AVERAGES OVER A PAIR OF CONVEX SURFACES

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Let C_1 denote a convex curve in the plane and r the distance from a fixed point in the region bounded by C_1 to a variable point on C_1. In [3], Sachs obtained a result equivalent to the assertion that the average of r^2 with respect to arc length on C_1 is at least as large as the average of r^2 with respect to angle. We generalize this result in several directions.

First we replace r^2 with any increasing function of r; then we replace angular measure (which represents arc length on the unit circle) with arc length in a second convex curve; finally, we extend the result to convex surfaces in d-dimensional Euclidean space R^d.

The proof rests on the following theorem (see [4, page 146]): If A and B are convex bodies and $A \subseteq B$, then the measure of the surface of A is less than or equal to the measure of the surface of B. (If A is properly contained in B, the inequality is strict.)

In Section 2, we discuss nonconvex surfaces, in particular, the pedal curve of a convex curve.

1. AVERAGES OF A MONOTONIC FUNCTION OVER CONVEX SURFACES

If $X \subseteq R^d$ and t is a positive real number, the set $\{tx \mid x \in X\}$ will be denoted by tX. Let K_1 and K_2 be convex bodies in R^d that contain the origin as an interior point, and let surface measures m_1 and m_2 be defined on their surfaces S_1 and S_2, respectively. Let $p: S_1 \to S_2$ denote radial projection from the origin. The norm of $x \in R^d$ we shall denote by $|x|$.

THEOREM 1. *Let* K_1 *and* K_2 *be convex bodies in* R^d *containing the origin as an interior point. Let* $f: R^1 \to R^1$ *be a monotonically increasing function. Then*

$$(1) \qquad \frac{1}{m(S_1)} \int_{S_1} f(|x_1|/|p(x_1)|)\, dm_1 \geq \frac{1}{m(S_2)} \int_{S_2} f(|p^{-1}(x_2)|/|x_2|)\, dm_2 .$$

Proof. For convenience, we shall assume that $f(0) = 0$, that f is continuous from the right, and that $m(S_1) = m(S_2)$. Define $g: S_1 \to R^1$ by setting

$$g(x_1) = f(|x_1|/|p(x_1)|).$$

Set $S_1(t) = \{x \mid x \in S_1, g(x) \geq t\}$. In view of Fubini's theorem, it suffices to show that $m(S_1(t)) \geq m[p(S_1(t))]$ for each nonnegative real number t. Now,

$$S_1(t) = \{x_1 \mid x_1 \in S_1, |x_1|/|p(x_1)| \geq t^*\},$$

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where $t^* = \min\{u \mid f(u) \geq t\}$. The condition $|x_1|/|p(x_1)| \geq t^*$ is equivalent to $x_1 \notin \mathrm{Int}\, t^* K_2$. Let $L_2(t^*) = K_1 \cap t^* S_2$. Since $t^* K_2 \cap K_1 \subseteq K_1$, we see that $m(S_1(t)) \geq m(L_2(t^*))$. Also, since $t^* K_2 \cap K_1 \subseteq t^* K_2$, we have the inequality $m(S_1 \cap t^* K_2) \leq m(t^* S_2 - (K_1 \cap t^* S_2))$.

From these inequalities it follows that

$$\frac{m(S_1(t))}{m(S_1)} = \frac{m(S_1(t))}{m(S_1 \cap t^* K_2) + m(S_1(t))} \geq \frac{m(L_2(t^*))}{m(S_1 \cap t^* K_2) + m(L_2(t^*))}$$

$$\geq \frac{m(L_2(t^*))}{m(t^* S_2 - (K_1 \cap t^* S_2)) + m(L_2(t^*))} = \frac{m(L_2(t^*))}{m(t^* S_2)} = \frac{m[p_1(S_1(t))]}{m(S_2)};$$

hence

$$m(S_1(t)) \geq m(p_1(S_1(t))), \tag{2}$$

and the proof is complete.

The question when equality occurs in (1) is easily answered. Note that equality holds in (2) if and only if equality holds in each of the sequence of inequalities leading to (2). In particular, if $m(S_1 \cap t^* K_2)$ is positive, then $m(L_2(t^*)) = m(S_1(t))$, that is, $t^* K_2 \supseteq K_1$. Thus, if t^* is a number such that $m(S_1 \cap t^* K_2) > 0$ and $t^* K_2 \not\supseteq K_1$, then inequality (2) is strict. We may therefore conclude that if f is a strictly increasing function, equality holds in (1) if and only if K_1 is of the form $t^* K_2$ for some number t^*.

If we let S_2 in Theorem 1 denote the unit sphere in R^d whose center is at the origin, we obtain the following corollary.

COROLLARY 1. *Let* K *be a convex body in* R^d *containing the origin as an interior point, and let* S *be its surface. Let* f *be a strictly increasing function. Then*

$$\frac{1}{m(S)} \int_S f(|x|)\, dm \geq \frac{1}{\Omega} \int_S f(|x|)\, d\omega, \tag{3}$$

where Ω *is the measure of the solid angle of a sphere and* $d\omega$ *denotes solid angular measure. Equality holds in* (3) *if and only if* K *is a ball with center at the origin.*

In particular, if $n > 0$ and C is a convex plane curve surrounding the origin, then

$$\frac{1}{L} \int_C r^n\, ds \geq \frac{1}{2\pi} \int_0^{2\pi} r^n\, d\theta; \tag{4}$$

this includes Sachs' theorem.

Let T be a rigid motion of R^d that leaves the origin fixed, and let $K_2 = T(K_1)$. In this case, Theorem 1 takes a special form; the following corollary illustrates this for the case where $d = 2$.

COROLLARY 2. *Let* C *be a plane convex curve surrounding the origin, let its equation in polar coordinates be* $r = f(\theta)$, *and let* α *be a number. Then*

$$\int_C [f(\theta)/f(\theta - \alpha)]\, ds \geq \int_C [f(\theta + \alpha)/f(\theta)]\, ds.$$

2. AVERAGES OF A MONOTONIC FUNCTION OVER NONCONVEX SURFACES

If we delete the assumption of convexity, Theorem 1 is no longer valid. We can see this easily from (4) by considering a curve C that oscillates strongly near the origin. However, we can say something about arcs or portions of surfaces that may not be convex.

THEOREM 2. *Let* S_1 *and* S_2 *be smooth* $(d-1)$*-dimensional surfaces in* R^d, *not passing through the origin, and starlike with respect to the origin. For* $x \in S_2$, *let* $\rho(x)$ *denote the set derivative* $\lim_{A \downarrow x} m_1(p^{-1}(A))/m_2(A)$. *Assume that if* $x, y \in S_2$ *and* $|p^{-1}(x)|/|x| \geq |p^{-1}(y)|/|y|$, *then* $\rho(x) \geq \rho(y)$. *Let* f *be a monotonically increasing function. Then*

$$(5)\qquad \frac{1}{m_1(S_1)} \int_{S_1} f(|x|/|p(x)|)\, dm_1 \geq \frac{1}{m_2(S_2)} \int_{S_2} f(|p^{-1}(x)/|x|)\, dm_2 .$$

Proof. Though this is easily proved with the aid of Fubini's theorem, it also follows from the Tchebychef inequality [2, page 168, Theorem 236], which in the present case, applied to the functions $f(|x|/|p(x)|)$ and $\rho(x)$, yields the relation

$$(6)\quad \int_{S_2} 1\, dm_2 \int_{S_2} f(|p^{-1}(x)|/|x|)\rho(x)\, dm_2 \geq \int_{S_2} f(|p^{-1}(x)|/|x|)\, dm_2 \int_{S_2} \rho(x)\, dm_2 .$$

Since $\int_{S_2} 1\, dm_2 = m(S_2)$, $\int_{S_2} \rho(x)\, dm_2 = m_1(S_1)$, and

$$\int_{S_2} f(|p^{-1}(x)|/|x|)\rho(x)\, dm_2 = \int_{S_1} f(|x|/|p(x)|)\, dm_1 ,$$

(6) reduces to (5).

COROLLARY 3. *Let* g *be a function of type* C^2, *and let* $r = g(\theta)$ $(\alpha \leq \theta \leq \beta)$ *describe an arc* C *in the plane. Assume that* $g'(\theta) \geq 0$ *and* $g(\theta) + g''(\theta) \geq 0$ *for all* $\theta \in [\alpha, \beta]$, *and let* $n \geq 0$. *Then*

$$\frac{\int_C r^n\, ds}{\int_C ds} \geq \frac{\int_C r^n\, d\theta}{\int_C d\theta} .$$

Proof. In Theorem 2, let $S_1 = C$, and let S_2 be the radial projection of S_1 onto the unit circle. The function ρ is now $\sqrt{g^2 + (g')^2}$. A straightforward computation shows that ρ satisfies the hypothesis of Theorem 2.

Note that Corollary 3 applies to a line segment whose interior does not contain the foot of the perpendicular from the origin to it. It also applies to an arc of the pedal curve of a convex curve, if the sign of $g'(\theta)$ does not change on the arc. In this case, $g(\theta) + g''(\theta) \geq 0$, since g is the support function of a convex curve whose radius of curvature is $g(\theta) + g''(\theta)$.

3. A QUESTION

It may be that the analogue of Theorem 1 with reversed inequalities holds for support functions and nth-power functions f. For instance, let S_1 be a convex curve C in the plane, enclosing the origin, and of length L, and let S_2 be the unit circle. Let p be the support function of C relative to the origin, and let $n \geq 0$. Can we assert that

$$\frac{1}{L}\int_C p^n\,ds \leq \frac{1}{2\pi}\int_0^{2\pi} p^n\,d\theta\ ? \tag{7}$$

For $n = 1$, this is precisely the isoperimetric inequality, expressed in the form $2A/L \leq L/2\pi$. (Note that equality holds in this case if and only if C is a circle; its center need not be the origin.) As G. D. Chakerian has pointed out in a conversation, the analogue of (7) for $n = 1$ and higher dimensions holds, and it is a consequence of known inequalities.

A different inequality involving p^n was conjectured by Chern [1] for $d = 3$, namely $B_n^2 - A_n C_n \geq 0$, where $A_n = \int_S p^n\,dS$, $B_n = \int_S p^n H\,dS$ (H = mean curvature), and $C_n = \int_S p^n\,d\omega$.

REFERENCES

1. S.-S. Chern, *Some formulas in the theory of surfaces.* Bol. Soc. Mat. Mexicana 10 (1953), 30-40.

2. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Second Edition. Cambridge Univ. Press, Cambridge, 1952.

3. H. Sachs, *Ungleichungen für Umfang, Flächeninhalt und Trägheitsmoment konvexer Kurven.* Acta Math. Acad. Sci. Hungar. 11 (1960), 103-115.

4. F. A. Valentine, *Convex sets.* McGraw-Hill, New York, 1964.

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