SUMS OF SMALL NUMBERS OF IDEMPOTENTS

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1. INTRODUCTION

Recently, Stampfli [16] showed that every (bounded, linear) operator on a separable, infinite-dimensional Hilbert space $\mathscr{H}$ is the sum of 8 idempotent operators. Using this fact, Fillmore [8] gave an elementary proof that every operator on $\mathscr{H}$ is the sum of 64 operators each having square zero, and he also showed that every operator is a linear combination of 257 projections (that is, Hermitian idempotents). These results at first seem somewhat surprising, and since the proofs involve some rather intricate constructions, they do not clearly reveal why the theorems are true.

It is the purpose of this paper to introduce techniques that seem to provide a more natural way of looking at such questions. Using these techniques, which are based partly on the theory of commutators [2], [9], we are able to improve the above-mentioned results considerably, and at the same time to give arguments that are relatively transparent.

In Section 2, we show that most operators on $\mathscr{H}$ (more precisely, every operator in class (F) of [2]) can be written as the sum of four idempotents, and that every operator on $\mathscr{H}$ can be written as the sum of five idempotents. We also show that every operator on $\mathscr{H}$ is the sum of five operators having square zero, and that every Hermitian operator on $\mathscr{H}$ is a real linear combination of eight projections. All of these results remain valid, moreover, on nonseparable spaces.

In Section 3 we take up the question as to which of these constructions can be carried out in the framework of von Neumann algebras, and we show that essentially all of the above results are valid in every properly infinite von Neumann algebra (that is, in every algebra without direct summand of finite type). Our sharpest results are obtained in the case of a factor of type III acting on a separable Hilbert space, where our knowledge of commutators is complete [3]. In such a factor, every nonscalar operator can be expressed as the sum of four idempotents, and also as the sum of four operators each having square zero.

Finally, in Section 4 we consider a certain class of von Neumann algebras, and we show that each algebra of the class can be generated by various small sets of special operators. This is related to the result of Davis [4] that a I_∞-factor on a separable space can be generated by three projections.

2. IDEMPOTENTS, SQUARE-ZERO OPERATORS, AND PROJECTIONS

Throughout Sections 2 and 3, $\mathscr{H}$ will denote a complex, infinite-dimensional, but not necessarily separable, Hilbert space. We denote by $\mathscr{L}(\mathscr{H})$ the algebra of all bounded, linear operators on $\mathscr{H}$. As in [2], we denote by (F) the class of operators obtained by removing from $\mathscr{L}(\mathscr{H})$ all operators of the form $\lambda I + C$, where λ is a

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scalar and C belongs to the unique maximal norm-closed ideal of $\mathscr{L}(\mathscr{H})$. (For a separable space $\mathscr{H}$, this ideal is precisely the ideal of compact operators.)

Throughout this section, we suppose that $\mathscr{H} = \mathscr{K} \oplus \mathscr{K}$, where $\mathscr{K}$ is a fixed Hilbert space, and we remind the reader that this decomposition leads to an isomorphism between $\mathscr{L}(\mathscr{H})$ and the 2×2 matrix algebra over $\mathscr{L}(\mathscr{K})$. In particular, every operator on $\mathscr{H}$ can be viewed as a 2×2 matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with entries from $\mathscr{L}(\mathscr{K})$ and acting on $\mathscr{K} \oplus \mathscr{K} = \mathscr{H}$ in the usual fashion.

We shall first prove that every operator of class (F) on $\mathscr{H}$ can be written as the sum of four idempotents; for this, the following lemma is needed.

LEMMA 2.1. *Every operator* T *of class* (F) *on* $\mathscr{H}$ *is similar to a matrix*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A + D$ *is of class* (F) *on* $\mathscr{K}$.

Proof. By Corollary 3.4 of [2], T is similar to an operator of the form

$$\begin{pmatrix} A & V \\ B & 0 \end{pmatrix},$$

where V is an isometry on $\mathscr{K}$ of deficiency equal to the dimension of $\mathscr{K}$. If A is already of class (F), the proof is complete. If not, we have the equation

$$\begin{pmatrix} I & 0 \\ -V^* & I \end{pmatrix}\begin{pmatrix} A & V \\ B & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ V^* & I \end{pmatrix} = \begin{pmatrix} A + VV^* & V \\ * & -I \end{pmatrix}.$$

Since V has deficiency equal to the dimension of $\mathscr{K}$, it is clear that $VV^* - I$ is of type (F), from which it follows that $A + (VV^* - I)$ is of type (F).

THEOREM 1. *Every operator on* $\mathscr{H}$ *of class* (F) *is the sum of four idempotents, and every operator on* $\mathscr{H}$ *is the sum of five idempotents.*

Proof. Let T be written as the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

and consider first the case that T is of class (F). It obviously suffices to show that some operator similar to T is the sum of four idempotents. By Lemma 2.1, we may assume that $A + D$ is of class (F). We now invoke Theorem 4 of [2] to exhibit $A + D - 4I$ (which is also of class (F)) as a commutator, say $A + D - 4I = XY - YX$. Next we define

$$Z = XY - A + 2I, \qquad U = B - X + Z - I, \qquad V = C + (YX - I)Y - Z.$$

Then the equation

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} XY & X \\ (I - YX)Y & I - YX \end{pmatrix} + \begin{pmatrix} I - Z & I - Z \\ Z & Z \end{pmatrix} + \begin{pmatrix} I & U \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ V & I \end{pmatrix}$$

represents T as the sum of four idempotents.

Turning now to the case that T is arbitrary, we may suppose that $A + D$ fails to be of class (F), for otherwise the above reasoning applies. Let E be a projection on $\mathscr{K}$ whose null space and range each have the same dimension as $\mathscr{K}$. Then E is an operator of class (F), as is $A + D - E$. Writing

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A - E & B \\ C & D \end{pmatrix} + \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix},$$

we see that the first term on the right can be expressed as the sum of four idempotents, and that the last term is a projection.

Theorem 1 comes close to being best possible, because Stampfli has shown [16, Theorem 1] that not every operator can be written as the sum of three idempotents. It seems likely that every operator is the sum of four idempotents, but we have been unable to prove this for operators outside the class (F).

Next we examine the situation arising when operators of square zero are summed.

THEOREM 2. *Every operator on $\mathscr{H}$ of class* (F) *is the sum of four operators each having square zero, and every operator on $\mathscr{H}$ is the sum of five such operators.*

Proof. As in the proof of Theorem 1, let T be the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

and consider first the case that $T \in (F)$. Applying a similarity transformation, if necessary, we may (by Lemma 2.1) assume that $A + D$ is of class (F). By Theorem 4 of [2], we may choose operators X and Y such that $A + D = XY - YX$. Next we define

$$Z = XY - A, \qquad U = B + Z - X, \qquad V = C + YXY - Z.$$

Then the equation

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} XY & X \\ -YXY & -YX \end{pmatrix} + \begin{pmatrix} -Z & -Z \\ Z & Z \end{pmatrix} + \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix}$$

represents T as the sum of four operators each having square zero.

Turning to the case that T is arbitrary, we may suppose that $A + D$ is not of class (F). Since the space $\mathscr{K}$ is infinite-dimensional, we can choose a partial isometry W on $\mathscr{K}$ whose initial and final spaces are the orthogonal complements of each other. Then $W^2 = 0$, and it is easy to see that W must be of class (F) on $\mathscr{K}$. Furthermore, the operator

$$\begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}$$

is of class (F) on $\mathscr{H}$, and its square is zero. Now

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A - W & B \\ C & D \end{pmatrix} + \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix},$$

and since $A + D - W$ is of class (F), the first part of the proof allows us to write the first term on the right as the sum of four operators whose squares are zero, as required.

The idempotent operators appearing in Theorem 1 are not generally Hermitian. If one considers a similar problem for projections, one must clearly treat linear combinations instead of sums, since the sum of any number of projections is positive semidefinite. We proceed now to show that $\mathscr{L}(\mathscr{H})$ is the complex linear span of its projections. Our basic tool is again commutator theory, but the following lemma is used to solve some operator equations.

LEMMA 2.2. *Let* B *be any Hermitian operator with* $\|B\| < 1/2$. *Then there exists a Hermitian contraction* A *such that* $A(I - A^2)^{1/2} = B$, *where* $(I - A^2)^{1/2}$ *is the positive semidefinite square root of* $I - A^2$. *Furthermore,* A *can be chosen from any von Neumann algebra containing* B.

Proof. Elementary calculus shows that the function $f(x) = x(1 - x^2)^{1/2}$ is strictly increasing on the interval $[-1/\sqrt{2}, 1/\sqrt{2}]$, and clearly $f(-1/\sqrt{2}) = -1/2$, $f(1/\sqrt{2}) = 1/2$. It follows that there exists a real-valued continuous function $g(x)$ defined on the interval $[-1/2, 1/2]$ such that for $x \in [-1/2, 1/2]$, $f(g(x)) = x$. Define $A = g(B)$, and note that A is a Hermitian contraction belonging to any von Neumann algebra containing B. Since $f(A) = f(g(B)) = B$, the result follows.

THEOREM 3. *Every Hermitian operator on* $\mathscr{H}$ *is a real linear combination of eight projections.*

Proof. Any Hermitian operator T on $\mathscr{H}$ can be written as a matrix

$$\begin{pmatrix} K & M^* \\ M & L \end{pmatrix}$$

on $\mathscr{K} \oplus \mathscr{K}$ with K and L Hermitian.

We shall construct eight contraction operators $A_1, \cdots, A_8$, which we shall then use to build the eight projections needed for the proof. For later notational convenience, we formally define $S_i = (I - A_i A_i^*)^{1/2}$ $(i = 1, \cdots, 8)$.

By Theorem 1 of [9], it is possible to choose operators B_1 and B_2 such that

$$(B_1^* B_1 - B_1 B_1^*) + (B_2 B_2^* - B_2^* B_2) = K + L.$$

Now choose $\lambda > \max(\|B_1\|, \|B_2\|)$, define $A_i = \lambda^{-1} B_i$ $(i = 1, 2)$, and note that $\|A_i\| \leq 1$ $(i = 1, 2)$. Letting

$$H = K - \lambda^2(A_1^* A_1 - A_2^* A_2),$$

we decompose $H = H^+ - H^-$ into its positive and negative semidefinite parts, and we choose $\mu > \max(\|H^+\|, \|H^-\|)$. Define $A_7 = (\mu^{-1} H^+)^{1/2}$, $A_8 = (\mu^{-1} H^-)^{1/2}$, and note that A_7 and A_8 are positive contractions.

Now consider the operator

$$C = M - \lambda^2(S_1 A_1 - S_2 A_2) - \mu(S_7 A_7 - S_8 A_8),$$

and let $C = C_1 + iC_2$ be the decomposition of C into its real and imaginary parts. Choose $\sigma > \max(\|C_1\|, \|C_2\|)$. Then by Lemma 2.2 there exists a Hermitian contraction A_3 such that $S_3 A_3 = (2\sigma)^{-1} C_1$, and a skew-Hermitian contraction A_5 such that $S_5 A_5 = (2\sigma)^{-1} i C_2$. Finally, define $A_4 = -A_3$ and $A_6 = A_5^*$.

We are now ready to determine the real coefficients needed to form the linear combination in question. These are

$$\alpha_1 = \lambda^2, \quad \alpha_2 = -\lambda^2, \quad \alpha_3 = \alpha_5 = \sigma, \quad \alpha_4 = \alpha_6 = -\sigma, \quad \alpha_7 = \mu, \quad \alpha_8 = -\mu.$$

As was pointed out earlier, each A_i $(i = 1, \cdots, 8)$ is a contraction, and if we define $T_i = (I - A_i^* A_i)^{1/2}$ $(i = 1, \cdots, 8)$, then it is not hard to see that $S_i A_i = A_i T_i$ $(i = 1, \cdots, 8)$. Using this last relation, we can easily verify that each operator

$$E_i = \begin{pmatrix} A_i^* A_i & A_i^* S_i \\ S_i A_i & S_i^2 \end{pmatrix}$$

is a projection. Assembling the above information, we obtain the equations

$$\sum_{i=1}^{8} \alpha_i A_i^* A_i = K, \qquad \sum_{i=1}^{8} \alpha_i S_i^2 = L, \qquad \sum_{i=1}^{8} \alpha_i S_i A_i = M,$$

and by taking the adjoint of the third equation, we see that

$$\sum_{i=1}^{8} \alpha_i A_i^* S_i = M^*,$$

which shows that $T = \sum_{i=1}^{8} \alpha_i E_i$, as desired.

COROLLARY 2.3. *Every operator on* $\mathcal{H}$ *is a complex linear combination of sixteen projections.*

3. PROPERLY INFINITE VON NEUMANN ALGEBRAS

In this section we shall extend most of the results of the preceding section to operators in an arbitrary properly infinite von Neumann algebra. (A von Neumann algebra $\mathscr{A}$ is *properly infinite* if it contains no nonzero finite central projection. For the general theory of von Neumann algebras, see [5].) If $\mathscr{A}$ is properly infinite, then [11, Corollary, p. 41] there exists a projection $E \in \mathscr{A}$ such that $I \sim E \sim I - E$. Moreover, if E is any such projection, the projection $I - E$ can be written as the sum $I - E = F_1 + F_2 + \cdots$ of an infinite sequence of orthogonal projections $F_n \in \mathscr{A}$, each equivalent to E. Suppose now that a properly infinite algebra $\mathscr{A}$ and a projection $E \in \mathscr{A}$ with the above properties are fixed. It is standard algebra that the equivalence $E \sim I - E$ can be used to obtain a spatial isomorphism ϕ of $\mathscr{A}$ onto the 2×2 matrix algebra $M_2(E\mathscr{A}E)$ over $E\mathscr{A}E$, and we suppose this done. (Strictly speaking, to fix such an isomorphism, one must also specify a partial isometry implementing the above equivalence.) In addition, the relations

$$E + F_1 + F_2 + \cdots = I \quad \text{and} \quad E \sim F_1 \sim F_2 \sim \cdots$$

can be used (together with an appropriate choice of partial isometries) to construct a spatial isomorphism ψ of $\mathscr{A}$ onto the algebra $M_\infty(E\mathscr{A}E)$ of all $\aleph_0 \times \aleph_0$ matrices over $E\mathscr{A}E$ that act as bounded operators; we suppose this done. (Since $E \sim I$, $E\mathscr{A}E$ is spatially isomorphic to $\mathscr{A}$ and thus is also properly infinite; but we do not yet use this information.) The principal fact we need at present regarding ϕ and ψ is that if $T \in \mathscr{A}$ is carried by ϕ onto a 2×2 matrix in $M_2(E\mathscr{A}E)$ of the form

$$\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix},$$

then T is carried by ψ onto a matrix of the form

$$(*) \qquad \begin{pmatrix} A_1 & 0 & 0 & \cdots \\ B_1 & 0 & 0 & \cdots \\ B_2 & 0 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{pmatrix}$$

The following lemma gets our program under way.

LEMMA 3.1. *Let $\mathscr{A}$ be a properly infinite von Neumann algebra, and let $T \in \mathscr{A}$. Then there exist operators $A, B, C, D \in \mathscr{A}$ such that*

$$T = (AB - BA) + (CD - DC).$$

Proof. As was pointed out above, we may write T as the sum of two operators $T = T_1 + T_2$, where

$$\phi(T_1) = \begin{pmatrix} K & 0 \\ L & 0 \end{pmatrix} \quad \text{and} \quad \phi(T_2) = \begin{pmatrix} 0 & M \\ 0 & N \end{pmatrix}.$$

The isomorphism ψ carries the operator T_1 onto a matrix of the form (*), and the construction in Section 1 of [15] can be applied to this matrix to show that it is a commutator of operators from the algebra. The operator T_2 can be handled similarly, and the proof is complete.

We are now prepared to prove the analogues of Theorems 1 and 2 in properly infinite von Neumann algebras, but the incompleteness of knowledge concerning commutators in von Neumann algebras (see [2], [3]) forces us to modify the proofs somewhat.

THEOREM 4. *Every operator in a properly infinite von Neumann algebra* $\mathscr{A}$ *is the sum of five idempotents in* $\mathscr{A}$.

Proof. In view of the isomorphism ϕ, an arbitrary operator $T \in \mathscr{A}$ can be regarded as a matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with entries from the properly infinite algebra $E\mathscr{A}E$. By Lemma 3.1, there exist operators $R, S, X, Y \in E\mathscr{A}E$ such that $(RS - SR) + (XY - YX) = A + D - 5I$. Next we define

$$Z = RS + XY - A + 2I, \qquad U = B + Z - R - X - I, \qquad V = C + (SR - I)S + (YX - I)Y - Z.$$

Then the equation

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} RS & R \\ (I - SR)S & I - SR \end{pmatrix} + \begin{pmatrix} XY & X \\ (I - YX)Y & I - YX \end{pmatrix} + \begin{pmatrix} I - Z & I - Z \\ Z & Z \end{pmatrix}$$
$$+ \begin{pmatrix} I & U \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ V & I \end{pmatrix}$$

represents T as the sum of five idempotents in $\mathscr{A}$.

THEOREM 5. *Every operator in a properly infinite von Neumann algebra* $\mathscr{A}$ *is the sum of five operators in* $\mathscr{A}$ *each having square zero.*

Proof. As before, let $T \in \mathscr{A}$ be regarded as the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with entries from $E\mathscr{A}E$. By Lemma 3.1, we may choose operators R, S, X, Y in $E\mathscr{A}E$ so that $A + D = (RS - SR) + (XY - YX)$. Next we define

$$Z = RS + XY - A, \qquad U = B + Z - R - X, \qquad V = C + SRS + YXY - Z.$$

Then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} RS & R \\ -SRS & -SR \end{pmatrix} + \begin{pmatrix} XY & X \\ -YXY & -YX \end{pmatrix} + \begin{pmatrix} -Z & -Z \\ Z & Z \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix},$$

and therefore T is the sum of five operators in $\mathcal{A}$ each having square zero.

In order to generalize Theorem 3 to operators in an arbitrary properly infinite algebra, we need the following lemmas.

LEMMA 3.2. *Let $\mathcal{A}$ be a properly infinite von Neumann algebra, and suppose that* F *is a projection in $\mathcal{A}$ such that* $F \sim I - F$. *Then* $F \sim I$, *and* $I - F$ *can be written as the sum* $I - F = G_1 + G_2 + \cdots$ *of an infinite sequence of orthogonal projections* $G_i \in \mathcal{A}$, *each equivalent to* I.

Proof. By [11, Corollary, p. 41], $\mathcal{A}$ contains a projection E with $I \sim E \sim I - E$. Using Generalized Comparability [11, Lemma 4, p. 43], we can find a central projection $H \in \mathcal{A}$ such that $EH \precsim FH$ and $(I - E)(I - H) \precsim (I - F)(I - H)$. Then

$$E = EH + E(I-H) \precsim FH + E(I-H) \sim FH + (I-E)(I-H) \sim (I-F)H + (I-E)(I-H)$$

$$\precsim (I-F)H + (I-F)(I-H) = I - F \sim F,$$

so that $E \precsim F$. By symmetry, $F \precsim E$; thus $F \sim E \sim I$, and the first statement is proved.

Since $I - F \sim I$, we see that $I - F$ is a properly infinite projection, and the second statement follows immediately from [5, Corollary 2, p. 319].

LEMMA 3.3. *If $\mathcal{A}$ is a properly infinite von Neumann algebra, and* H *is any Hermitian operator in $\mathcal{A}$, then there exist operators* $A, B \in \mathcal{A}$ *such that*

$$H = (A^*A - AA^*) + (B^*B - BB^*).$$

Proof. By [7, Theorem 3] there is a projection $E \in \mathcal{A}$ such that $EH = HE$ and $E \sim I - E$. By Lemma 3.2, $E \sim I$, and there exists a sequence $\{F_n\}_{n=1}^{\infty}$ of orthogonal projections $F_n \in \mathcal{A}$, each equivalent to I, such that $I - E = F_1 + F_2 + \cdots$. Since $H = EH + (I - E)H$, it suffices to prove that EH and $(I - E)H$ are self-commutators in $\mathcal{A}$, and by symmetry it suffices to deal with one of these operators, say EH. Let G and K be orthogonal spectral projections of EH such that $G + K = E$, GH is positive semidefinite, and KH is negative semidefinite. Then $G, K \in \mathcal{A}$, and we define four projections in $\mathcal{A}$ as follows:

$$P_1 = G + F_1, \quad P_2 = F_3, \quad P_3 = K + F_2, \quad P_4 = F_4 + F_5 + \cdots.$$

It is clear that the P_i are mutually orthogonal and have sum I. Furthermore, since $F_i \sim I$ for all i, each $P_i \sim I$, so that the P_i are mutually equivalent. Using the family $\{P_i\}_{i=1}^{4}$ and an appropriate collection of partial isometries implementing the equivalence of the P_i, we obtain a spatial isomorphism τ from $\mathcal{A}$ onto $M_4(\mathcal{A})$ such that the operator EH is carried by τ onto a matrix of the form

$$\tau(EH) = \begin{pmatrix} H_1 & & & \\ & 0 & & \\ & & H_2 & \\ & & & 0 \end{pmatrix},$$

where $H_1 \in \mathscr{A}$ is positive semidefinite and $H_2 \in \mathscr{A}$ is negative semidefinite. Since $\tau(EH)$ can be viewed as the *direct sum* of the operators

$$\begin{pmatrix} H_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} H_2 & 0 \\ 0 & 0 \end{pmatrix}$$

in $M_2(\mathscr{A})$, we have reduced the problem to showing that every operator in $M_2(\mathscr{A})$ of the form

$$\begin{pmatrix} H_1 & 0 \\ 0 & 0 \end{pmatrix}$$

[where H_1 is a positive (or negative) semidefinite operator in $\mathscr{A}$] is a self-commutator in $M_2(\mathscr{A})$. To complete the proof, we can now carry out the construction of [9, Lemmas 1 and 2] in $M_2(\mathscr{A})$, using the fact that $M_2(\mathscr{A})$ is spatially isomorphic to $M_\infty(\mathscr{A})$.

THEOREM 6. *Every Hermitian operator in a properly infinite von Neumann algebra is a real linear combination of eight projections in the algebra.*

Sketch of the proof. Let $\mathscr{A}$ be an arbitrary properly infinite algebra. By employing the isomorphism ϕ of $\mathscr{A}$ onto $M_2(E\mathscr{A}E)$ as above, we can regard a Hermitian operator $H \in \mathscr{A}$ as a 2×2 matrix

$$\begin{pmatrix} K & M^* \\ M & L \end{pmatrix}$$

with entries from the properly infinite algebra $E\mathscr{A}E$. One now simply copies the proof of Theorem 3, noting that all the constructions required can be carried out in the ring $E\mathscr{A}E$. Of course, Lemma 3.3 is used in place of [9, Theorem 1]. We leave the details to the interested reader.

The success of Theorems 1 and 2 is due partly to the availability of an exact characterization of commutators in the algebra $\mathscr{L}(\mathscr{H})$. Although existing information about commutators in a general von Neumann algebra is sketchy, Brown and Pearcy [3] have completely described the situation for factors of one other type. The next theorem is a direct application of their description.

THEOREM 7. *Let $\mathscr{A}$ be a von Neumann factor of type* III *that acts on a separable Hilbert space. Then each nonscalar operator in $\mathscr{A}$ is the sum of four idempotents in $\mathscr{A}$, and also the sum of four operators in $\mathscr{A}$ each having square zero.*

In a factor of type III, the nonzero commutators are precisely the nonscalar operators, that is, the operators that are not scalar multiples of the identity operator [3, Theorem 1]. Knowing this, one easily proves Theorem 7 by adapting the techniques employed in the proofs of Theorems 1 and 2.

We remark that Theorems 4 and 5 fail in any von Neumann algebra $\mathscr{A}$ of finite type (and hence in any algebra having a finite direct summand). In the first place, idempotents are similar, in $\mathscr{A}$, to projections [11, Theorem A, p. 20], and thus have nonnegative (center-valued) trace. Consequently, any sum of idempotents has nonnegative trace, so that negative semidefinite operators in $\mathscr{A}$, for example, cannot be obtained by summing idempotents in $\mathscr{A}$.

On the other hand, any operator $T \in \mathscr{A}$ such that $T^2 = 0$ must have central trace zero, which implies that not every operator in $\mathscr{A}$ can be a sum of operators with square zero. To see this, consider the polar decomposition $T = V(T^*T)^{1/2}$. Then $V \in \mathscr{A}$ is a partial isometry whose initial space is the orthocomplement of the null space of T and whose final space is the closure of the range of T. Let $E = VV^*$, and note that $TE = 0$, since the null space of T contains its range. Further, $ET = T$, so that $\operatorname{tr}(T) = \operatorname{tr}(ET) = \operatorname{tr}(TE) = 0$.

It is perhaps appropriate to note also that Theorem 6 fails in many finite von Neumann algebras of type I. In particular, suppose that $\mathscr{A}$ is an n-homogeneous algebra whose center $\mathscr{Z}$ contains infinitely many orthogonal nonzero projections. Let $\mathscr{X}$ be an extremely disconnected compact Hausdorff space such that $\mathscr{Z}$ is C^*-isomorphic to $C(\mathscr{X})$, and recall that $\mathscr{A}$ is C^*-isomorphic to the algebra $M_n(\mathscr{X})$ of all $n \times n$ matrices over $C(\mathscr{X})$. If $A \in \mathscr{A}$ is a finite linear combination of projections from $\mathscr{A}$, then the function $A(\cdot)$ in $M_n(\mathscr{X})$ corresponding to A has the property that the trace of $A(x)$ can assume only finitely many values as x ranges over $\mathscr{X}$. Since there are functions in $C(\mathscr{X})$ that take on infinitely many values, it follows easily that not every operator in $\mathscr{A}$ can be a finite linear combination of projections.

We do not know whether a von Neumann algebra of type II_1 is the linear span of its projections.

4. GENERATORS FOR CERTAIN VON NEUMANN ALGEBRAS

Throughout this section, we assume that $\mathscr{H}$ is a *separable*, infinite-dimensional Hilbert space. A von Neumann algebra $\mathscr{A}$ acting on $\mathscr{H}$ is said to be *generated* by a family $\{T_i\}$ of operators if $\mathscr{A}$ is the smallest weakly closed *-algebra of operators containing each T_i.

We shall consider von Neumann algebras $\mathscr{A}$ on $\mathscr{H}$ with the following property:

(G) $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$ has a single generator and is *-isomorphic to $M_2(\mathscr{A})$.

The purpose of this section is to exhibit various small sets of generators for von Neumann algebras with property (G). It is known that every properly infinite von Neumann algebra of type I and every hyperfinite II_1-factor have property (G) [13], [17]. Furthermore, examples of von Neumann algebras of types II_∞ and III with property (G) have been given [14], [17]. The following elementary lemma was first noted by Douglas and Topping [6].

LEMMA 4.1. *Let $\mathscr{A}$ be a von Neumann algebra having a single generator* A. *Then the operators*

$$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$$

generate $M_2(\mathscr{A})$.

COROLLARY 4.2. *Let* $\mathscr{A}$ *be a von Neumann algebra having a single generator* A. *Then the idempotent operators*

$$\begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I & A \\ 0 & 0 \end{pmatrix}$$

generate $M_2(\mathscr{A})$.

The proofs of these two propositions are easy computations, which we omit.

LEMMA 4.3. *If* A *is an invertible operator on* $\mathscr{H}$, *then the operator*

$$T = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$$

on $\mathscr{H} \oplus \mathscr{H}$ *generates a von Neumann algebra of type* I_2.

Proof. By [1, Lemma 7.1], T is a binormal operator, and therefore [1, Theorem 1] generates a von Neumann algebra of the form $\mathscr{B} \oplus \mathscr{C}$, where $\mathscr{B}$ is of type I_2 and $\mathscr{C}$ is abelian. But the normal kernel [1, p. 420] of T is $\mathscr{N} \cap \mathscr{N}^*$, where $\mathscr{N}$ and $\mathscr{N}^*$ are the null spaces of T and T*, respectively, and it is easy to see that the invertibility hypothesis on A guarantees that this normal kernel is the zero subspace. Thus $\mathscr{C}$ acts on the zero subspace, and $\mathscr{B} \oplus \mathscr{C} = \mathscr{B}$, as was to be proved.

LEMMA 4.4. *A von Neumann algebra* $\mathscr{B}$ *of type* I_2 *acting on* $\mathscr{H}$ *is always generated by two of its projections.*

Proof. By [1, Theorem 2], $\mathscr{B}$ is spatially isomorphic with a von Neumann algebra $M_2(\mathscr{Z})$ of all 2×2 matrices with entries from an abelian von Neumann algebra $\mathscr{Z}$. By a well-known result of von Neumann [12], $\mathscr{Z}$ has a single Hermitian generator, say A. It is clear that if α and λ are any positive scalars such that $A + \lambda I$ is invertible, then $\alpha(A + \lambda I)$ also generates $\mathscr{Z}$, so that we may assume that A is positive definite and has norm less than one. Now define

$$E = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} A & B \\ B & I - A \end{pmatrix},$$

where $B = (A(I - A))^{1/2} \in \mathscr{Z}$. Note that B is invertible, and that E and F are projections in $\mathscr{B}$. We assert that the von Neumann algebra $\mathscr{D}$ generated by E and F is exactly $\mathscr{B}$. To see this, observe first that the three matrices

$$EFE = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad EF = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, \quad EF - EFE = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$$

all belong to $\mathscr{D}$, as does the matrix

$$\begin{pmatrix} B^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

since A generates $\mathscr{Z}$. The algebra $\mathscr{D}$ also contains the matrices

$$\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix},$$

which, by Lemma 4.1, generate $\mathscr{B}$. Thus $\mathscr{D} = \mathscr{B}$.

THEOREM 8. *Let* $\mathscr{A}$ *be a von Neumann algebra having property* (G). *Then* $\mathscr{A}$ *can be generated by each of the following:*

a) *one partial isometry,*

b) *two operators with square zero,*

c) *two idempotents,*

d) *four projections.*

Proof. To show that a von Neumann algebra $\mathscr{A}$ with property (G) satisfies the conclusion of the theorem, it suffices to show that $M_2(\mathscr{A})$ does, since a *-isomorphism between von Neumann algebras is an ultraweak homeomorphism. Thus part (a) follows immediately from [14, Lemma 1]. Part (b) follows from Lemma 4.1, and part (c) is a consequence of Corollary 4.2. To prove part (d), note that by translating the original generator by a suitable scalar, we may assume that $\mathscr{A}$ is generated by an invertible operator A. Then the matrices

$$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$$

together generate $M_2(\mathscr{A})$, by Lemma 4.1. Each matrix individually, however, generates a subalgebra of type I_2, according to Lemma 4.3. Moreover, each of these subalgebras of type I_2 is generated by two projections, in accordance with Lemma 4.4. Thus $M_2(\mathscr{A})$ is generated by four projections, and the proof is complete.

Added in Proof. Recently, Saito [*Generators of hyperfinite factors,* to appear] has shown that every von Neumann algebra with property (G) can be generated by three projections; this improves part (d) of Theorem 8 and extends the result of Davis [4]. For completeness, we sketch a modified proof of Saito's theorem. Let $\mathscr{A}$ be a von Neumann algebra having a single invertible generator A with norm less than 1, and define $S = (I - AA^*)^{1/2}$, $T = (I - A^*A)^{1/2}$. Then the three projections

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix}, \quad \begin{pmatrix} AA^* & SA \\ A^*S & T^2 \end{pmatrix}$$

generate $M_2(\mathscr{A})$ (use Lemma 4.1), and they are unitarily equivalent in $M_2(\mathscr{A})$.

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