ε-MAPPINGS AND GENERALIZED MANIFOLDS, II

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All spaces considered in this paper are compact metric spaces. A map $f: X \to Y$ of a space X onto Y is an ε-*map* ($\varepsilon > 0$) provided diam $f^{-1}(y) < \varepsilon$, for each $y \in Y$. If Π is a class of polyhedra, we say that X is Π-*like* provided for each $\varepsilon > 0$ there exists a polyhedron $P \in \Pi$ and an ε-mapping $f: X \to P$ onto P (P and f depend on ε) (see Definition 1 in [5]). By an n-*manifold* we mean a closed connected triangulable manifold of dimension n. We are interested in Π-like continua, where Π is a class of n-manifolds. The following is our main result.

THEOREM 1. *Let* X *be a* Π-*like* n-*dimensional absolute neighborhood retract, where* Π *is a class of* n-*manifolds. Then* X *is a locally orientable,* n-*dimensional generalized closed manifold over every principal ideal domain* L (n-gcm$_L$). *If* Π *is a class of orientable* n-*manifolds, then* X *is also orientable.*

For the definitions of these notions, see [6] and [1] (see also [9]).

Theorem 1 was proved in [6] for the case where Π is a class of orientable n-manifolds. The consequences stated there for the orientable case are now established without this restriction.

Theorem 1 follows from Theorem 1 in [6] and the following result.

THEOREM 2. *Let* X *be a* Π-*like,* n-*dimensional absolute neighborhood retract, where* Π *is a class of nonorientable* n-*manifolds* P. *Let* $\tilde{\Pi}$ *denote the class of orientable* n-*manifolds* $\tilde{P}$ *that are the* 2-*fold covering spaces of* P. *Then* X *admits a* 2-*fold covering space* $\tilde{X}$ *that is a* $\tilde{\Pi}$-*like continuum.*

Remark. Recall that every (triangulable) nonorientable n-manifold P has a uniquely determined 2-fold covering space $\tilde{P}$ that is a (triangulable) orientable n-manifold (see for example [7, pp. 271-272]).

To see that Theorem 2 and [6] imply Theorem 1, consider a Π-like, n-dimensional ANR X, where Π is a class of n-manifolds. By a theorem of T. Ganea [3], there exists an $\varepsilon > 0$ such that all ε-maps of X onto an n-manifold are homotopy equivalences. Therefore, there exists a subclass $\Pi_0 \subset \Pi$ each of whose members is of the same homotopy type as X, and X is Π_0-like. Consequently, either all manifolds in Π_0 are orientable, or all are nonorientable. In the first case, X must be an orientable n-gcm$_L$, by Theorem 1 of [6]. In the second case, we apply Theorem 2 to obtain a 2-fold covering space $\tilde{X}$ of X that is a $\tilde{\Pi}_0$-like continuum. The spaces $\tilde{X}$ and X are locally homeomorphic, and therefore $\tilde{X}$ inherits the local properties of X. By a theorem of K. Borsuk [2], a compact metric space is an n-dimensional ANR if and only if it is n-dimensional and locally contractible. Since the latter properties are local, we may conclude that $\tilde{X}$ is also an n-dimensional ANR. The class $\tilde{\Pi}_0$ consists of orientable n-manifolds, and so Theorem 1 of [6] implies that $\tilde{X}$ is an orientable (and hence locally orientable) n-gcm$_L$. Since local orientability is a local property, we conclude that X is a locally orientable n-gcm$_L$.

Received July 18, 1966.

During this research, S. Mardešić was visiting the University of Washington on leave from the University of Zagreb. Both authors were supported by National Science Foundation Grant NSFG - GP 3902.

The proof of Theorem 2 is based on the following lemma.

LEMMA. *Let* X *and* Y *be pathwise connected spaces, and let* $f: X \to Y$ *be a homotopy equivalence. Let* $(\tilde{X}, p)$ *and* $(\tilde{Y}, q)$ *be* m-*fold regular coverings* (m *finite*) *such that the isomorphism* $f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ $(y_0 = f(x_0))$ *of the fundamental groups satisfies*

$$(1) \qquad f_* p_* \pi_1(\tilde{X}, \tilde{x}_0) = q_* \pi_1(\tilde{Y}, \tilde{y}_0), \qquad \textit{where } \tilde{x}_0 \in p^{-1}(x_0) \textit{ and } \tilde{y}_0 \in q^{-1}(y_0).$$

Furthermore, let $\tilde{f}: \tilde{X} \to \tilde{Y}$ *be a map that lifts* f, *in other words, satisfies the condition* $q\tilde{f} = fp$. *Then*

(i) $\tilde{f}(\tilde{x}) \neq \tilde{f}(\tilde{x}')$ *if* $\tilde{x}, \tilde{x}' \in \tilde{X}$, $\tilde{x} \neq \tilde{x}'$, *and* $p(\tilde{x}) = p(\tilde{x}')$,

(ii) $\tilde{f}(\tilde{X}) = \tilde{Y}$ *if* $f(X) = Y$,

(iii) $H_n(\tilde{Y}; Z) \neq 0$ *implies* $H_n(\tilde{X}; Z) \neq 0$ (H_n *is the* n*th homology group, and* Z *denotes the group of integers*).

Proof. Notice that the subgroup $p_* \pi_1(\tilde{X}, \tilde{x}_0)$ is independent of the choice of the point $\tilde{x}_0 \in p^{-1}(x_0)$, since the covering space $(\tilde{X}, p)$ is regular. The same comment applies to $(\tilde{Y}, q)$. Furthermore, if (1) holds for one pair of base points x_0, $y_0 = f(x_0)$, the corresponding equality holds for any other choice of base points $x_1 \in X$, $y_1 = f(x_1) \in Y$.

In the proof of (i), we may therefore assume, without loss of generality, that $\tilde{x} = \tilde{x}_0$, $\tilde{x}' = \tilde{x}_0'$, and $p(\tilde{x}) = p(\tilde{x}') = x_0$. Let $\tilde{v}$ be a path in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_0'$, and let $v = p\tilde{v}$. Then v is a loop based at x_0. The element $\{v\}$ of the fundamental group $\pi_1(X, x_0)$ determined by v cannot belong to $p_* \pi_1(\tilde{X}, \tilde{x}_0)$, because the loop v lifts to the path $\tilde{v}$, which is not closed (see for example [4, p. 251]). By (1) and the fact that f_* is an isomorphism, we conclude that fv is a loop based at y_0 and that $\{fv\}$ does not belong to $q_* \pi_1(\tilde{Y}, \tilde{y}_0)$. Consequently, fv lifts to paths that are not closed. Since $\tilde{f}\tilde{v}$ is such a path, it follows that $\tilde{f}(\tilde{x}_0) \neq \tilde{f}(\tilde{x}_0')$.

To prove (ii), consider any point $\tilde{y} \in \tilde{Y}$, and let $y = q(\tilde{y})$. Since f maps X onto Y, there is a point $x \in X$ such that $f(x) = y$. By (i), the map $\tilde{f}$ takes $p^{-1}(x)$ into $q^{-1}(y)$ in a one-to-one manner, and since these two sets are both of finite cardinality m, it follows that

$$\tilde{y} \in q^{-1}(y) = \tilde{f} p^{-1}(x) \subset \tilde{f}(\tilde{X}).$$

Proof of (iii). Let $g: Y \to X$ be a homotopy inverse of f. There is no loss of generality in assuming that $g(y_0) = x_0$ (for if $g(y_0) \neq x_0$, the homotopy extension theorem yields a map $g' \simeq g$ with $g'(y_0) = x_0$). The induced homomorphism $g_*: \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is then the inverse of f_*, and therefore

$$g_* q_* \pi_1(\tilde{Y}, \tilde{y}_0) = p_* \pi_1(\tilde{X}, \tilde{x}_0).$$

Therefore we can lift g to a map $\tilde{g}: \tilde{Y} \to \tilde{X}$. The composite map $\tilde{f}\tilde{g}: \tilde{Y} \to \tilde{Y}$ is a lifting of the map $fg: Y \to Y$. Since $fg \simeq 1$, the covering homotopy theorem (see [8, p. 50]) yields a homotopy between $\tilde{f}\tilde{g}$ and a map $\tilde{i}: \tilde{Y} \to \tilde{Y}$ that lifts the identity $i: Y \to Y$. Denoting by $\tilde{f}_{*n}$, $\tilde{g}_{*n}$, $\tilde{i}_{*n}$ the induced homomorphisms on the nth homology groups, we obtain

$$(2) \qquad \tilde{f}_{*n} \tilde{g}_{*n} = \tilde{i}_{*n}.$$

It readily follows from (i) and (ii) that every lifting of the identity $i: Y \to Y$ is a homeomorphism, and therefore $\tilde{i}_{*n}: H_n(\tilde{Y}; Z) \to H_n(\tilde{Y}, Z)$ is an isomorphism. Since $H_n(\tilde{Y}; Z) \neq 0$, (2) implies that $\tilde{g}_{*n} \neq 0$, and therefore $H_n(\tilde{X}; Z) \neq 0$.

Proof of Theorem 2. Let Π be a class of nonorientable n-manifolds, and let X be a Π-like n-dimensional ANR. By Theorem 1 of [5], X is the inverse limit of an inverse sequence $\{X_i; f_{ij}\}$ $(i = 1, 2, \cdots)$ of nonorientable n-manifolds $X_i \in \Pi$ with bonding maps $f_{ij}: X_j \to X_i$ mapping onto X_i $(i \leq j)$. The projections $f_i: X \to X_i$ map X onto X_i and satisfy

$$f_{ij}f_j = f_i \quad (i \leq j). \tag{3}$$

For each $\varepsilon > 0$, the projections $f_i: X \to X_i$ are ε-maps for sufficiently large i. Therefore, in view of the previously mentioned result of Ganea [3], there is no loss of generality in assuming that all the projections f_i are homotopy equivalences. It then follows from (3) that all the maps f_{ij} are also homotopy equivalences. If $x_0 \in X$ and $x_i = f_i(x_0) \in X_i$, then

$$f_{ij}(x_j) = x_i \quad (i \leq j),$$

and the maps f_{ij} and f_i induce isomorphisms

$$f_{ij*}: \pi_1(X_j, x_j) \to \pi_1(X_i, x_i) \ (i \leq j) \quad \text{and} \quad f_{i*}: \pi_1(X, x_0) \to \pi_1(X_i, x_i), \text{ respectively}.$$

Since X_1 is a (triangulable) nonorientable n-manifold, there exists a unique 2-fold covering space $(\tilde{X}_1, p_1)$ of X_1 such that $\tilde{X}_1$ is a (triangulable) orientable n-manifold.

Let $N_1 = p_{1*}\pi_1(\tilde{X}_1, \tilde{x}_1)$, where $\tilde{x}_1 \in p_1^{-1}(x_1)$. Note that since the covering space is 2-fold, it is regular.

Define subgroups $N_i \subset \pi_1(X_i, x_i)$ and $N_0 \subset \pi_1(X, x_0)$ by

$$N_i = f_{1i*}^{-1}(N_1), \qquad N_0 = f_{1*}^{-1}(N_1). \tag{4}$$

We now construct covering spaces $(\tilde{X}_i, p_i)$ of $\tilde{X}_i$ $(i \geq 2)$ and $(\tilde{X}, p)$ of X, respectively, in such a way that

$$p_{i*}\pi_1(\tilde{X}_i, \tilde{x}_i) = N_i \quad (i \geq 2),$$

$$p_*\pi_1(\tilde{X}, \tilde{x}_0) = N_0.$$

These covering spaces are 2-fold (and hence regular), because the subgroups N_i, N_0 are of index 2.

It follows from (4) that the maps f_{ij} and f_i can be lifted to maps $\tilde{f}_{ij}: \tilde{X}_j \to \tilde{X}_i$ and $\tilde{f}_i: \tilde{X} \to \tilde{X}_i$ such that $\tilde{f}_{ij}(\tilde{x}_j) = \tilde{x}_i$ and $f_i(\tilde{x}_0) = \tilde{x}_i$. We thus obtain an inverse sequence $\{\tilde{X}_i; \tilde{f}_{ij}\}$ $(i = 1, 2, \cdots)$.

It follows from (ii) of the lemma that $\tilde{f}_{ij}$ and $\tilde{f}_i$ are mappings onto $\tilde{X}_i$. Since $\tilde{X}_i$ is a 2-fold covering of an n-manifold, it follows that it is also an n-manifold. To prove that $\tilde{X}_i$ is orientable, it suffices to show that $H_n(\tilde{X}_i; Z) \neq 0$. This follows by induction, if we use (iii) of the lemma and the fact that $\tilde{X}_1$ is an orientable n-manifold. Since $X_i \in \Pi$, we now see that $\tilde{X}_i \in \tilde{\Pi}$.

The proof of Theorem 2 will be complete if we show that $\tilde{X}$ is the inverse limit of the sequence $\{\tilde{X}_i; \tilde{f}_{ij}\}$. First, we notice that the maps $\tilde{f}_{ij}\tilde{f}_j$ and $\tilde{f}_i$ both lift the map $f_{ij}f_j = f_i$. Since $\tilde{f}_{ij}\tilde{f}_j(\tilde{x}_0) = \tilde{f}_i(\tilde{x}_0)$, it follows that

$$\tilde{f}_{ij}\tilde{f}_j = \tilde{f}_i \qquad (i \leq j).$$

Furthermore, we already know that the maps $\tilde{f}_i$ are onto and that $\tilde{X}$ is a compactum. So we need only show that the maps $\tilde{f}_i$ distinguish points of $\tilde{X}$. Let $\tilde{x}, \tilde{x}' \in \tilde{X}$ be such that

$$\tilde{f}_i(\tilde{x}) = \tilde{f}_i(\tilde{x}') \qquad (i = 1, 2, \cdots).$$

Since $p_i\tilde{f}_i = f_i p$, we conclude that

$$f_i(p\tilde{x}) = f_i(p\tilde{x}') \qquad (i = 1, 2, \cdots),$$

and therefore $p(x) = p(x')$. Now (i) of the lemma implies that $\tilde{x} = \tilde{x}'$. This concludes the proof of Theorem 2.

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