ON THE ORDER OF A SIMPLY CONNECTED DOMAIN

Donald Sarason

In an earlier paper [5] dealing with a problem about bounded analytic functions, I was led to associate with each bounded simply connected domain in the plane a certain countable ordinal number, called the *order* of the domain. The question was left open whether there actually exist domains of all possible orders. The purpose of the present note is to answer this question in the affirmative. The appropriate construction turns out to be almost embarrassingly easy. It nevertheless seems worth presenting, because it provides easily visualized examples of an interesting phenomenon connected with weak-star topologies.

In Section 1, we recall the basic definitions and describe the construction. The application to weak-star topologies is given in Section 2.

1. Let G be a bounded, simply connected domain in the plane. The *Carathéodory hull* (or *$\mathscr{C}$-hull*) of G is by definition the interior of the polynomially convex hull of G, that is, the interior of the set of points z_0 in the plane such that for all polynomials p,

$$|p(z_0)| \leq \sup\{|p(z)|: z \in G\}.$$

The $\mathscr{C}$-hull of G can be described in purely topological terms as the complement of the closure of the unbounded component of the complement of the closure of G. The components of a $\mathscr{C}$-hull are always simply connected.

If E is a bounded, simply connected domain containing G, then the *relative hull of* G *in* E, or the E-*hull* of G, is by definition the interior of the set of points z_0 in E such that for all functions f bounded and analytic in E,

$$|f(z_0)| \leq \sup\{|f(z)|: z \in G\}.$$

Relative hulls can also be described in purely topological terms: the E-hull of G is the interior of the set of points in E that cannot be separated from G by a crosscut of E (see [5]). The components of a relative hull are always simply connected.

For each countable ordinal number α we now define a simply connected domain G^α containing G. First we let G^1 be the component of the $\mathscr{C}$-hull of G that contains G. We then proceed by induction, assuming G^β has been defined for all $\beta < \alpha$. If α is not a limit ordinal, we let G^α be the component of the $G^{\alpha-1}$-hull of G that contains G. If α is a limit ordinal, we let G^α be the component of the interior of $\bigcap_{\beta<\alpha} G^\beta$ that contains G. From the topological description of relative hulls given above, it follows that if the inclusion $G^{\alpha+1} \subset G^\alpha$ is proper, then $G^\alpha - G^{\alpha+1}$ has interior points. Hence the inclusion is proper for at most countably many ordinals α, so that there is a least ordinal γ for which $G^\gamma = G^{\gamma+1}$. The ordinal γ is called the *order* of G.

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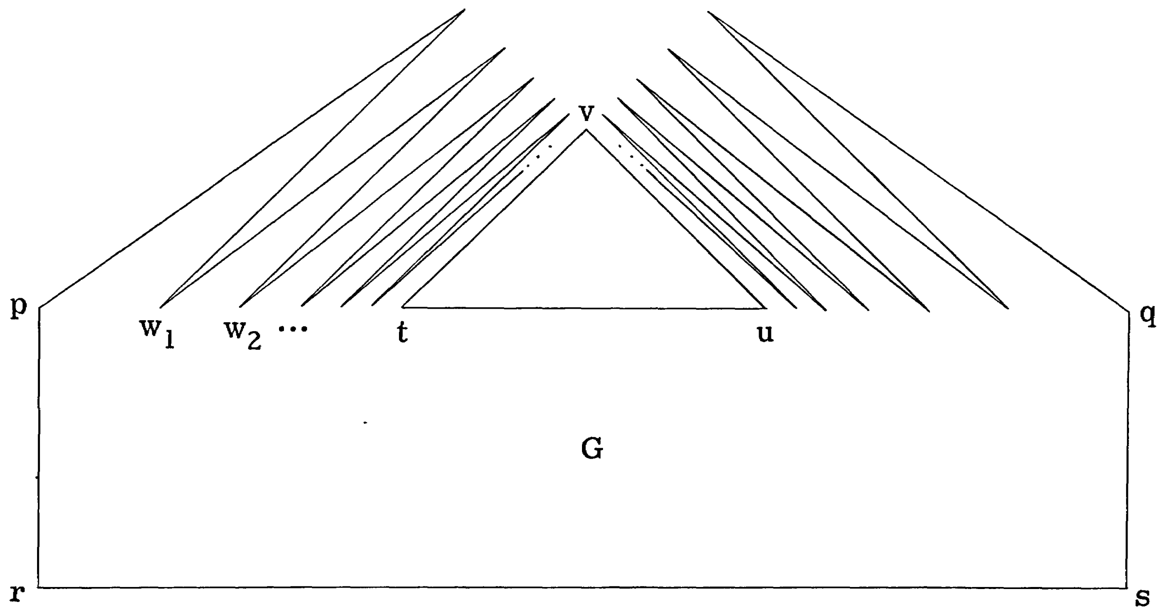

Figure 1.

In Figure 1 we depict a domain of order 2, and by modifying this domain we shall obtain domains of all possible orders. To avoid any possible ambiguities, I shall describe in detail how the domain of Figure 1 was constructed.

We start with three sides pr, rs, and sq of a rectangle prsq. On the missing side pq we draw in the closed middle third, tu. With tu as base we construct an isosceles triangle tuv, say with a height of one-half the length of its base. On the missing segment pt we choose a sequence $\{w_n\}_1^\infty$ converging monotonically to t, and we set $w_0 = p$. For each $n > 0$ we draw the segment parallel to tv connecting w_n with the extension of uv; we then connect the upper extremity of this segment with w_{n-1}. Similar segments are drawn on the other side of the triangle tuv. When we finish, we have drawn the boundary of a simply connected domain G. Clearly, G^1 is obtained from G by adding the open segment tu and the interior of the triangle tuv. Thus $G^2 = G$; that is, G has order 2.

The exact proportions used in constructing the domain of Figure 1 are not important; they were chosen for convenience in drawing. The important property is that the tips of the spikes on either side of the triangle tuv converge to the vertex v.

By an obvious modification of the domain of Figure 1 we can obtain a domain of order 3; such a domain is shown in Figure 2. It is only slightly more difficult to construct domains of arbitrary orders. I shall describe the construction for the case of a limit ordinal. This will suffice, because if G is a domain of order γ and α is an ordinal less than γ, then G^α has order α.

Let γ be a countable limit ordinal. As before, we start with three sides pr, rs, and sq of a rectangle prsq. We mark the midpoint, o, of the missing side pq. We assume the rectangle is positioned so that o is the origin of the plane and pq lies on the real axis (so that $p = -q$).

We now use the well-known fact that every countable well-ordered set can be realized as a subset of the line. Thus, we can choose on the open segment po a well-ordered set $A = \{a_\alpha\}_{\alpha<\gamma}$ of type γ (where the notation is chosen so as to make the map $\alpha \to a_\alpha$ order-preserving). We may assume, moreover, that o is a limit point of A, and that the order topology of A coincides with the topology that A inherits from the plane. For each $\alpha < \gamma$ we draw the segment with endpoints a_α

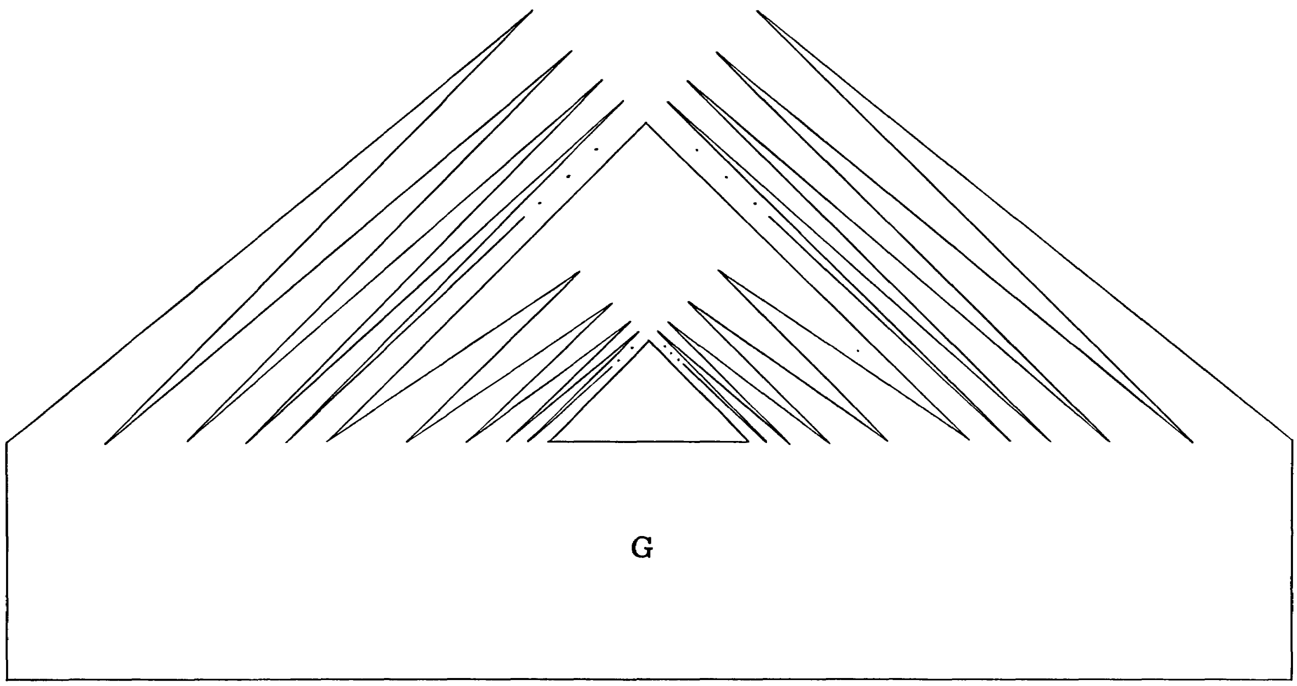

Figure 2.

and $-ia_\alpha$, and the segment with endpoints $-a_\alpha$ and $-ia_\alpha$. We obtain a transfinite sequence of tents, each covering its successors. Outside the first tent we draw spikes, as indicated in Figure 3, just as we did in constructing the domain of Figure 1. Between the αth and $(\alpha + 1)$st tents we draw more spikes, as indicated in Figure 4. We do this for each $\alpha < \gamma$, and we thus obtain the boundary of a simply connected domain G. A simple induction argument shows that for each $\alpha < \gamma$, the domain G^α is obtained by adding to G the open segment with endpoints a_α and $-a_\alpha$, and the interior of the triangle with vertices a_α, $-a_\alpha$, and $-ia_\alpha$. Thus $\bigcap_{\alpha<\gamma} G^\alpha = G \cup \{o\}$; that is, $G^\gamma = G$. Hence G has order γ, as desired.

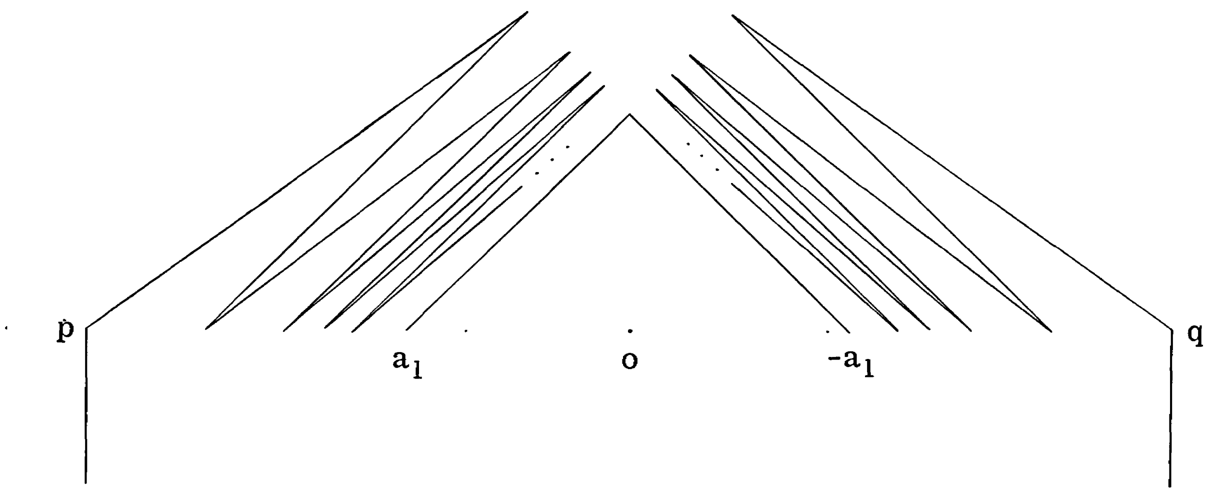

Figure 3.

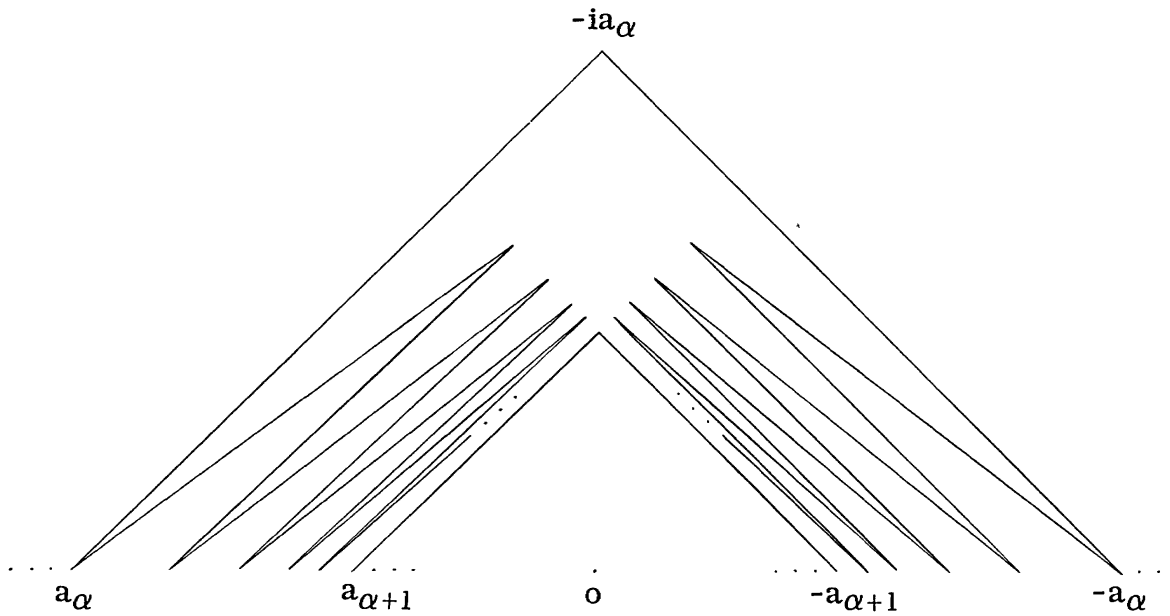

Figure 4.

2. Let B be a separable Banach space. A theorem of Banach states that for a linear manifold in B^* to be weak-star closed, it is necessary and sufficient that it be weak-star sequentially closed [1, p. 124]. However, if a linear manifold in B^* is not weak-star closed, then in order to form its weak-star closure it does not suffice, in general, merely to add to it all the limits of its weak-star convergent sequences. The first example of this phenomenon was given by Mazurkiewicz [3] in the space ℓ^1 ($= (c_0)^*$). To form the weak-star closure of a linear manifold M in B^*, one must take the union of the so-called derived sets of M. Namely, let $M^0 = M$, and for any countable ordinal number α, define M^α inductively to be the set of limits of weak-star convergent sequences in $\bigcup_{\beta<\alpha} M^\beta$. The manifold M^α is called the αth *derived set* of M. The manifold $\bigcup_{\alpha<\Omega} M^\alpha$ is clearly weak-star sequentially closed, and therefore it is the weak-star closure of M. Moreover, the manifolds M^α eventually become constant; that is there exists a least ordinal γ such that $M^\gamma = M^{\gamma+1}$ [1, p. 213]. Hence the weak-star closure of M is actually M^γ. The ordinal γ is called the *order* of M.

In [1, pp. 209-213], using a construction based partly on the one of Mazurkiewicz, Banach shows that there exist linear manifolds in ℓ^1 of all finite orders. He then goes on to assert more, namely, that there exist linear manifolds in ℓ^1 of arbitrarily high orders [1, p. 213]. However, the paper to which he refers for the proof was never published, and I know of no proof in the literature either of Banach's assertion itself or of the analogous assertion with ℓ^1 replaced by some other dual space. (A Fourier-analytic proof of Banach's assertion has recently been found by Carruth McGehee [4].)

The connection between all this and what is discussed above emerges when one takes B^* to be the space H^∞ of bounded analytic functions in the open unit disk. (The corresponding space B is then a certain quotient space of L^1 of the unit circle.) Let G be a bounded simply connected domain, and ϕ a conformal map of the unit disk onto G. Let M be the set of polynomials in ϕ. Then M is a linear manifold in H^∞, and the following theorem, proved in [5], describes explicitly the derived sets of M: *For each countable ordinal α, the manifold M^α consists precisely of all functions ψ in H^∞ such that $\psi \circ \phi^{-1}$ is the restriction of a function* *bounded and analytic in* G^α. In particular, the orders of M and G coincide. On the basis of the above construction, we can therefore assert that *there exist in* H^∞ *linear manifolds of all possible orders.*

The problem of proving a comparable theorem for other familiar dual spaces seems interesting. On the basis of the result for H^∞, it is easy to show that *in the space* ℓ^∞ $(= (\ell^1)^*)$, *there exist linear manifolds of all possible orders.* In fact, let $\{z_n\}_1^\infty$ be a dominating sequence for H^∞, that is, a sequence of points in the open unit disk with the property that

$$\sup_n |\psi(z_n)| = \sup_{|z|<1} |\psi(z)|$$

for all ψ in H^∞. (A dense sequence will do.) We can then map H^∞ isometrically into ℓ^∞ by sending a function ψ in the former onto the sequence $\{\psi(z_n)\}$ in the latter. It is easy to show that this map is a weak-star homeomorphism, so that it sends a linear manifold in H^∞ onto one in ℓ^∞ of the same order. The desired conclusion follows.

In closing, I raise two questions.

1. Can one construct in ℓ^∞ weak-star dense linear manifolds of all possible orders?

2. What orders occur for linear manifolds in the Hardy space H^1 of the unit circle? (This is the dual of a quotient space of the space of continuous functions on the circle; see [2, p. 137].)

REFERENCES

1. S. Banach, *Théorie des opérations linéaires.* Chelsea, New York, 1955.
2. K. Hoffman, *Banach spaces of analytic functions.* Prentice-Hall, Englewood Cliffs, N. J., 1962.
3. S. Mazurkiewicz, *Sur la dérivée faible d'un ensemble de fonctionnelles linéaires.* Studia Math. 2 (1930), 68-71.
4. O. C. McGehee, *A proof of a statement of Banach about the weak* topology.* Michigan Math. J. 15 (1968), 135-140.
5. D. Sarason, *Weak-star generators of* H^∞. Pacific J. Math. 17 (1966), 519-528.

University of California, Berkeley