TWO HOMOMORPHIC BUT NONISOMORPHIC MINIMAL SETS

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Let (X, T, π) be a transformation group with compact Hausdorff phase space [5]. We say that (X, T, π) is a minimal set if and only if for every point x in X the orbit closure $\overline{O(x, T)} = \overline{\{xt: t \in T\}}$ is the space X. The classification of minimal sets is one of the important problems in topological dynamics. Although significant progress has recently been made [4], the problem is far from solved. In a forthcoming paper [2], J. Auslander classifies the minimal sets by means of homomorphisms. A homomorphism $\theta: (X, T) \to (Y, T)$ is a continuous map from X into Y such that $x\theta t = xt\theta$ for all $t \in T$, $x \in X$. In regard to such classifications, the following question naturally arises: *If* (X, T) *and* (Y, T) *are compact minimal sets having homomorphisms* $\theta: (X, T) \to (Y, T)$ *and* $\phi: (Y, T) \to (X, T)$, *does there exist an isomorphism from* (X, T) *onto* (Y, T)?

In this note, we shall show by an example that the answer to this question is negative. Our minimal sets are based on minimal sets given by R. Ellis (see [4, Example 4] or [1, p. 613]); we shall describe these first.

1. Let Y denote the additive group of real numbers modulo 1, let Y_1 and Y_2 be two disjoint copies of Y, and let $X = Y_1 \cup Y_2$. For each $y \in Y$, corresponding points in Y_1 and Y_2 will be written as (y, 1) and (y, 2), respectively. Topologize X by specifying an open-closed neighborhood system for each point. If $\varepsilon > 0$, let

$$N_\varepsilon(y, 1) = \{(y+t, 1): 0 \leq t \leq \varepsilon\} \cup \{(y+t, 2): 0 < t < \varepsilon\}$$

be an open-closed neighborhood of $(y, 1) \in Y_1$, and let

$$N_\varepsilon(y, 2) = \{(y+t, 2): 0 \geq t \geq -\varepsilon\} \cup \{(y+t, 1): 0 > t > -\varepsilon\}$$

be an open-closed neighborhood of $(y, 2) \in Y_2$. For i = 1, 2, let $\tau: X \to X$ be defined by the formula $(y, i)\tau = (y + \alpha, i)$, where α is a real number. Then τ is a self-homomorphism of X.

1.1 LEMMA. *Let* T *denote the group generated by* τ *and topologized by the discrete topology. Then*

(X, T) *is a transformation group, and*

X *is a compact, separable Hausdorff space satisfying the first countability axiom.*

1.2 *Definition.* Let us call the real number α associated with τ the *rotation constant* of τ (or of T). Then (X, T) is a minimal set when the rotation constant is an irrational number. In this case, two points u and v in Y are proximal [4] if and only if u = (y, i) and v = (y, j) for some $y \in Y$.

Now we shall proceed to construct our minimal sets.

2. Let α, β, and γ be three real numbers such that α, β, γ, and 1 are rationally independent. Let (X, T) be the Ellis minimal set with rotation constant α, and let

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(A, T) be the Ellis minimal set with rotation constant β. In order to distinguish the elements of X and A, we write (a, i) and (b, j) (i, j = 1, 2) for the elements of A, where a and b belong to the circle group B (equivalently, to an isomorphic disjoint copy of Y).

2.1 Let T denote an infinite discrete cyclic group generated by t. For points in $X \times A$, we use the symbol [x, a] instead of (x, a), and we define

$$[(y, i), (a, j)]t = [(y + \alpha, i), (a + \beta, j)].$$

Then $(X \times A, T)$ is a compact minimal set. The proof of minimality is easy if we use the fact α and β are rationally independent.

2.2 For any $y \in Y$ and $a \in B$, the points

$$[(y, 1), (a, 1)], \quad [(y, 1), (a, 2)], \quad [(y, 2), (a, 1)], \quad [(y, 2), (a, 2)]$$

are proximal. This can be proved either directly or by the use of results in [6] and [3].

2.3 Define $[(y, i), (a, j)]\rho = [(y + \gamma, i), (a + \gamma, j)]$ for $[(y, i), (a, j)] \in X \times A$. Then ρ is a self-isomorphism of $(X \times A, T)$.

2.4 Choose fixed elements $y \in Y$, $a \in B$. Let R_1 denote the relation on $X \times A$ induced by identifying each of the four points

$$[(y, 1), (a, 1)]\rho^m t^n, \quad [(y, 2), (a, 1)]\rho^m t^n, \quad [(y, 1), (a, 2)]\rho^m t^n,$$

$$[(y, 2), (a, 2)]\rho^m t^n \qquad (m \geq 1, \ n \in I).$$

Let R_2 denote the relation induced by identifying the points of each pair $[(y, 1), (a, 1)]\rho^m t^n$ and $[(y, 2), (a, 1)]\rho^m t^n$ $(m \geq 0, \ n \in I)$.

2.5 LEMMA. $[(y, i), (a, j)]\rho^m = [(y, k), (a, \ell)]t^n$ *if and only if* $m = 0$, $n = 0$, $i = k$, $j = \ell$.

Proof. $[(y, i), (a, j)]\rho^m = [(y + m\gamma, i), (a + m\gamma, j)] = [(y, k), (a, \ell)]t^n$
$= [(y + n\alpha, k), (a + n\beta, \ell)]$.

Since α, β, and γ are rationally independent, the assertion follows.

2.6 LEMMA. R_1 *is a closed* T-*invariant equivalence relation.*

Proof. It is easy to show that R_1 is a T-invariant equivalence relation. Corresponding to any convergent sequence $\{X_n\}$ with

$$X_n \in R_1 \subseteq (X \times A) \times (X \times A),$$

we consider the following two cases.

Case 1. There exists a subsequence of $\{X_n\}$ whose terms do not belong to the set

$$W = \{[(y, i), (a, j)]\rho^m t^n \colon m \geq 1, \ n \in I\}.$$

It is clear that the limit of this sequence is in the diagonal of $(X \times A) \times (X \times A)$.

Case 2. $X_n \in W$, except for finitely many indices. Here we may assume that $X_n \in W$ for all n. There exists a subsequence that (after reindexing) can be written as $([(z_n, i), (b_n, j)], [(z_n, k), (b_n, \ell)])$. If there exists a subsequence of $\{z_n\}$ whose terms are equal to a constant, then by 2.5 the corresponding b_n is also constant, and *a fortiori* the limit is in R_1. If all z_n are distinct, except for possibly finitely many indices, then so are the b_n. If z_n converges to z and b_n converges to b, then the limit of $\{x_n\}$ is $([(z, p), (b, q)], [(z, p), (b, q)])$, where p and q are equal to 1 or 2, depending on how the z_n converge to z (see [1, p. 613]). Thus we know that R_1 is closed.

2.7 LEMMA. *R_2 is a closed T-invariant equivalence relation.*

Proof. Use the same technique as for 2.6.

2.8 LEMMA. *If $R_0 = R_1 \cup R_2$, then R_0 is a closed T-invariant equivalence relation.*

Proof. Use Lemma 2.5 and straightforward computation. One can show that R_0 is an equivalence relation. The other assertions follow from Lemma 2.6 and Lemma 2.7.

2.9 LEMMA. *$R_0\rho \subseteq R_1$, where it is understood that*

$$([(x, i), (b, j)], [(x', i'), (b', j')])\rho = ([(x, i), (b, j)]\rho, [(x', i'), (b', j')]\rho).$$

The proof is obvious.

2.10 THEOREM. *$\left(\frac{X \times A}{R_0}, T\right)$, $\left(\frac{X \times A}{R_1}, T\right)$ are minimal sets. There exist homomorphisms*

$$\theta: \left(\frac{X \times A}{R_0}, T\right) \rightarrow \left(\frac{X \times A}{R_1}, T\right) \quad \text{and} \quad \phi: \left(\frac{X \times A}{R_1}, T\right) \rightarrow \left(\frac{X \times A}{R_0}, T\right),$$

but the two minimal sets are not isomorphic.

Proof. The first assertion is clear. The inclusion relation $R_1 \subseteq R_0$ induces a homomorphism

$$\theta: \left(\frac{X \times A}{R_0}, T\right) \rightarrow \left(\frac{X \times A}{R_1}, T\right).$$

Because $R_0\rho \subseteq R_1$, ρ induces a homomorphism

$$\phi: \left(\frac{X \times A}{R_1}, T\right) \rightarrow \left(\frac{X \times A}{R_0}, T\right).$$

$\left(\frac{X \times A}{R_0}, T\right)$ and $\left(\frac{X \times A}{R_1}, T\right)$ are not isomorphic, since $\left(\frac{X \times A}{R_0}, T\right)$ contains a point z such that the cardinality of P(z) is 3, whereas the cardinality of P(z) is 1 or 4 if $z \in \frac{X \times A}{R_1}$.

REFERENCES

1. J. Auslander, *Endomorphisms of minimal sets,* Duke Math. J. 30 (1963), 605-614.

2. ———, *Regular minimal sets, I,* (to appear).

3. J. P. Clay, *Proximity relations in transformation groups,* Trans. Amer. Math. Soc. 108 (1963), 88-96.

4. R. Ellis and W. H. Gottschalk, *Homomorphisms of transformation groups,* Trans. Amer. Math. Soc. 94 (1960), 258-271.

5. W. H. Gottschalk and G. A. Hedlund, *Topological dynamics,* Amer. Math. Soc. Colloquium Publications 36, American Mathematical Society, Providence, R. I. (1955).

6. T. S. Wu, *Proximal relations in topological dynamics,* Proc. Amer. Math. Soc. 16 (1965), 513-514.

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