = 0$. By the primitivity of M, this would imply that $T = 0$, a contradiction. Hence we see that $[T, \Gamma]N = N$. Likewise, $[S, \Gamma]N = N$. Thus, we obtain the relation

$$0 = [S\Gamma T, \Gamma]N = [S, \Gamma][T, \Gamma]N = [S, \Gamma]N = N,$$

and this is again a contradiction.

Next we shall consider Γ-rings having minimal left ideals. As we pointed out earlier, for these Γ-rings, one-sided primitivity implies two-sided primitivity.

THEOREM 3.4. *If* M *is a* Γ-*ring having minimal left ideals, then* M *is primitive if and only if it is prime.*

Proof. By Theorem 3.3, primitivity implies primeness.

Now assume that M is prime. Let I be a minimal left ideal of M. Clearly, I is an irreducible left L-module. We shall show that I is faithful. Since $I\Gamma I \neq 0$, $I = M\gamma e$, where $e\gamma e = e$. Suppose $\sum_i [x_i, \gamma_i]I = 0$. Then the relation

$$\sum_i x_i\gamma_i M\Gamma M\gamma e \subseteq \sum_i [x_i, \gamma_i]I$$

implies that $\left(\sum_i x_i\gamma_i M\right)\Gamma(M\gamma e) = 0$. By Corollary 3.1, $\sum_i x_i\gamma_i M = 0$ or $\sum_i [x_i, \gamma_i] = 0$. Thus I is a faithful irreducible left L-module, and L is a left primitive ring. Moreover, if $x\Gamma M = 0$, then $x\Gamma M\Gamma x = 0$. Again by Corollary 3.1, $x = 0$. Therefore M is a primitive Γ-ring.

Finally, let us consider the Γ-rings with minimum condition on left ideals.

LEMMA 3.1. *If* M *is a primitive* Γ-*ring with minimum condition on left ideals, then*

$$M = M\gamma_1 e_1 + M\gamma_2 e_2 + \cdots + M\gamma_n e_n \quad \textit{(direct sum)},$$

where $e_i\gamma_i e_i = e_i$ *and* $e_i\gamma_j e_j = 0$ *if* $i > j$, *and where the* $M\gamma_i e_i$ *are minimal left ideals of* M.

Proof. Let $I_1 = M\gamma_1 e_1$ be a minimal left ideal of M, where $e_1\gamma_1 e_1 = e_1$, and let $M_1 = \{x \in M: x\gamma_1 e_1 = 0\}$. Clearly, M_1 is a left ideal of M, and each $a \in M$ has the form $a = a\gamma_1 e_1 + (a - a\gamma_1 e_1)$, where $a - a\gamma_1 e_1 \in M_1$. Hence

$$M = M\gamma_1 e_1 + M_1 \quad \text{(direct sum)}.$$

If $M_1 \neq 0$, then by the minimum condition, M_1 contains a minimal left ideal $M\gamma_2 e_2$ of M, where $e_2\gamma_2 e_2 = e_2$ and $e_2\gamma_1 e_1 = 0$. Consequently,

$$M = M\gamma_1 e_1 + M\gamma_2 e_2 + M_2 \quad \text{(direct sum)},$$

where $M_2 = \{x \in M_1: x\gamma_2 e_2 = 0\}$. Continuing this process, we find that $M_n = 0$ for some positive integer n. Thus,

$$M = M\gamma_1 e_1 + M\gamma_2 e_2 + \cdots + M\gamma_n e_n \quad \text{(direct sum)},$$

as was to be proved.

LEMMA 3.2. *Let* M *be a left primitive* Γ-*ring, and let* I *be a nonzero left ideal of* M. *If* $e\gamma e = e \neq 0$, *where* $e \in M$ *and* $\gamma \in \Gamma$, *then* $e\gamma I \neq 0$.

Proof. Let N be a faithful irreducible left L-module, where L is the left operator ring of M. By the primitivity of M, $[I, \Gamma] \neq 0$, and hence $[I, \Gamma]N = N$.

Now suppose that, contrary to the lemma, $e\gamma I = 0$. Then

$$[e, \gamma]N = [e, \gamma][I, \Gamma]N = [e\gamma I, \Gamma]N = 0.$$

Since N is faithful, $[e, \gamma] = 0$. This leads to the contradiction that $e = e\gamma e = [e, \gamma]e = 0$. Therefore, $e\gamma I \neq 0$.

THEOREM 3.5. *If* M *is a primitive* Γ-*ring with minimum condition on left ideals, then* M *is simple.*

Proof. By Lemma 3.1,

$$M = M\gamma_1 e_1 + \cdots + M\gamma_n e_n \qquad \text{(direct sum)},$$

where $e_i \gamma_i e_i = e_i$ and $e_i \gamma_j e_j = 0$ if $i > j$, and where the $M\gamma_i e_i$ are minimal left ideals of M.

Let I be a nonzero ideal of M. Each $x \in I$ has the form

$$x = x_1 \gamma_1 e_1 + \cdots + x_n \gamma_n e_n,$$

where $x_i \in M$ $(i = 1, 2, \cdots, n)$. Assume that $x_k \gamma_k e_k + \cdots + x_n \gamma_n e_n \in I$, where $1 \leq k < n$. Then $(x_k \gamma_k e_k + \cdots + x_n \gamma_n e_n)\gamma_k e_k \in I$; hence, $x_k \gamma_k e_k \in I$, so that $x_{k+1} \gamma_{k+1} e_{k+1} + \cdots + x_n \gamma_n e_n \in I$. Hence, by induction, $x_k \gamma_k e_k \in I$ $(k = 1, 2, \cdots, n)$. But $x_k \gamma_k e_k = (x_k \gamma_k e_k)\gamma_k e_k \in I\gamma_k e_k$, so that $I \subseteq I\gamma_1 e_1 + \cdots + I\gamma_n e_n$. Since I is a two-sided ideal, $I\gamma_k e_k \subseteq I$, and hence $I = I\gamma_1 e_1 + \cdots + I\gamma_n e_n$.

We assert now that $I\gamma_k e_k \neq 0$ for each k. For otherwise,

$$(e_k \gamma_k I)\Gamma(e_k \gamma_k I) = e_k \gamma_k (I\Gamma e_k)\gamma_k I = 0,$$

while by Lemma 3.2 $e_k \gamma_k I \neq 0$; hence $e_k \gamma_k I$ is a nonzero, strongly nilpotent right ideal of M. This contradicts the fact that a primitive Γ-ring has no nonzero, strongly nilpotent, one-sided ideals. Consequently, $I\gamma_k e_k = M\gamma_k e_k$, since $M\gamma_k e_k$ is a minimal left ideal of M. Therefore,

$$I = M\gamma_1 e_1 + \cdots + M\gamma_n e_n = M,$$

and M is simple.

Theorems 3.2, 3.4, and 3.5 immediately imply the following.

THEOREM 3.6. *For a* Γ-*ring* M *with minimum condition on left ideals, the three conditions*

(i) M *is prime,*

(ii) M *is primitive,*

(iii) M *is simple*

are equivalent.

However, none of the three conditions implies the complete primeness, even for a finite Γ-*ring.*

Example 3.1. Let M be the ring of 2×2 matrices over the field GF(2), and let $\Gamma = \{\zeta, \varepsilon\}$ be the additive group of order two with ζ as the identity element. For all $a, b \in M$, we define $a\zeta b = 0$ and $a\varepsilon b = ab$ (the ordinary product of the matrices a and b). It is easy to verify that M forms a finite Γ-ring that is prime but not completely prime.

For a Γ-ring having minimal left ideals, complete primeness (hence primeness and primitivity) does not imply simplicity.

Example 3.2. Let $M = \mathscr{L}(V, V')$ and $\Gamma = \mathscr{F}(V', V)$ be defined as in Theorem 2.1. Then M is a completely prime Γ-ring, but it is not simple.

For a general Γ-ring, neither primeness nor complete primeness implies primitivity.

Example 3.3. Let M = Γ be the ring of integers. Considered as a Γ-ring, M is completely prime as well as prime, but it is not primitive.

The following example shows that for a Γ-ring, simplicity does not imply primitivity.

Example 3.4. Let M be a simple radical ring (the existence of such rings has been shown by Sasiada [5]). We regard M as a Γ-ring with Γ = M. Clearly, M is a simple Γ-ring.

We shall show that M is not left primitive. Suppose to the contrary that a faithful irreducible left L-module N exists, where L is the left operator ring of M. Since $M^2 = M$, each $a \in M$ has the form $a = \sum_i a_i \alpha_i$ for some $a_i, \alpha_i \in M$. For each $x \in N$, define $ax = \sum_i [a_i, \alpha_i]x$. We see that N is a faithful irreducible left M-module. This is a contradiction, since M is not a left primitive associative ring.

4. SOCLES OF Γ-RINGS

Let M be a Γ-ring. The sum S_ℓ (S_r) of all minimal left (right) ideals of M is called the *left (right) socle* of M. It is understood that if M has no minimal left (right) ideals, then the left (right) socle of M is 0.

In this section we shall show that a one-sided socle of a Γ-ring M is an ideal of M, and that if M has no strongly nilpotent ideals other than 0, then the left socle and the right socle of M coincide.

LEMMA 4.1. *Let* M *be a* Γ*-ring. If* I *is a minimal left ideal of* M, *then, for each* $\gamma \in \Gamma$ *and each* $x \in M$, $I\gamma x$ *is either zero or a minimal left ideal of* M.

Proof. If $I\gamma x \neq 0$ and J is a nonzero left ideal of M contained in $I\gamma x$, then there exists $a \in I$ with $0 \neq a\gamma x \in J$. Let $H = \{z \in I: z\gamma x \in J\}$. H is a nonzero left ideal of M contained in I. The minimality of I implies that H = I, so that $I\gamma x \subseteq J$. It follows that $I\gamma x = J$ and that $I\gamma x$ is a minimal left ideal of M.

THEOREM 4.1. *If* M *is a* Γ*-ring, then the left socle and the right socle of* M *are ideals of* M.

Proof. By symmetry, we need only prove that the left socle S_ℓ of M is an ideal of M. It is clear that S_ℓ is a left ideal of M. Assume that $\gamma \in \Gamma$, $x \in M$, $s \in S_\ell$, and $s \in I_1 + \cdots + I_n$, where I_i are minimal left ideals of M. Then

$$s\gamma x \in I_1\gamma x + \cdots + I_n\gamma x .$$

By Lemma 4.1, $I_i\gamma x$ is either 0 or a minimal left ideal of M. Hence $s\gamma x \in S_\ell$, and S_ℓ is a right ideal of M.

The following extends Lemma 3.1 to simple rings with minimal one-sided ideals.

THEOREM 4.2. *If* M *is a simple* Γ*-ring having minimal left ideals, then* M *is a direct sum of minimal left ideals.*

Proof. Since the left socle of M is M itself, M is a sum of minimal left ideals of M. Consider the family $\mathscr{A}$ of all independent sets of minimal left ideals of M. Here a set $\{I_\alpha : \alpha \in A\}$ of minimal left ideals of M is said to be *independent* if $I_\alpha \cap \sum_{\beta \neq \alpha} I_\beta = 0$ for each $\alpha \in A$. The family $\mathscr{A}$ is partially ordered by inclusion. Applying Zorn's lemma, we can obtain a maximal independent set in $\mathscr{A}$, say $\{I_\alpha : \alpha \in B\}$. By the maximality of this set, $I \cap \sum_{\alpha \in B} I_\alpha \neq 0$ for each minimal left ideal I of M, so that

$$I \cap \sum_{\alpha \in B} I_\alpha = I \quad \text{and} \quad I \subseteq \sum_{\alpha \in B} I_\alpha .$$

Therefore, $M = \sum_{\alpha \in B} I_\alpha$ (direct sum).

THEOREM 4.3. *Let* M *be a* Γ*-ring. If* M *has no nonzero strongly nilpotent ideals, then the left socle* S_ℓ *and the right socle* S_r *of* M *coincide.*

Proof. We recall that a Γ-ring M without nonzero strongly nilpotent ideals has minimal left ideals if and only if it has minimal right ideals. Moreover, every minimal left ideal is of the form $M\gamma e$, where $e\gamma e = e$, and $M\gamma e$ is a minimal left ideal if and only if $e\gamma M$ is a minimal right ideal.

Let $S_\ell = \sum_i M\gamma_i e_i$, where the $M\gamma_i e_i$ are minimal left ideals of M and $e_i \gamma_i e_i = e_i$. Since $e_i \gamma_i M$ are minimal right ideals of M, $\sum_i e_i \gamma_i M \subseteq S_r$. But $e_i \in S_r$, and S_r is an ideal of M, so that $M\gamma_i e_i \subseteq S_r$. It follows that $S_\ell \subseteq S_r$. By symmetry, $S_r \subseteq S_\ell$. Hence $S_\ell = S_r$, as was to be proved.

Remark: If M is a Γ-ring with the properties described in Theorem 2.1, then the left (right) socle of M is $\mathscr{F}(V, V')$.

5. SIMPLE Γ-RINGS HAVING MINIMAL LEFT IDEALS

In this section we shall prove a generalization of the Litoff theorem for Γ-rings. First, we need two lemmas.

LEMMA 5.1. *Let* (V, W) *be a pair of dual vector spaces over a division ring* D, *let* V_0 *be a finite-dimensional vector subspace of* V, *and let* W_0 *be a finite-dimensional vector subspace of* W. *Then there exist finite-dimensional vector subspaces* V_1 *and* W_1 *such that*

$$V_0 \subseteq V_1 \subseteq V \quad \textit{and} \quad W_0 \subseteq W_1 \subseteq W,$$

and such that (V_1, W_1) *is a dual pair relative to the given bilinear form.*

For a proof, see [2, p. 90].

Let G be an additive group. We shall denote by $G_{m,n}$ the additive group of all m-by-n matrices over the group G. For $1 \leq h \leq m$, $1 \leq k \leq n$, and $g \in G$, let $gE_{(h,k)}$ denote the matrix having g at the hth row and kth column, and 0 elsewhere. Let M be a Γ-ring. Consider the groups $M_{m,n}$ and $\Gamma_{n,m}$. For $(a_{ij}), (b_{ij}) \in M_{m,n}$ and $(\gamma_{ij}) \in \Gamma_{n,m}$, define $(a_{ij})(\gamma_{ij})(b_{ij}) = (c_{ij})$, where

$$c_{ij} = \sum_{k=1}^{m} \sum_{h=1}^{n} a_{ih}\gamma_{hk} b_{kj} .$$

Then $M_{m,n}$ forms a $\Gamma_{n,m}$-ring.

LEMMA 5.2. *Let* M *be a* Γ*-ring such that* $x \in M\Gamma x\Gamma M$ *for every* $x \in M$. *Then the ideals of the* $\Gamma_{n,m}$*-ring* $M_{m,n}$ *are of the form* $U_{m,n}$, *where* U *is an ideal of* M.

Proof. Let I be an ideal of $M_{m,n}$, and let

$$U = \{a_{pq}: (a_{ij}) \in I\}.$$

Clearly, $I \subseteq U_{m,n}$.

To show that $U_{m,n} \subseteq I$, it will suffice to show that, for all h and k $(1 \leq h \leq m,\ 1 \leq k \leq n)$ and all $u \in U$ with $u = a_{pq}$ and $(a_{ij}) \in I$, the m-by-n matrix $uE_{(h,k)}$ is in I.

We note that, for all $a, b \in M$ and $\gamma, \delta \in \Gamma$,

$$(aE_{(i,1)})(\gamma E_{(1,p)})(a_{ij})(\delta E_{(q,1)})(bE_{(1,j)}) = (a\gamma a_{pq}\delta b)E_{(i,j)} \in I.$$

Since $u \in M\Gamma u\Gamma M$, $u = \sum_i a_i\gamma_i u\delta_i b_i$ for some $a_i, b_i \in M$ and $\gamma_i, \delta_i \in \Gamma$. Thus,

$$uE_{(h,k)} = \Big(\sum_i a_i\gamma_i u\delta_i b_i\Big)E_{(h,k)} = \sum_i (a_i\gamma_i u\delta_i b_i)E_{(h,k)} \in I.$$

This completes the proof.

THEOREM 5.1. *If* M *is a simple* Γ*-ring, then, for all positive integers* m *and* n, $M_{m,n}$ *is a simple* $\Gamma_{n,m}$*-ring.*

Proof. Since M is simple, $M\Gamma a\Gamma M = M$ for each nonzero element a in M. Hence $a \in M\Gamma a\Gamma M$. Let I be an arbitrary ideal of the $\Gamma_{n,m}$-ring $M_{m,n}$. Then, by Lemma 5.2, $I = U_{m,n}$ for some ideal U of M. However, M is simple, so that $U = 0$ or $U = M$. Therefore, $I = 0$ or $I = M_{m,n}$. Also, it is evident that $M_{m,n}\Gamma_{n,m}M_{m,n} \neq 0$. Hence $M_{m,n}$ is a simple $\Gamma_{n,m}$-ring.

Now assume that M is a simple Γ-ring in the sense of Nobusawa. For all $x, y, z \in M$ and $\alpha, \beta_1, \beta_2 \in \Gamma$,

$$x(\alpha y(\beta_1 + \beta_2))z = x\alpha(y(\beta_1 + \beta_2)z) = x\alpha(y\beta_1 z + y\beta_2 z) = x\alpha(y\beta_1 z) + x\alpha(y\beta_2 z)$$

$$= x(\alpha y\beta_1)z + x(\alpha y\beta_2)z = x(\alpha y\beta_1 + \alpha y\beta_2)z;$$

therefore, by condition (4'), $\alpha y(\beta_1 + \beta_2) = \alpha y\beta_1 + \alpha y\beta_2$ in the definition of Γ-rings in the sense of Nobusawa. Likewise,

$$(\alpha_1 + \alpha_2)y\beta = \alpha_1 y\beta + \alpha_2 y\beta, \qquad \alpha(y_1 + y_2)\beta = \alpha y_1\beta + \alpha y_2\beta,$$

and

$$(\alpha x\beta)y\gamma = \alpha(x\beta y)\gamma = \alpha x(\beta y\gamma),$$

for all $\alpha_1, \alpha_2, \alpha, \beta, \gamma \in \Gamma$, and $x, y, y_1, y_2 \in M$. Moreover, $\Gamma x\Gamma = 0$ implies that $M\Gamma x\Gamma M = 0$ and $x = 0$. Therefore, if M is a simple Γ-ring in the sense of Nobusawa, then Γ is a Γ'-ring in the sense of Nobusawa, where $\Gamma' = M$.

A Γ-ring M in the sense of Nobusawa will be called *strongly simple* if M is a simple Γ-ring and Γ is a simple Γ'-ring, where $\Gamma' = M$.

Now, we are ready to prove an analogue of the Litoff theorem for Γ-rings.

THEOREM 5.2. *Let* M *be a strongly simple* Γ-*ring in the sense of Nobusawa. If* M *has minimal left ideals, then there exists a division ring* D *such that, for each finite subset* M_0 *of* M *and each finite subset* Γ_0 *of* Γ, *there exist* $M_1 \subseteq M$ *and* $\Gamma_1 \subseteq \Gamma$, *satisfying the following three conditions.*

(i) M_1 *is a* Γ_1-*ring with respect to the composition defined in the* Γ-*ring* M.

(ii) $M_0 \subseteq M_1$, $\Gamma_0 \subseteq \Gamma_1$.

(iii) *The* Γ_1-*ring* M_1 *is isomorphic to a* $D_{n,m}$-*ring* $D_{m,n}$; *that is, there exist group isomorphisms* ϕ *of* M_1 *onto* $D_{m,n}$, *and* θ *of* Γ_1 *onto* $D_{n,m}$, *with* $(x\gamma y)\phi = (x\phi)(\gamma\theta)(y\phi)$ *for all* $x, y \in M_1$ *and* $\gamma \in \Gamma_1$.

Proof. According to Theorems 3.2 and 3.4, M is a primitive Γ-ring and Γ is a primitive Γ'-ring, where Γ' = M. By Theorem 2.1 and the strong simplicity of M, there exist two pairs of dual vector spaces (V, W) and (V', W') over division rings D and D', respectively, where D and D' are isomorphic, such that $M = \mathscr{F}(V, V')$ and $\Gamma = \mathscr{F}(V', V)$. Since every element of M (of Γ) is a finite sum of elements of M (of Γ) of rank one, we may without loss of generality assume that all elements of M_0 and of Γ_0 are of rank one. Let σ be the isomorphism of D onto D', and for $d \in D$, denote by d^σ the image of d under σ. Let $M_0 = \{a_1, a_2, \cdots, a_s\}$, $\Gamma_0 = \{\gamma_1, \gamma_2, \cdots, \gamma_t\}$, where

$$a_i\colon v \to (v, w_i)^\sigma v_i', \qquad \gamma_j\colon v' \to (v', w_j')^{\sigma^{-1}} v_j$$

for all $v \in V$ and $v' \in V'$ (see [3]). By Lemma 5.1, there exist two pairs of finite-dimensional dual vector spaces (V_1, W_1) and (V_1', W_1') over D and D', respectively, relative to the given bilinear forms, such that

$$\{v_1, v_2, \cdots, v_t\} \subseteq V_1 \subseteq V, \qquad \{w_1, w_2, \cdots, w_s\} \subseteq W_1 \subseteq W,$$

$$\{v_1', v_2', \cdots, v_s'\} \subseteq V_1' \subseteq V', \qquad \{w_1', w_2', \cdots, w_t'\} \subseteq W_1' \subseteq W'.$$

Let $\{u_1, u_2, \cdots, u_m\}$ and $\{x_1, x_2, \cdots, x_m\}$ be biorthogonal bases of V_1 and W_1 over D, respectively, and let $\{u_1', u_2', \cdots, u_n'\}$ and $\{x_1', x_2', \cdots, x_n'\}$ be biorthogonal bases of V_1' and W_1' over D'. Let M_1 be the subgroup of $M = \mathscr{F}(V, V')$ consisting of all transformations x of the form

$$x\colon v \to \sum_{i,j} (v, x_i)^\sigma d_{ij}^\sigma u_j',$$

and let Γ_1 be the subgroup of $\Gamma = \mathscr{F}(V', V)$ consisting of all transformations γ of the form

$$\gamma\colon v' \to \sum_{i,j} (v', x_i')^{\sigma^{-1}} f_{ij} u_j,$$

where $d_{ij}, f_{ij} \in D$. Then M_1 forms a Γ_1-ring and $M_0 \subseteq M_1$, $\Gamma_0 \subseteq \Gamma_1$.

It remains to show that the Γ_1-ring M_1 is isomorphic to the $D_{n,m}$-ring $D_{m,n}$. Consider the mappings $\phi\colon M_1 \to D_{m,n}$ and $\theta\colon \Gamma_1 \to D_{n,m}$ defined by $x\phi = (d_{ij})$, $\gamma\theta = (f_{ij})$. By straightforward computation, we can see easily that ϕ and θ are isomorphic and onto, and that $(x\gamma y)\phi = (x\phi)(\gamma\theta)(y\phi)$ for all $x, y \in M_1$, $\gamma \in \Gamma_1$. The proof is therefore complete.

6. ONE-SIDED IDEALS OF STRONGLY SIMPLE Γ-RINGS WITH MINIMAL ONE-SIDED IDEALS

Let M be a strongly simple Γ-ring. As in the discussion in the proof of Theorem 5.2, we may assume that M is a Γ-ring of continuous semilinear transformations of finite rank on certain vector spaces. Throughout this section, let (V, W) and (V', W') be two pairs of dual vector spaces over division rings D and D', respectively, let σ be the isomorphism of D onto D', and let $M = \mathscr{F}(V, V')$, $\Gamma = \mathscr{F}(V', V)$. We shall completely determine the one-sided ideals of the Γ-ring M. Our technique is analoguous to that in ring theory for a simple ring having minimal one-sided ideals (see [2, p. 91]).

If U' is a subspace of V' over D', we denote by $\hat{U}'$ the left ideal of M consisting of all elements of M whose range is contained in U'. More precisely,

$$\hat{U}' = \left\{ x \in M: vx = \sum_i (v, w_i)^\sigma u_i', \; u_i' \in U', w_i \in W, v \in V \right\}.$$

We shall show that every left ideal of M is of this form.

THEOREM 6.1. *If* I *is a left ideal of* M, *then* $I = \hat{U}'$ *for some vector subspace* U' *of* V'.

Proof. If I = 0, then clearly $I = \hat{U}'$, where U' is the zero subspace of V'. We assume now that $I \neq 0$. Let $U' = VI = \left\{ \sum_i v_i x_i: v_i \in V, x_i \in I \right\}$. Clearly, $I \subseteq \hat{U}'$. On the other hand, each element in $\hat{U}'$ is a finite sum of y's satisfying the condition

$$vy = (v, w_0)^\sigma u_1' \qquad \text{for all } v \in V,$$

where $w_0 \in W$ and $u_1' \in Vx$ with $x \in I$. Hence, to show that $\hat{U}' \subseteq I$, it suffices to show that for each $x \in I$, each $w_0 \in W$, and each nonzero $u_1' \in Vx$, the mapping

$$y: v \to (v, w_0)^\sigma u_1'$$

is an element in I.

Let $\{u_1', u_2', \cdots, u_m'\}$ be a basis of the range of x, and let

$$x: v \to \sum_{i=1}^m (v, w_i)^\sigma u_i',$$

where $w_i \in W$.

We assert that $w_1, w_2, \cdots, w_m$ are linearly independent over D. Otherwise, there would exist $d_1, d_2, \cdots, d_m$ in D, not all zero (say $d_1 \neq 0$), such that

$$w_1 d_1 + w_2 d_2 + \cdots + w_m d_m = 0.$$

This would imply that, for all $v \in V$,

$$vx = \sum_{i=1}^m (v, w_i)^\sigma u_i' - \sum_{i=1}^m (v, w_i)^\sigma d_i^\sigma (d_1^{-1})^\sigma u_1' = \sum_{i=2}^m (v, w_i)^\sigma (u_i' - d_i^\sigma (d_1^{-1})^\sigma u_1'),$$

and that the dimension of the range of x is less than m, a contradiction. Thus by the nondegeneracy of the bilinear forms, there exist $v_1 \in V$, $v'_1 \in V'$, $w'_1 \in W'$, such that $(v_1, w_1) = 1$, $(v_1, w_i) = 0$ for $i = 2, 3, \cdots, m$, and $(v'_1, w'_1) = 1$. We define $\gamma \in \Gamma$, $z \in M$ by

$$v'\gamma = (v', w'_1)^{\sigma^{-1}} v_1 \quad \text{for all } v' \in V', \qquad vz = (v, w'_0)^{\sigma} v'_1 \quad \text{for all } v \in V.$$

Then it is easy to see that $y = z\gamma x$ and hence $y \in I$.

Similarly, for each subspace U of W, we denote by $\hat{U}$ the right ideal of M consisting of all elements x of M whose adjoints x^* have ranges contained in U. We note that if $x: v \to \sum_i (v, w_i)^{\sigma} v'_i$ for all $v \in V$, where $w_i \in U$ and $v'_i \in V'$, then

$$x^*: w' \to \sum_i w_i (v'_i, w')^{\sigma^{-1}}$$

for all $w' \in W'$. From this, we can prove that

$$\hat{U} = \left\{ x \in M: vx = \sum_i (v, w_i)^{\sigma} v'_i,\ w_i \in U,\ v'_i \in V',\ v \in V \right\}.$$

THEOREM 6.2. *If* I *is a right ideal of* M, *then* $I = \hat{U}$ *for some vector subspace* U *of* W.

The proof is similar to that of Theorem 6.1, and we omit it.

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Wright State University
Dayton, Ohio 45431
and
North Carolina State University
Raleigh, North Carolina 27607