TWO PROBLEMS IN THE THEORY OF GENERALIZED MANIFOLDS

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Dedicated to Professor R. L. Wilder on the occasion of his seventieth birthday.

At the R. L. Wilder Conference in Topology at the University of Michigan, held in honor and appreciation of R. L. Wilder's contributions to topology, I discussed three problems in the theory of generalized manifolds that have interested me. This is a report on two of these problems.

1. DIMENSION OF GENERALIZED MANIFOLDS

We adopt the conventions of [1] and call a locally orientable generalized n-manifold M over a principal ideal domain L a *cohomology* n-*manifold over* L (M is an n-cm over L). This differs slightly from the terminology of Wilder [12] in that no assumption on the covering dimension of M is made. It is known that the cohomological dimension of M over L, to be denoted by $\dim_L M$, is exactly n. Whether or not the covering dimension of M is finite is still unknown. Several other interesting questions concerning the covering dimension of generalized manifolds have been stated by Wilder in [12, p. 382]. The recent reprinting of [12] (1963) contains a discussion of the present status of these questions.

PROBLEM 1. *Let* M *be an* n-cm *over* L. *Is* $\dim_{Z_p} M = \dim_Q M = \dim_L M$, *for all primes* p?

If Z_p or Q are L-modules, then M is also an n-cm over the respective Z_p or Q. Also, if $n \leq 2$ and M is separable metric, then M is locally Euclidean by a theorem of Wilder [12, pp. 275-280]. Hence, the answer to Problem 1 is partially known. However, the question is unanswered in the following special situation.

(i) *Let* M *be an* n_p-cm *over* Z_p *for each prime* p *and the rational field* (p = 0). *Is* n_p *independent of* p? *Is* M *a* cm *over* Z?

It may be possible to answer affirmatively the second part of (i) with the added assumption that M is clc over Z (see [8, p. 1375]).

An affirmative answer to either question in (i) would imply that every compact effective group of homeomorphism of a manifold (or separable metric cohomology manifold) is a Lie group (see [7] for more details). This problem, of course, is the generalized Fifth Problem of Hilbert. The answer is still unknown. We can indicate a feeling for the connections between these two problems by considering a special case. Assume that the p-adic group A_p operates freely on an orientable n-cm M over Z. Consider the space $(M \times \Sigma_p)/A_p$, where the action of A_p is the diagonal action and Σ_p is the p-adic solenoid. This space can be fibered over the circle with fiber M. Hence $(M \times \Sigma_p)/A_p = M'$ is an (n + 1)-cm over Z. The p-adic solenoid now operates freely on M', so that M'/Σ_p is homeomorphic to M/A_p, the space

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that we want to discuss. Since Σ_p is acylic, if we use cohomology with Z_p-coefficients, the Vietoris mapping theorem implies that $\dim_{Z_p} M/A_p$ is at least $n+1$. In fact, we can use Wilder's monotone mapping theorem [13] and actually conclude that M/A_p is an $(n+1)$-cm over Z_p. On the other hand, if we take coefficients in Z_q (q prime to p), then M/A_p is an n-cm over Z_q (q may be 0). This is easily proved by means of the transfer homomorphism (see [7]). Thus, if such an action exists, the answer to Problem 1 must be negative.

I shall give several other unusual properties of the orbit map $\pi\colon M \to M/A_p$ and the orbit space M/A_p, with the hope that they might either provide a guide to the construction of an action or lead to a proof that no action exists.

In [14] and [2] it was shown that $\dim_Z M/A_p = n+2$ if $\dim_Z M = n$. Let C be a closed subspace of M/A_p; then

$$\dim_Z C = n+2 \Rightarrow \dim_{Z_p} C = n+1,\ \dim_{Z_q} C = n,$$

$$\dim_Z C = n+1 \Rightarrow \dim_{Z_p} C = n,\ \dim_{Z_q} C = n-1,$$

$$\dim_Z C = n \Rightarrow \begin{cases} \dim_{Z_p} C = n,\ \dim_{Z_q} C = n-1 \text{ or} \\ \dim_{Z_p} C = n-1,\ \dim_{Z_q} C < n-1, \end{cases}$$

$$\dim_Z C = n-1 \Rightarrow \dim_{Z_p} C \leq n-1,\ \dim_{Z_q} C < n-1.$$

(Here q may take the value 0.) These facts are consequences of [8, 3.1 Lemma] and the property (see [6, p. 9]) that for a cm, closed subspaces of (cohomological) codimension 0 (respectively, codimension 1) are identical with the subspaces having nonempty interior (respectively, the subspaces that locally separate the cm).

It follows from [8, 3.1 Corollary] that $\dim_Z (C \times C) = 2\dim_Z C - 1$ in the first two cases. Thus, each of these subspaces is a space of Boltjanskiĭ type. The remaining two cases are also not dimensionally full-valued (see also [10, Section 6]).

C. N. Lee has made the observation (unpublished) that the mapping π must be a fibering in the sense of Hurewicz. (It is known that all Hurewicz fiberings of a manifold, with totally disconnected fibers, onto weakly locally contractible spaces are genuine covering maps, [9, 2.10].) Suppose that C is a closed subspace of M/A_p and that C can be deformed, in M/A_p, to a point. A generalization of the argument used in the proof of Proposition 4 of [5] shows that $\pi\colon \pi^{-1}(C) \to C$ has a cross-section. Hence, C can be imbedded as a closed subspace of M, an n-cm over Z. In a cm over Z, $\dim_L C$ is independent of L if the (cohomological) codimension is at most 1. Thus we have the following proposition, which is a variation in this specific case, of an unpublished result of R. F. Williams.

PROPOSITION. *No closed subspace* C *of* M/A_p *with* $\dim_Z C > n-2$ *can be deformed to a point, in* M/A_p.

2. BORDISM GROUPS

Consider continuous maps f: $M^n \to X$, where M^n is a closed triangulated homology n-manifold over L and X is any topological space. We say that (M^n, f) is equivalent to (N^n, g) if and only if there exists a triangulated homology $(n + 1)$-manifold with boundary W^{n+1} and a map h: $W^{n+1} \to X$ such that the boundary of W^{n+1} is $M^n \cup N^n$ and $h \mid M^n = f$, $h \mid N^n = g$ ($\cup$ denotes disjoint union). That is, we define a bordism homology theory in analogy to the differentiable case by using triangulable homology manifolds instead of differentiable manifolds. With care, one can actually show that this defines a homology theory on the category of CW complexes (in fact over all topological spaces, by certain extension techniques; see [4, Section 5] for details in the differentiable case).

Let $\mathcal{M}_n$, $\mathcal{P}\mathcal{L}_n$, and $\mathcal{P}_n$ denote the nth bordism functor whose antecedent spaces are differentiable n-manifolds, piecewise linear n-manifolds, or triangulated homology n-manifolds. Certain obvious natural transformations

$$\begin{array}{ccc} & \mathcal{P}\mathcal{L}_n & \\ \nearrow & & \searrow \\ \mathcal{M}_n & \longrightarrow & \mathcal{P}_n \end{array}$$

are equivalences if $n \leq 3$.

PROBLEM 2. *What are the coefficient groups for the generalized homology theory* $\mathcal{P}_*$? *Can one also compute the ring structure of* $\mathcal{P}_*$ (point)?

By suitably restricting our antecedent triangulable homology manifolds, (for example: take only triangulated homology manifolds imbedded in some triangulated Euclidean space, where the triangulations of R^n are compatible with those of R^{n+1}; and assume that the maps have range in the finite subcomplexes of a countably infinite simplex) we may assert that the cohomology theory associated to this homology theory is representable [3]. This yields the analogue of the Thom spectra. Problem 2 calls for the determination of the stable homotopy groups of this spectrum. Of course, there are at least two such theories, the oriented and unoriented theories. In the unoriented theory, where all modules are over Z_2,

$$\mathcal{P}_n(X) \approx \sum_j \mathcal{P}_{n-j}(\text{point}) \otimes H_j(X).$$

One may prove this by using (among other things) the injectivity of $\mathcal{M}_n \to \mathcal{P}_n$, together with the spectral-sequence argument employed in [4, Section 8] for the differentiable case. Well-behaved analogues of tangent and normal bundles do exist for triangulated homology manifolds. One can also define Stiefel-Whitney classes of a homology manifold. Furthermore, the Stiefel-Whitney numbers must agree if two homology manifolds are to be bordant. However, it is likely that many more "exotic characteristic classes" exist and must be recognized before a theorem analogous to Thom's [11] can be proved.

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