ABELIAN ACTIONS ON 2-MANIFOLDS

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Dedicated to R. L. Wilder on his seventieth birthday.

Let G be a finite group. We shall consider the problem of classifying the effective actions of G on closed oriented surfaces $\mathfrak{M}$. If two such actions are equivalent, their orbit spaces are homeomorphic, and so it is natural to study the totality A of actions having a fixed orbit space M. The problem is (1) to determine the equivalence classes of A in the sense of putting them into one-to-one correspondence with the equivalence classes of some algebraic system and (2) to compute, using the algebraic scheme, the number of equivalence classes in A for different M's and G's. J. Nielsen [4] gave a solution of (1) for the case where G is cyclic, but he did not consider (2) explicitly. We give here a solution of (1) for $G = Z_p \times \cdots \times Z_p$ (p a prime) by showing that in this case the equivalence classes are in one-to-one correspondence with the equivalence classes of certain matrices over Z_p under certain operations on the columns. Actually, a solution for (1) can be given for arbitrary finite abelian groups. But in this more general case we have little information relative to (2), whereas in the case considered we do solve (2) in some simple instances.

A CLASSIFICATION THEOREM FOR FREE ACTIONS

A homeomorphism $X \to Y$ or $(X, x) \to (Y, y)$ is always "onto." A homeomorphism between oriented manifolds always preserves orientation.

1. We consider actions $a = (G, \mathfrak{X})$, where G is a fixed discrete group and $\mathfrak{X}$ a topological space. We denote by $\mu(a)$ the space in which the action a takes place: if $a = (G, \mathfrak{X})$, then $\mu(a) = \mathfrak{X}$. An action $(G, \mathfrak{X})$ is *effective* if the identity is the only element of G that leaves all points of $\mathfrak{X}$ fixed; it is *free* if no point of $\mathfrak{X}$ is left fixed by any element of $G - \{1\}$. Two actions $a = (G, \mathfrak{X})$ and $a' = (G, \mathfrak{X}')$ are *equivalent* (notation: $a \sim a'$) if there exists a homeomorphism $t: \mathfrak{X} \to \mathfrak{X}'$ such that $t(g\mathfrak{x}) = g(t\mathfrak{x})$ for $g \in G$, $\mathfrak{x} \in \mathfrak{X}$. To indicate that t defines an equivalence, we refer to it as an *equivalence map*.

A space X will be called *allowable* provided it is arcwise connected and semi-locally arcwise connected (so that the theory of coverings as described by paths is valid; see [2, pp. 89-97]), and provided further that it has the following "isomorphism replacement" property: if x, x' are distinct points of X, and u is a path in X from x to x' and u_π the isomorphism $\pi_1(X, x') \to \pi_1(X, x)$ induced by u, then there exists a homeomorphism $t: (X, x') \to (X, x)$ such that $t_\pi = u_\pi$, where t_π is the isomorphism of fundamental groups induced by t. A connected manifold admitting a differentiable structure is allowable. In particular, compact 2-manifolds are allowable.

Call an action $(G, \mathfrak{X})$ *allowable* if its orbit space X is allowable and if the pair $(\mathfrak{X}, \psi)$, where ψ is the natural map of $\mathfrak{X}$ onto X, is a covering of X.

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For the present we consider only free actions. Let X be an allowable space. We denote by $A_f(X, G)$ the totality of free allowable actions whose orbit spaces are homeomorphic to X. For each a, we choose a definite identification of its orbit space with X. In this way, the natural map of $\mu(a)$ onto the orbit space of a becomes a map $\phi_a: \mu(a) \to X$ such that $\phi_a(g\mathfrak{x}) = \phi_a(\mathfrak{x})$ for $g \in G$, $\mathfrak{x} \in \mu(a)$ and $(\mu(a), \phi_a)$ is a covering of X. The maps ϕ_a will be denoted simply by ϕ.

2. Let K, G be groups. We denote Hom[K, G] by [K, G], and the set of epimorphisms in [K, G] by $[K, G]_e$. If f_1, f_2 are elements of [K, G], we write $f_1 \doteq f_2$ if there exists an inner automorphism s of G such that $f_1 = sf_2$.

Let X be an allowable space, and let $\pi_x = \pi_1(X, x)$ $(x \in X)$. Let G be a group. We introduce an equivalence into $\bigcup_{s\in X} [\pi_s, G]_e$ as follows. Let $f_{x_i} \in [\pi_{x_i}, G]_e$ $(i = 1, 2)$. Then $f_{x_1} \sim f_{x_2}$ means that there exists a homeomorphism $t: (X, x_1) \to (X, x_2)$ such that $f_{x_1} \doteq f_{x_2} t_\pi$. Let $a \in A_f(X, G)$, and let $\mathfrak{X} = \mu(a)$. With each pair $(a, \mathfrak{x})$ ($\mathfrak{x}$ a point of $\mathfrak{X}$) we associate an element $f^a_\mathfrak{x} \in \bigcup_s [\pi_s, G]_e$ as follows. Let $x = \phi\mathfrak{x}$, let u be a loop representing an element q of π_x, and let $\mathfrak{u}$ be the cover of u that begins at $\mathfrak{x}$ (that is, the path obtained by lifting u). The terminal point of $\mathfrak{u}$ covers x and hence equals $g\mathfrak{x}$ for some $g \in G$. The element g depends only on $\mathfrak{x}$ and q. Define $f^a_\mathfrak{x} q$ to be g.

We hold a fixed for the moment and write $f_\mathfrak{x}$ for $f^a_\mathfrak{x}$. It is easily seen that $f_\mathfrak{x}$ is surjective. We prove that $f_\mathfrak{x}$ is a homomorphism, hence an element of $[\pi_x, G]_e$. Let v be a loop representing $l \in \pi_x$, and $\mathfrak{v}$ the cover of v that begins at $\mathfrak{x}$; v ends at $h\mathfrak{x}$, where $h = f_\mathfrak{x} l$. Now $\mathfrak{u}(g\mathfrak{v})$ is a cover of uv, and it begins at $\mathfrak{x}$. Hence its terminal point $gh\mathfrak{x}$ equals $(f_\mathfrak{x}(ql))\mathfrak{x}$. Hence $f_\mathfrak{x}(ql) = gh = f_\mathfrak{x}(q) f_\mathfrak{x}(l)$.

It follows immediately from the construction of $f^a_\mathfrak{x}$ that

$$\ker f^a_\mathfrak{x} = \phi_* \pi_1(\mathfrak{X}, \mathfrak{x}). \tag{2.1}$$

We wish to compare $f^a_\mathfrak{x}$ with $f^a_\mathfrak{y}$. Let $\mathfrak{v}$ be a path in $\mathfrak{X}$ from $\mathfrak{x}$ to $\mathfrak{y}$, and let $x = \phi\mathfrak{x}$, $y = \phi\mathfrak{y}$, $v = \phi\mathfrak{v}$. One verifies readily that

$$f^a_\mathfrak{y} = f^a_\mathfrak{x} v_\pi,$$

where v_π is the isomorphism $\pi_y \to \pi_x$ induced by v.

Case 1. Suppose $x = y$, so that $\{\mathfrak{x}, \mathfrak{y}\} \subset \phi^{-1}x$. Then v is a loop and represents, say, $l \in \pi_x$, and v_π is conjugation of π_x by l. Hence for $q \in \pi_x$,

$$f^a_\mathfrak{y}(q) = h f^a_\mathfrak{x}(q) h^{-1}, \qquad \text{where } h = f^a_\mathfrak{x}(l) \in G, \tag{2.2}$$

so that $f^a_\mathfrak{y} \doteq f^a_\mathfrak{x}$.

Case 2. $x \neq y$. In this case, $v_\pi = t_\pi$ for some homeomorphism $t: (X, x) \to (X, y)$ (Section 1).

Thus $f^a_\mathfrak{x} \sim f^a_\mathfrak{y}$ in both cases. Hence, *for fixed* a, *the maps* $f^a_\mathfrak{x}$ $(\mathfrak{x} \in \mu(a))$ *belong to one and the same equivalence class of* $\bigcup_s [\pi_s, G]_e$.

(2.3) PROPOSITION. *Let* $a, a' \in A_f(X, G)$, *and let* $\mathfrak{x} \in \mathfrak{X} = \mu(a)$, $\mathfrak{x}' \in \mathfrak{X}' = \mu(a')$. *Then* $a \sim a'$ *if and only if* $f^a_\mathfrak{x} \sim f^{a'}_{\mathfrak{x}'}$.

Proof. Suppose $a \sim a'$. Let $\mathfrak{t}: \mathfrak{X} \to \mathfrak{X}'$ be an equivalence map. Let $\mathfrak{x}_1' = \mathfrak{t}\mathfrak{x}$. Since $f_{\mathfrak{x}'}^{a'} \sim f_{\mathfrak{x}_1'}^{a'}$, it is sufficient to show that $f_{\mathfrak{x}}^{a} \sim f_{\mathfrak{x}_1'}^{a'}$. The map $\mathfrak{t}$ induces a homeomorphism $t: (X, x) \to (X, x_1')$, where $x = \phi\mathfrak{x}$ and $x_1' = \phi(\mathfrak{x}_1')$. Clearly, the construction that defines $f_{\mathfrak{x}}^a(q)$ $(q \in \pi_x)$ is carried over by the maps $\mathfrak{t}$ and t into the construction that defines $f_{\mathfrak{x}_1'}^{a'}(q')$ $(q' = t_\pi q)$, so that $f_{\mathfrak{x}}^a(q) = f_{\mathfrak{x}_1'}^{a'}(q)$. Suppose conversely that $f_{\mathfrak{x}}^a \sim f_{\mathfrak{x}'}^{a'}$. Then there exists a homeomorphism $t: (X, x) \to (X, x')$ such that $f_{\mathfrak{x}}^a \doteq f_{\mathfrak{x}'}^{a'} t_\pi$. Hence there exists $g \in G$ such that

$$g f_{\mathfrak{x}}^a(q) g^{-1} = f_{\mathfrak{x}'}^{a'} t_\pi(q) \qquad \text{for each } q \in \pi_x.$$

Now the element 1 in formula (2.2) depends on $\mathfrak{x}$, $\mathfrak{y}$, and the path $\mathfrak{v}$ from $\mathfrak{x}$ to $\mathfrak{y}$. If $\mathfrak{x}$ is fixed and $\mathfrak{y}$ ranges over $\phi^{-1}x$, and if for each $\mathfrak{y}$, $\mathfrak{v}$ ranges over the paths from $\mathfrak{x}$ to $\mathfrak{y}$, then 1 takes on all values in π_x, and hence $h = f_{\mathfrak{x}}^a(1)$ takes on all values in G. Hence there is a $\mathfrak{y} \in \phi^{-1}(x)$ such that $f_{\mathfrak{y}}^a(q) = g f_{\mathfrak{x}}^a(q) g^{-1}$. Hence $f_{\mathfrak{y}}^a = f_{\mathfrak{x}'}^{a'} t_\pi$. It follows that

$$t_\pi \ker f_{\mathfrak{y}}^a \subset \ker f_{\mathfrak{x}'}^{a'},$$

and therefore $t_\pi \phi_* \pi_1(\mathfrak{X}, \mathfrak{y}) \subset \phi_* \pi_1(\mathfrak{X}', \mathfrak{x}')$. Hence [2, Theorem 16.4] there exists a unique homeomorphism $\mathfrak{t}: (\mathfrak{X}, \mathfrak{y}) \to (\mathfrak{X}', \mathfrak{x}')$ that covers t and therefore maps orbits onto orbits. Hence, if h is a given element of G, there exists a function $j = j(\mathfrak{x})$ with values in G such that $h\mathfrak{t}\mathfrak{x} = \mathfrak{t} j \mathfrak{x}$ $(\mathfrak{x} \in \mathfrak{X})$. Easy considerations of continuity show that $j(\mathfrak{x})$ is constant. We assert that in fact $j(\mathfrak{x}) = h$. We may assume $\mathfrak{x} = \mathfrak{y}$. Let q be an element of π_x such that $f_{\mathfrak{y}}^a(q) = h$. Then $f_{\mathfrak{x}'}^{a'} t_\pi(q) = h$. Now, by the definition of f, $h\mathfrak{y}$ is the terminal point of a path $\mathfrak{u}$ whose projection u represents q. Also, $h\mathfrak{x}'$ is the terminal point of a path $\mathfrak{u}'$ covering u' representing $t_\pi q$. We may suppose that $\mathfrak{u}' = \mathfrak{t}\mathfrak{u}$ and $u' = tu$. The terminal point $h\mathfrak{x}'$ is therefore the $\mathfrak{t}$-image of the terminal point of $\mathfrak{u}$, namely $\mathfrak{t}h\mathfrak{y}$. Then $\mathfrak{t}h\mathfrak{y} = h\mathfrak{x}' = h\mathfrak{t}\mathfrak{y}$. Hence $\mathfrak{t}h\mathfrak{y} = \mathfrak{t}j\mathfrak{y}$, which implies $h\mathfrak{y} = j\mathfrak{y}$, $h = j$. Since h is an arbitrary element of G, we have proved that $a \sim a'$.

(2.4) PROPOSITION. *Let* f *be an element of* $[\pi_x, G]_e$. *Then* $f = f_{\mathfrak{x}}^a$ *for some* $a = (G, \mathfrak{X})$ *in* A_f *and some* $\mathfrak{x} \in \phi^{-1}x$.

Proof. Let $(\mathfrak{X}, \phi)$ be the covering of X constructed by the paths in X emanating from x taken modulo ker f. (See [2, Section 17].) There is a natural action $(\pi_x, \mathfrak{X})$ in which the stability group of each point of $\mathfrak{X}$ is precisely ker f. Hence the rule $g\mathfrak{x} = q\mathfrak{x}$, $g = f(q)$ induces a free action $a = (G, \mathfrak{X})$. Let $\mathfrak{x}$ be the point of $\mathfrak{X}$ represented by the constant path x, so that $\phi\mathfrak{x} = x$. It follows immediately that $f_{\mathfrak{x}}^a = f$. Obviously, a is allowable.

Propositions (2.3) and (2.4) imply the following:

(2.5) THEOREM. *The map that associates with each* $a = (G, \mathfrak{X})$ *in* $A_f(X, G)$ *the subset* $\{f_{\mathfrak{x}}^a\}_{\mathfrak{x} \in \mathfrak{X}}$ of $\bigcup_{s \in X} [\pi_s, G]_e$ *defines a one-to-one correspondence between the equivalence classes of* $A_f(X, G)$ *and those of* $\bigcup_s [\pi_s, G]_e$.

3. Let $\mathfrak{u}$ be an oriented, simple closed curve in $\mu(a)$ $(a \in A(X, G))$, and let $G_\mathfrak{u} = \{g \in G,\ g\mathfrak{u} = \mathfrak{u}\}$ (the stability group of $\mathfrak{u}$). Now let u be an oriented, simple closed curve in X, and $\mathfrak{x}$ a point in $\phi^{-1}u$. Let q be the element of π_x $(x = \phi\mathfrak{x})$ represented by u. Let $\mathfrak{u}$ be the component (an oriented simple closed curve) of $\phi^{-1}u$ that contains $\mathfrak{x}$.

(3.1) *The stability group* G_u *is trivial if and only if* $f_{\mathfrak{x}}^a(q) = 1$.

Indeed, starting at $\mathfrak{x}$ and proceeding along u in the direction of the orientation of u, let $\mathfrak{x}' = g\mathfrak{x}$ be the first point of $G_u \mathfrak{x}$ encountered after leaving $\mathfrak{x}$. The arc $\mathfrak{x}\mathfrak{x}'$ thus traversed is precisely that cover of u that begins at $\mathfrak{x}$. Therefore $g = f_{\mathfrak{x}}^a(q)$, and $g = 1$ if and only if $G_u = \{1\}$.

4. Let V and G be groups, and let $\mathcal{A}$ be a group of automorphisms of V. Call two elements f_1 and f_2 of $[V, G]$ *$\mathcal{A}$-equivalent* if there exists an $\alpha \in \mathcal{A}$ such that $f_2 = f_1\alpha$. In particular, suppose V is the additive group of a vector space over a field F and G is the additive group of F. Then f_1 and f_2 are elements of the dual space V^* and are $\mathcal{A}$-equivalent if and only if one is the image of the other under the dual of some element of $\mathcal{A}$.

Now assume that G is abelian. Let $a = (G, \mathfrak{X})$ be an element of $A_f(X, G)$, and for $x \in X$, let τ_x be the canonical epimorphism $\pi_x \to H_1(X)$. Let $\mathfrak{x} \in \phi^{-1}x$. Two elements of π_x with equal images under τ_x differ by a commutator, hence have equal images under $f_{\mathfrak{x}}^a$ (since G is abelian). Hence the formula

$$h_{\mathfrak{x}}^a = f_{\mathfrak{x}}^a \tau_x^{-1} \qquad (x = \phi\mathfrak{x}) \tag{4.1}$$

defines an epimorphism $h_{\mathfrak{x}}^a \in [H_1(X), G]_e$.

If h is an epimorphism $H_1(X) \to G$, there exists an $f_{\mathfrak{x}}^a$ such that the corresponding $h_{\mathfrak{x}}^a$ is h. In fact, let f be the epimorphism $\pi_x \to G$ defined by $f = h\tau_x$. Let a and $\mathfrak{x}$ be such that $f_{\mathfrak{x}}^a = f$ (2.4). Then $h_{\mathfrak{x}}^a = f_{\mathfrak{x}}^a \tau_x^{-1} = h\tau_x \tau_x^{-1} = h$.

(4.2) THEOREM. *The correspondence* $f_{\mathfrak{x}}^a \to h_{\mathfrak{x}}^a$ *defines a bijective map from the equivalence classes of* $\bigcup_s [\pi_s, G]_e$ *to the* $\mathcal{A}$*-equivalence classes of* $[H_1(X), G]_e$, *where* $\mathcal{A}$ *is the group of automorphisms of* $H_1(X)$ *induced by homeomorphisms* $X \to X$.

It will be sufficient to show that $f_{\mathfrak{x}}^a \sim f_{\mathfrak{x}'}^{a'}$ if and only if $h_{\mathfrak{x}}^a$ and $h_{\mathfrak{x}'}^{a'}$ are $\mathcal{A}$-equivalent.

(4.3) LEMMA. *Let* x *and* x' *be points in an allowable space* X, *and let* t *be a homeomorphism* $X \to X$. *There exists a homeomorphism* $t'\colon (X, x) \to (X, x')$ *such that* $t_H = t'_H$, *where* t_H *and* t'_H *are the induced automorphisms of* $H_1(X)$.

Proof. Let $x_1 = tx$. If $x_1 = x'$, there is nothing to prove. Suppose $x_1 \neq x'$, and let u be a path in X from x_1 to x'. Let s be a homeomorphism $(X, x') \to (X, x_1)$ such that $s_\pi = u_\pi$ (Section 1), and let $t' = st$. Then $t'_H = s_H t_H$. The lemma will be proved if we show that s_H is the identity. The canonical projections τ_x and $\tau_{x'}$ satisfy the conditions $\tau_x = \tau_{x'} u_\pi$, $\tau_{x'} s_H = s_H \tau_x$. Hence

$$\tau_x = \tau_{x'} u_\pi = \tau_{x'} s_\pi = s_H \tau_x;$$

this implies $s_H = \mathrm{id}$, since τ_x is surjective.

Now suppose $f_{\mathfrak{x}}^a \sim f_{\mathfrak{x}'}^{a'}$. Then there is a homeomorphism $t\colon (X, x) \to (X, x')$ such that $f_{\mathfrak{x}}^a = f_{\mathfrak{x}'}^{a'} t_\pi$. Now (4.1) implies that $h_{\mathfrak{x}}^a \tau_x = f_{\mathfrak{x}}^a$ and $h_{\mathfrak{x}'}^{a'} \tau_{x'} = f_{\mathfrak{x}'}^{a'}$. Hence

$$h_{\mathfrak{x}}^a \tau_x = f_{\mathfrak{x}'}^{a'} \tau_{x'} t_\pi = h_{\mathfrak{x}'}^{a'} t_H \tau_x.$$

Since τ_x is surjective, $h_{\mathfrak{x}}^a = h_{\mathfrak{x}'}^{a'} t_H$, that is, $h_{\mathfrak{x}}^a \sim h_{\mathfrak{x}'}^{a'}$. Conversely, suppose there is a homeomorphism $t\colon X \to X$ such that $h_{\mathfrak{x}}^a = h_{\mathfrak{x}'}^{a'} t_H$. By Lemma (4.3), we may assume that $tx = x'$. Then

$$h_{\mathfrak{x}}^{a}\,\tau_x = h_{\mathfrak{x}'}^{a'}\,t_H\,\tau_x = h_{\mathfrak{x}'}^{a'}\,\tau_{x'}\,t_\pi ;$$

therefore $f_{\mathfrak{x}}^{a} = f_{\mathfrak{x}'}^{a'} t_\pi$, that is, $f_{\mathfrak{x}}^{a} \sim f_{\mathfrak{x}'}^{a'}$.

5. In certain cases, Theorem (4.2) is valid when coefficients other than integers are used for homology. Let $G = Z_p^r = Z_p \times \cdots \times Z_p$ (p a prime), and suppose that $H_1(X)$ has a free basis $e_1, \cdots, e_n$. Then $1 \otimes e_1, \cdots, 1 \otimes e_n$ is a basis for the Z_p-module $H_1(X, Z_p) = Z_p \otimes H_1(X)$. With each element h of $[H_1(X), G]_e$ we associate an element $h^{(p)}$ in $[H_1(X, Z_p), G]_e$ by the rule

$$h^{(p)}(1 \otimes e_i) = h(e_i).$$

The correspondence defined in this way is one-to-one. The homeomorphisms $t: X \to X$ induce automorphism groups $\mathscr{B}$, $\mathscr{A}$ of $H_1(X)$, $H_1(X, Z_p)$, and two elements in $[H_1(X), G]_e$ are $\mathscr{B}$-equivalent if and only if the corresponding elements in $[H_1(X, Z_p), G]_e$ are $\mathscr{A}$-equivalent. We now have the following modification of (4.2):

(5.1) THEOREM. *If* $G = Z_p^r$ *and if* $H_1(X)$ *is free and finitely generated, then the correspondence* $f_{\mathfrak{x}}^{a} \to (h_{\mathfrak{x}}^{a})^{(p)}$ *defines a bijective map from the equivalence classes of* $A_f(X, G)$ *to the* $\mathscr{A}$*-equivalence classes of* $[H_1(X, Z_p), G]_e$, *where* $\mathscr{A}$ *consists of the automorphisms of* $H_1(X, Z_p)$ *induced by homeomorphisms* $X \to X$.

We state also a modification of (3.1):

(5.2) *Let* $G = Z_p^r$, *and let* $H_1(X)$ *be free and finitely generated. Let* $f \in [H_1(X, Z_p), G]_e$. *Let* u *be an oriented, simple closed curve in* X *representing the element* c *of* $H_1(X, Z_p)$, *and let* $\mathfrak{u}$ *be any component of* $\phi^{-1}u$. *Then* $f(c) = 0$ *if and only if the stability group of* $\mathfrak{u}$ *is trivial.*

SYMPLECTIC AUTOMORPHISMS

6. For $n \geq 1$, let U^n be the totality of sequences $(x, y) = (x_1, \cdots, x_n, y_1, \cdots, y_n)$, where the x's and y's are elements of a vector space over a field F, the dimension of which will be clear in each context.

Let $E_1, \cdots, E_5$ be the sets $\{E_{1i}\}, \cdots, \{E_{5ij}\}$ of maps $U^n \to U^n$, where $E_{1i}, \cdots$ are defined as follows:

$$E_{1i}: x_i \to x_i + \lambda y_i,$$

$$E_{2i}: y_i \to y_i + \lambda x_i,$$

$$E_{3ij}: x_i \to x_i + \lambda x_j,\ y_j \to y_j - \lambda y_i \quad (i \neq j),$$

$$E_{4ij}: x_i \to x_i + \lambda y_j,\ x_j \to x_j + \lambda y_i \quad (i \neq j),$$

$$E_{5ij}: y_i \to y_i + \lambda x_j,\ y_j \to y_j + \lambda x_i \quad (i \neq j),$$

where λ is an element of F. For example, E_{1i} is the map that, in each (x, y), replaces x_i by $x_i + \lambda y_i$, leaving the remaining elements unchanged. Let $E = E_1 \cup \cdots \cup E_5$. Each $e \in E$ has a $2n \times 2n$—matrix over F that is independent of the space in which the sequences (x, y) are taken. If A is the matrix of an element $e \in E$, then ${}^tA$ is the matrix of an element of E denoted by ${}^te$.

Let V be a vector space of dimension 2n $(n \geq 1)$ over F, and let $u \cdot v$ be a nondegenerate, alternating bilinear form on V. The form having been chosen, V is *symplectic* over F, and the automorphisms of V that leave the form invariant are the symplectic automorphisms of V, and they form a group that we shall denote by Sp(n, F). Let $(a, b) = (a_1, \cdots, a_n, b_1, \cdots, b_n)$ be a *symplectic basis* for V, in other words, a basis such that

$$a_i \cdot b_j = \delta_{ij}, \quad a_i \cdot a_j = 0, \quad b_i \cdot b_j = 0 \quad (i, j = 1, \cdots, n).$$

The dual V^* of V is symplectic with symplectic dual basis (a^*, b^*), and the symplectic automorphisms of V^* are the duals of those of V.

Let V be a vector space of dimension 2n $(n > 0)$ over F, and let (a, b) be a basis. For a given operation $e \in E$, there is an automorphism $T: V \to V$ uniquely defined by $T(a, b) = e(a, b)$. Call T an automorphism of type E relative to the basis (a, b). The matrix of E relative to (a, b) is precisely the matrix of e. Suppose now that V and (a, b) are symplectic. Then it can immediately be verified that the automorphisms of type E relative to (a, b) are symplectic; let $Sp^o(n, F, a, b)$ be the subgroup of Sp(n, F) that they generate.

(6.1) PROPOSITION. *Let* (a, b) *be a symplectic basis for a symplectic space* V *over* F, *and let* $v_1, \cdots, v_r$ *be linearly independent elements of* V. *There exists an automorphism* $T \in Sp^o(n, F, a, b)$ *such that the component matrix of* $Tv_1, \cdots, Tv_r$ *relative to* (a, b) *is* (J, Q), *where* J *is an* $r \times n$-*matrix with* 1's *in the main diagonal and* 0's *elsewhere, and where* $Q = (q_{ij})$ *is an* $r \times n$-*matrix over* F *with* $q_{ij} = 0$ *when* $i \leq j$. *The elements* q_{ij} *with* $i > j$ *are uniquely determined by the elements* v, *by the relations*

$$q_{ij} = v_j \cdot v_i \quad (i > j). \tag{6.2}$$

Proof. Assuming that T exists, we obtain 6.2 by a trivial computation.

Let $(x, y) = (x_1, \cdots, x_n, y_1, \cdots, y_n)$ be the component vector of an element v of V. Then, if T is the element of $Sp^o(n, F, a, b)$ defined by $e \in E$, the component vector of Tv is ${}^te(x, y)$. More generally, if $(X, Y) = (X_1, \cdots, X_n, Y_1, \cdots, Y_n)$ is the component matrix of $v_1, \cdots, v_r$ (where $X_1, \cdots$ are columns), then the component matrix of $Tv_1, \cdots, Tv_n$ is ${}^te(X, Y)$. Hence, it is sufficient to prove that (X, Y) can be reduced to the form (J, Q) by a finite number of operations $e \in E$ on columns.

It is easy to verify that (X, Y) can be reduced to, say, (J, Y') by elements of E_3 on columns, *provided* X *is of rank* r, and that (J, Y') can then be reduced to the form (J, Q) by elements of $E_2 \cup E_5$. It is therefore sufficient to show that (X, Y) can be reduced to, say, (X', Y'), by elements $e \in E$, where rank $X' = r$.

Let M be an $r \times 2n$-matrix. If C is an $r \times r$-submatrix of M, denote by C^M the matrix consisting of the columns of C that lie in the right half of M; if there are none, write $C = \emptyset$. Similarly, ${}^MC$ consists of the columns of C that lie in the left half of M.

Now let $M = (X, Y)$, and assume that rank $M = r$. Let E(M) be the totality of matrices obtained from M by operations $e \in E$ on columns. The matrices of E(M) are of rank r. Let the number of columns of a matrix be denoted by k. There is an integer $k_0 \geq 0$ such that (1) for some member K of E(M) and some nonsingular $r \times r$-submatrix C of K, $k({}^KC) = k_0$ and (2) k_0 is maximal with respect to (1). *It is sufficient to prove that* $k_0 = r$.

We suppose $k_0 < r$ and force a contradiction. Choose K and C satisfying (1) and (2). Say K, given by its column vectors, is $(X_1, \cdots, X_n, Y_1, \cdots, Y_n)$. If $\gamma = (i, j, \cdots, \ell)$ is a subset of the ordered set $(1, \cdots, n)$, write X_γ and Y_γ for $(X_i, X_j, \cdots, X_\ell)$ and $(Y_i, Y_j, \cdots, Y_\ell)$. Let α, β be subsets of $(1, \cdots, n)$ such that ${}^K C = X_\alpha$ and $C^K = Y_\beta$, so that $C = (X_\alpha, Y_\beta)$.

Suppose $\alpha \cap \beta = \emptyset$. Let $j \in \beta$, and let L be the matrix obtained from K by replacing X_j by $X_j + Y_j$. Then $L \in E(M)$. We shall show that L contains a nonsingular $r \times r$-submatrix D such that $k({}^L D) = k + 1$, which contradicts the maximality of k_0. Let $\beta' = \beta - \{j\}$ and

$$D_1 = (X_\alpha, X_j + Y_j, Y_{\beta'}).$$

It will be seen that L contains an $r \times r$-submatrix D such that

$$D \sim D_1, \qquad {}^L D \sim (X_\alpha, X_j + Y_j),$$

where $D \sim D_1$ means that D_1 is obtainable from D by a permutation of the columns. Since $k({}^L D) = k_0 + 1$, it remains only to prove that $\det D \neq 0$. It is sufficient to show that $\det D_1 \neq 0$. We see that

$$\det D_1 = \det(X_\alpha, X_j, Y_{\beta'}) + \det(X_\alpha, Y_j, Y_{\beta'}).$$

Except for sign, the second determinant equals $\det C \neq 0$. As for the first, its columns are in K. Since $j \notin \alpha$, $(X_\alpha, X_j, Y_{\beta'})$ is a submatrix N of K and ${}^K N = (X_\alpha, X_j)$, $k({}^K N) = k_0 + 1$, hence $\det N = 0$ by maximality of k_0. It follows that $\det D_1 \neq 0$.

Suppose $\alpha \cap \beta \neq \emptyset$. Let $j \in \alpha \cap \beta$, $\alpha' = \alpha - \{j\}$, $\beta' = \beta - \{j\}$, and let $\ell \in \beta'$. Let L be obtained from K by replacing X_ℓ by $X_\ell + Y_j$ and X_j by $X_j + Y_\ell$. Then $L \in E(M)$. We shall show that L contains a nonsingular $r \times r$-submatrix D such that $k({}^L D) = k_0 + 1$, a contradiction. Let

$$D_1 = (X_{\alpha'}, X_\ell + Y_j, X_j + Y_\ell, Y_{\beta'}).$$

It will be seen that L contains an $r \times r$-submatrix D such that

$$D \sim D_1, \qquad {}^L D \sim (X_{\alpha'}, X_\ell + Y_j, X_j + Y_\ell).$$

Since $k({}^L D) = k_0 + 1$, it is sufficient to show that $\det D \neq 0$, hence that $\det D_1 \neq 0$.

We have the relation

$$\det D_1 = \det(X_j, X_\ell, X_j, Y_{\beta'}) + \det(X_{\alpha'}, Y_j, X_j, Y_{\beta'})$$
$$+ \det(X_{\alpha'}, X_\ell, Y_\ell, Y_{\beta'}) + \det(X_{\alpha'}, Y_j, Y_\ell, Y_{\beta'}).$$

The third and fourth determinants are zero, since Y_ℓ occurs twice in each. The first is zero since its columns are distinct columns of K and $k_0 + 1$ of them are in the left half of K. The second determinant equals $\pm\det C$. Hence $\det D_1 \neq 0$.

COROLLARY. $Sp^o(n, F, a, b) = Sp(n, F)$.

That is, $Sp(n, F)$ is generated by the elementary automorphisms; a proof of this is also given in [1]. Let $T \in Sp(n, F)$. The vectors $Ta_1, \cdots, Ta_n$ are linearly independent. Let S be an element of Sp^o such that the coefficient matrix of

$STa_1, \cdots, STa_n$ is (J, Q), which in this case is (I, Q) (I denotes the identity matrix). The coefficient matrix of the images of $a_1, \cdots, a_n$ under the identity automorphism is (I, 0). Hence, by the uniqueness of Q, we see that Q = 0, so that ST leaves each a_i of the basis (a, b) fixed. One verifies readily that ST is therefore a product of automorphisms of type E_1, hence $ST \in Sp^o$ and $T \in Sp^o$.

ACTIONS ON 2-MANIFOLDS

7. From here on we shall be concerned with actions of $Z^r = Z_p \times \cdots \times Z_p$ (where p is a prime) on 2-manifolds. As a *group*, Z_p denotes the additive group of the *field* $Z_p = Z/pZ$. Matrices and vector spaces *are always understood to be over the field* Z_p.

Let

$$0 \to C \xrightarrow{\sigma} V \xrightarrow{\tau} W \to 0 \tag{7.1}$$

be an exact sequence of vector spaces over Z_p, and let $c_1, \cdots, c_m$ $(m \geq 1)$ be elements that span C and satisfy the single relation $\sum c_i = 0$. Assume that W is symplectic of dimension 2n $(n \geq 1)$. Identify C with a subspace of V. An automorphism T of V will be called *canonical* if it permutes the vectors c (thus leaving C invariant) and induces a symplectic automorphism in W. The canonical automorphisms of V form a group K(V). Let $(a, b, c) = (a_1, \cdots, a_n, b_1, \cdots, b_n, c_1, \cdots, c_m)$ be elements of V such that $(\tau a, \tau b)$ is a canonical basis for W and the c are as above. The elements (a, b, c) span V, and we shall refer to them as a *canonical generating set*. In terms of a canonical generating set (a, b, c), a canonical automorphism T has the form

$$a_i \to \sum A_{ij} a_j + \sum B_{ij} b_j + \sum \lambda_{ik} c_k \qquad (i = 1, \cdots, n),$$

$$b_i \to \sum A'_{ij} a_j + \sum B'_{ij} b_j + \sum \lambda_{ik} c_k,$$

$$c_h \to c_{\sigma(h)} \qquad (h = 1, \cdots, m),$$

where σ is a permutation of $(1, \cdots, m)$ and the coefficients of the a's and b's form the matrix of a uniquely determined symplectic automorphism of a symplectic space of dimension 2n relative to a symplectic basis. (The λ are not unique, since the c are not linearly independent.)

We now extend the action of $e \in E$ to sequences

$$(x, y, z) = (x_1, \cdots, x_n, y_1, \cdots, y_n, z_1, \cdots, z_m)$$

by the rule e(x, y, z) = (e(x, y), z), and introduce new operations on such sequences:

$$E': \quad x_i \to x_i + \sum \lambda_{ij} z_j, \qquad y_i \to y_i + \sum \lambda'_{ij} z_j \qquad (i = 1, \cdots, n),$$

$$z_i \to z_i \qquad (i = 1, \cdots, m),$$

$$E'': \quad (x, y) \to (x, y) \qquad z_i \to Z_{\sigma(i)} \qquad (i = 1, \cdots, m),$$

where σ is a permutation of $(1, \cdots, m)$.

Let (a, b, c) be a canonical basis for V. Then, if $e \in E$, the correspondence $(a, b, c) \rightarrow e(a, b, c)$ defines a canonical automorphism T of V whose matrix relative to (a, b, c) is that of e. We shall say that T *is of type* E *relative to* (a, b, c). Similarly, we have canonical automorphisms of types E', E" relative to (a, b, c). From (6.1) and the general form for canonical autormophisms we see that every canonical automorphisms is the product of automorphisms of types E, E', E" relative to a given canonical basis.

Note that if e, e', e" are elements of E, E', E", then

$$ee'' = e''e, \qquad e'e'' = e''e_0', \tag{7.2}$$

where e_0' is a uniquely determined element of E'.

8. Let M consistently represent a compact oriented 2-manifold. It is easy to see that if $a \in A_f(M, G)$ and G is finite, then $\mu(a)$ is an oriented 2-manifold and each element of G preserves orientation on $\mu(a)$.

Since $H_1(M)$ is free and finitely generated, Theorem (5.1) is applicable.

Let n be the genus of M, and let $\gamma_1, \cdots, \gamma_m$ be the oriented boundary curves, and $c_1, \cdots, c_m$ the elements of $H_1(M, Z_p)$ represented by the γ. Let N be the closed oriented 2-manifold obtained from M by attaching an oriented disc at each γ, and let

$$V = H_1(M, Z_p), \qquad W = H_1(N, Z_p).$$

V and W are vector spaces over Z_p, and $\dim V = 2n + m - 1$ if $m > 0$ and $\dim V = 2n$ if $m = 0$, that is, if there are no boundary curves. The c_i satisfy the single relation $\sum c_i = 0$; let C be the subspace of V that they span.

Let j be the inclusion $M \rightarrow N$, and i_* the injection $C \rightarrow V$. The sequence

$$0 \longrightarrow C \xrightarrow{i_*} V \xrightarrow{j_*} W \longrightarrow 0$$

is exact. W carries a nondegenerate alternating bilinear form $u \cdot v$, namely the intersection number, hence is symplectic over Z_p. Thus we have associated with M the situation described in Section 7.

Let $(\alpha, \beta, \gamma) = (\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n, \gamma_1, \cdots, \gamma_m)$ be a system of oriented simple closed curves on M such that (1) the γ's are the boundary curves of M, (2) the only intersections between the curves are the points $\alpha_i \cap \beta_i$ $(i = 1, \cdots, n)$, (3) each intersection number $\alpha_i \cdot \beta_i$ is 1. Such systems exist; call them *canonical.* If (α, β, γ) is a canonical system on M, then the corresponding sequence (a, b, c) of elements of $H_1(M, Z_p)$ is a canonical generating set in V.

(8.1) *The canonical generating sets in* V *are precisely the sets of elements of* $H_1(M, Z_p)$ *represented by canonical systems of curves on* M. *The group* $\mathcal{K}$ *of canonical automorphisms* (Section 7) *of* V *is the group* $\mathcal{A}$ *of automorphisms induced by homeomorphisms* $M \rightarrow M$.

The proof of the first part is elementary, and we omit it. To prove the second part, let t be a homeomorphism $M \rightarrow M$. Then t permutes the boundary curves; since it preserves orientation of M (as agreed) it preserves that of the boundaries. Hence the automorphism t_* of V permutes the c's. Now t can be extended in an obvious manner to a homeomorphism $t: N \rightarrow N$, and $t_*': W \rightarrow W$ is independent of the extension. One sees that t'_* is the automorphism of W induced by t_*, and that it is symplectic because t' preserves the orientation of N and therefore preserves intersection numbers. Hence t_* is canonical.

Conversely, let T be a canonical automorphism of V. We need to prove the existence of a homeomorphism $t: M \to M$ such that $t_* = T$. Let (α, β, γ) be a canonical system of curves on M, and let (a, b, c) be the corresponding set of canonical generators. It is sufficient to assume (Section 7) that relative to (a, b, c) the automorphism T is of type E, E', or E".

LEMMA. *Let* (α, β, γ), $(\alpha', \beta', \gamma')$ *be canonical systems on* M. *There is a homeomorphism* $t: M \to M$ *such that* t *maps each curve of the first system homeomorphically onto the corresponding curve of the second.*

This follows from the fairly elementary fact that M is homeomorphic to a "standard" 2-manifold M^* in such a way that the curves of each canonical system correspond to the curves of a standard canonical system on M^*. We shall describe M^*, since we need to use it later. Let S^2 be the extended z-plane, and let ω_k $(k = 1, \cdots, n)$ be a circle of radius 1/4, center at $z = k + i$, and with clockwise orientation. Let $\overline{\omega}_k$ be the image of ω_k under $z \to \bar{z}$. Let ξ_k $(k = 1, \cdots, m)$ be a circle of radius 1/4, center at $z = n + k$. Let P_k be the point of ω_k nearest the axis of reals. Let δ_k be the oriented linear segment $\overline{P}_k P_k$. Now remove the interiors of all the circles, and orient the resulting manifold S_0^2 so that the orientations received by the ω's and ξ's as boundary curves are the original orientations reversed. Identify corresponding points of ω_k and $\overline{\omega}_k$ for each k. This gives an oriented manifold M^* in which the images of the ω's, the δ's, and the ξ's form a standard canonical system $(\alpha^*, \beta^*, \gamma^*)$.

Returning now to the proof of (8.1), suppose T is of type E_3, say

$$Ta_1 = a_1 + a_2, \qquad Tb_1 = b_i,$$

$$Ta_2 = a_2, \qquad Tb_2 = b_2 - b_1,$$

$$Ta_i = a_i \quad (i > 2), \qquad Tb_i = b_i \quad (i > 2).$$

By the lemma, it is sufficient to show the existence of a canonical system $(\alpha', \beta', \gamma')$ in M such that the corresponding canonical generators are (Ta, Tb, Tc). We may assume that (α, β, γ) are the "standard" curves $(\alpha^*, \beta^*, \gamma^*)$ described above. In S_0^2, let Y be the oriented line segment $P_1 P_2$, and let $\eta = Y - \overline{Y}$. The image of η in M^* is an oriented simple closed curve, call it β'_2. In S_0^2, let ζ be the simple closed curve consisting of the linear interval joining $z = 1/2$ and $z = 5/2$ and the vertical half-lines rising from these two points. Give ζ the clockwise orientation. Let α'_1 be the image of S. Let

$$\alpha'_i = \alpha_i \ (i \neq 1), \qquad \beta'_i = \beta_i \ (i \neq 2), \qquad \gamma'_i = \gamma_i \ (i = 1, \cdots, m).$$

It will be seen that $(\alpha', \beta', \gamma')$ is a canonical set of curves in M^*. Moreover, $a'_1 = a_1 + a_2$, $b'_2 = b_2 - b_1$. This last can be seen as follows. Keeping the end points fixed, deform η in S_0^2 to the arc traced by a point P that drops vertically from P_1 to the real axis, proceeds to the right, and rises vertically to P_2. Simultaneously, let $\overline{Y}$ undergo the corresponding deformation. The resulting deformation of η defines in M^* a deformation of η (whose homology class is b'_2) to a loop whose homology class if $b_2 - b_1$. Hence $b'_2 = b_2 - b_1$. We have now proved that for the case considered, canonical automorphisms are induced by homeomorphisms $M \to M$. The remaining cases can be treated in similar manner, but we omit the details.

To complete the proof of (8.1), we need to show that for each (a, b, c) there exists a canonical system (α, β, γ) on M that represents (a, b, c). Let $(\alpha', \beta', \gamma')$ be some canonical system of curves on M, and let (a', b', c') be the corresponding generating set. Let T be the automorphism of V defined by $a_i' \to a_i$, $b_i' \to b_i$, $c_j' \to c_j$. Since T is clearly canonical, it is induced, say, by $t: M \to M$. Let $\alpha_i = t\alpha_i'$, $\cdots$. Then (α, β, γ) is a canonical system on M, and the corresponding generating set is (a, b, c).

(8.2) COROLLARY. *The equivalence classes of* $A_f(M, Z_p)$ *are in one-to-one correspondence with the* $\mathcal{K}$*-equivalence classes of* $[H_1(M, Z_p), Z_p]_e$, *where* $\mathcal{K}$ *is the group of canonical automorphisms of* $V = H_1(M, Z_p)$.

9. In computations, it is more convenient to deal with bases than with generating sets.

Let

$$(9.1) \qquad 0 \to \hat{C} \xrightarrow{\hat{\sigma}} \hat{V} \xrightarrow{\hat{\tau}} \hat{W} \to 0$$

be an exact sequence, where $\hat{W}$ is a copy of W (Section 7), and let $\hat{c}_1, \cdots, \hat{c}_m$ be a basis for $\hat{C}$. Identify $\hat{C}$ with a subspace of $\hat{V}$, and define canonical automorphisms of $\hat{V}$ just as for V (Section 7). Let $\hat{\mathcal{K}}$ be the group of these automorphisms. Call $(\hat{a}, \hat{b}, \hat{c})$ a canonical basis for $\hat{V}$ if $(\hat{\tau}\hat{a}, \hat{\tau}\hat{b})$ is a canonical basis for $\hat{W}$. Now let (a, b, c) be a canonical generating set in V, and $(\hat{a}, \hat{b}, \hat{c})$ a canonical basis in V. Let θ be the epimorphism $\hat{V} \to V$ defined by $\hat{a}_i \to a_i$, $\hat{b}_i \to b_i$, $\hat{c}_i \to c_i$, and let θ' be the epimorphism $\hat{W} \to W$ defined by $\hat{\tau}\hat{a}_i \to \tau a_i$, $\hat{\tau}\hat{b}_i \to \tau b_i$. Then θ and θ' define a commutative diagram with (9.1) above, (7.1) below. To every canonical automorphism $\hat{T}$ of $\hat{V}$ there corresponds a unique canonical automorphism T of V such that $\theta\hat{T} = T\theta$. Let $[\hat{V}, Z_p^r]_e^0$ consist of the elements of $[\hat{V}, Z_p^r]$ that vanish on ker θ. The mapping

$$(9.2) \qquad [V, Z_p^r]_e \to [\hat{V}, Z_p^r]_e^0$$

defined by $f \to f\theta$ is bijective. Since $\hat{T}(\ker \theta) \subset \ker \theta$, $[\hat{V}, Z_p^r]_e^0$ is the union of $\hat{\mathcal{K}}$-equivalence classes.

(9.3) *The mapping* (9.2) *induces a one-to-one correspondence between the* $\mathcal{K}$*-equivalence classes of* $[V, Z_p^r]_e$ *and the* $\hat{\mathcal{K}}$*-equivalence classes of* $[\hat{V}, Z_p^r]_e^0$.

(9.4) COROLLARY. *The equivalence classes of* $A_f(M, Z_p^r)$ *are in one-to-one correspondence with the* $\hat{\mathcal{K}}$*-equivalence classes of* $[\hat{V}, Z_p^r]_e^0$, *where* $\hat{\mathcal{K}}$ *is the group of canonical automorphisms of* $\hat{V}$.

10. We shall formulate (9.4) in terms of operations on matrices.

We identify $[\hat{V}, Z_p^r]$ with $\hat{V}^* \times \cdots \times \hat{V}^*$ (r factors), where $\hat{V}^*$ is the dual of $\hat{V}$. Say $f = (\hat{v}_1^*, \cdots, \hat{v}_r^*)$. If we write out the components of each $\hat{v}_i^*$ relative to the dual $(\hat{a}^*, \hat{b}^*, \hat{c}^*)$ of the canonical basis $(\hat{a}, \hat{b}, \hat{c})$, we obtain, as matrix of f, an r-rowed matrix $(X, Y, Z) = (X_1, \cdots, X_n, Y_1, \cdots, Y_n, Z_1, \cdots, Z_m)$ in which $X_1, \cdots$ are the columns. The i-th row gives the components of $\hat{v}_i^*$ relative to the $\hat{a}^*$'s, the $\hat{b}^*$'s, and the $\hat{c}^*$'s, respectively. The condition that f be an epimorphism is equivalent to the condition that (X, Y, Z) be of rank r. The condition that f vanish on ker θ is equivalent to the condition that $Z = (Z_1, \cdots, Z_m)$ be a *null-matrix,* in other words, that the sum of the columns of Z be a column of zeros. Two elements $(\hat{u}_1^*, \cdots, \hat{u}_r^*)$ and $(\hat{v}_1^*, \cdots, \hat{v}_r^*)$ of $[V, Z_p^r]_e^0$ are $\mathscr{K}$-equivalent if and only if $\hat{u}_i^* = \hat{T}^* \hat{v}_i^*$, where $\hat{T}^*$ is the dual of a canonical automorphism $\hat{T}$. Now, in terms of a canonical basis (a, b, c) in V, $\hat{T}$ has the same form as T (see Section 7) and hence is the product of elementary automorphisms of types E, E', E" relative to (a, b, c). The matrix of $\hat{T}^*$ in terms of a dual basis is the transpose of the matrix of $\hat{T}$. But if $\hat{T}^*$ is expressed in terms of *components*, its matrix is the *same* as that of $\hat{T}$. It follows that (X', Y', Z') is the component matrix of $(\hat{T}^* \hat{v}_1^*, \cdots, \hat{T}^* \hat{v}_r^*)$ if and only if it can be obtained from (X, Y, Z) by elements of $E \cup E' \cup E''$ acting on columns.

For m, n not both zero, let $\Xi(n, m, p, r)$ be the totality of r-rowed matrices (X, Y, Z) of rank r, where X, Y are $r \times n$-matrices and Z is an $r \times m$-null-matrix. We understand that if $m = 0$, the elements of Ξ are of the form (X, Y), and that if $n = 0$ they are $r \times m$-null-matrices; in any case, they are of rank r. Evidently,

$$\Xi(n, m, p, r) = \emptyset \quad \text{if } r > 2n + m. \tag{10.1}$$

In view of (9.4), we can now state the following.

(10.2) THEOREM. *The number of equivalence classes of* $A_f(M, Z_p^r)$, *where* M *is an oriented compact* 2-*manifold of genus* n *and with* m *boundary curves, equals the number* $\xi(n, m, p, r)$ *of equivalence classes of* $\Xi(n, m, p, r)$ *under the operation* $e \in E \cup E' \cup E''$ *on columns.*

COMPUTATION OF ξ

11. Let Ω_p^r be the collection of subspaces of the vector space Z_p^r. Let $\Psi(n, p, r)$ consist of the matrices (X, Y), not necessarily of rank r, where X, Y are $r \times n$-matrices. Ψ_0 is empty if $n = 0$. For $W \in \Omega_p^r$ and $n > 0$, we introduce a set E'_W of operations on Ψ_0:

$$E'_W\colon X_i \to X_i + \xi_i, \qquad Y_i \to Y_i + \xi'_i \qquad (i = 1, \cdots, n),$$

where the subscripts denote columns and ξ_i, ξ'_i are elements of W. If $e'_W \in E'_W$ and $e \in E$, there exists $f'_W \in E'_W$ such that

$$e'_W e_W = e_W f'_W. \tag{11.1}$$

Now let $\Psi = \Psi(n, p, r)$ consist of the elements of Ψ_0 that have rank r. Call two elements of Ψ equivalent under $E \cup E'_W$ if one can be obtained from the other by a sequence of operations each of which is in $E \cup E'_W$. (Except for the first and last, the successive elements obtained during this process are not necessarily in Ψ.) Let $\eta_W(n, p, r)$ be the number of equivalence classes of $\Psi_W(n, p, r)$ under $E \cup E'_W$.

Let $\Upsilon = \Upsilon(m, p, r)$ be the totality of $r \times n$-null-matrices over Z_p (with $\Upsilon = \emptyset$ if $m = 0$), and let $\zeta = \zeta(m, p, r)$ be the number of equivalence classes of Υ under operations E". For $W \in \Omega_p^r$, let $\Upsilon_W = \Upsilon_W(m, p, r)$ consist of the elements Z of Υ for which $w(Z) = W$, *where* $w(Z)$ *is the element of* Ω_p^r *spanned by the columns of* Z (Υ is the disjoint union of the sets Υ_W, each of which is the union of equivalence classes of Υ under E"). Let $\zeta_W(m, p, r)$ be the number of equivalence classes in Υ_W. For any m, p, r,

$$\zeta = \sum \zeta_W \qquad (W \in \Omega_p^r). \tag{11.2}$$

For $W \in \Omega_p^r$, let $\Xi_W = \Xi_W(n, m, p, r)$ consist of those elements (X, Y, Z) of Ξ for which $Z \in \Upsilon_W$. Ξ_W is the union of equivalence classes of Ξ (under $E \cup E' \cup E''$); let their number be $\xi_W(n, m, p, r)$. Ξ is the disjoint union of the Ξ_W's and $\xi = \sum \xi_W$. Obviously $\xi(0, m, p, r) = \zeta(m, p, r)$. Hence

$$\xi(0, m, p, r) = \sum_W \zeta_W(m, p, r).$$

For any m, n, p, r *with* $m \geq 1$ *and* $n \geq 1$, *and for* $W \in \Omega_p^r$,

$$\xi_W = \eta_W \zeta_W. \tag{11.3}$$

Proof. Let L_1 be a set of η_W inequivalent elements (X, Y) in Ψ, and L_2 a set of ζ_W inequivalent elements Z in Υ_W. L_1 and L_2 are nonempty, since m and n are nonzero. It is sufficient to show that the set L_3 of elements (X, Y, Z), where $(X, Y) \in L_1$ and $Z \in L_2$, is a complete set of nonequivalent elements of Ξ_W. Let $(X, Y, Z) \in \Xi_W$. We show that (X, Y, Z) is equivalent to an element of L_3. We may assume that (X, Y) is of rank r, hence an element of Ψ, since the action on (X, Y, Z) by a suitable element of E' will replace (X, Y) by (X_1, Y_1), say, of rank r (see the proof of (6.1)). By (11.1), there exists a relation $be'_W(X, Y) \in L_1$, where $e'_W \in E'_W$ and b is a product of elements in E. Since $w(Z) = w(Z') = W$, we can regard e'_W as the restriction to Ψ of an element $e' \in E'$. There is a relation $e'' Z \in L_2$, and $e'' be'(X, Y, Z) \in L_3$.--Next, let (X, Y, Z) and (X', Y', Z') be elements of L_3, and suppose they are equivalent under $E \cup E' \cup E''$. We must show that they are identical. By (7.2), there is a relation $e'be''(X, Y, Z) = (X', Y', Z')$, where $e' \in E'$, $e'' \in E''$, and b is a product of elements in E. The elements Z and Z' are equivalent under E'', hence equal. Since $w(Z) = w(Z') = W$, we can regard e' as an element e'_W of E'_W, as far as the effect on (X, Y) is concerned. Therefore (X, Y) and (X', Y') are equivalent under $E \cup E'_W$, hence equal.

12. For $n \geq r$ and dim W = 0, we have the equation

$$\eta_W(n, p, r) = p^{r(r-1)/2}. \tag{12.1}$$

For, since W is the vector 0, operations E'_W are trivial, and the equivalence classes of $\Psi(n, p, r)$ under $E \cup E'_W$ are simply those under E. Each of the latter contains an element (J, Q) (6.1), where Q is uniquely determined by the class. But Q is also uniquely determined [(see (6.1)] by its $r(r - 1)/2$ elements q_{ij} $(i > j)$.

We shall evaluate $\eta_W(n, p, 3)$ $(n \geq 3)$. Let $\rho \subset \Psi(n, p, 3)$ be the totality of $3 \times 2n$-matrices (J, Q) as defined in (6.1). For $(J, Q) \in \rho$, call (q_{21}, q_{31}, q_{32}) the *characteristic* of (J, Q).

(12.2) *If* dim W = 2, *every element of* ρ *is* W-*equivalent to the element whose characteristic is* (0, 0, 0).

Proof. Let C, D be vectors in W. We perform the operation $e'_W \in E'_W$,

$$Y_i \to Y_i + \mu_i C + \nu_i D \qquad (i = 1, \cdots, n),$$

on (J, Q), obtaining (X', Y'), say, which is of rank 3. Let u'_1, u'_2, u'_3 be the vectors whose component matrix is (X', Y'). By operations $e \in E$ on columns, we transform (X', Y') to (J, Q') with characteristic set $(u'_1 \cdot u'_2, u'_1 \cdot u'_3, u'_2 \cdot u'_3)$. The elements $u'_1 \cdot u'_2$ and so forth are linear expressions in $\mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3, q_{21}, q_{31}, q_{32}$

in which the coefficients of the q_{ij} are 1 and the coefficients of $\mu_1, \cdots, \nu_3$ are given by the matrix

$$(12.3) \qquad \begin{pmatrix} c_2 & -c_1 & 0 & d_2 & -d_1 & 0 \\ c_3 & 0 & -c_1 & d_3 & 0 & -d_1 \\ 0 & c_3 & -c_2 & 0 & d_3 & -d_2 \end{pmatrix},$$

where ${}^t(c_1, c_2, c_3) = C$ and ${}^t(d_1, d_2, d_3) = D$. We wish to show that the μ_i and ν_i can be chosen so that $u_1' \cdot u_2' = u_1' \cdot u_3' = u_2' \cdot u_3' = 0$. A sufficient condition for this is that the rank of (12.3) be 3, which we show is the case if C and D are properly chosen. Suppose W contains W^1, where $W^1 = (Z_p, 0, 0)$. Since dim W = 2, we may take $C = {}^t(1, 0, 0)$ and $D = {}^t(0, d_2, d_3)$, where at least one of the d_i is not zero. Then the matrix consisting of columns 2, 3, 6 in (12.3) has determinant $-d_2$, and the matrix consisting of columns 2, 3, 5 has determinant d_3. Hence (12.3) is of rank 3. The argument is similar when W contains W^2 or W^3. Suppose then that $W \subset Z_p^3 - \{W^1, W^2, W^3\}$. Let C and D be nonzero vectors in $W \cap W^{12}$ and $W \cap W^{23}$, respectively, where $W^{ij} = W^i \times W^j$. We see that $C = {}^t(c_1, c_2, 0)$, where $c_1 \neq 0$ and $c_2 \neq 0$, since W would otherwise contain W^1 or W^2. Similarly, $D = {}^t(0, d_2, d_3)$ with $d_2 \neq 0$, $d_3 \neq 0$. The matrix consisting of columns 1, 4, 5 of (12.3) has determinant $c_2 d_3^2 \neq 0$.

(12.4) *Let* $C = {}^t(c_1, c_2, c_3)$ *be a vector in* W. *Every element of* ρ *is* W-*equivalent to an element whose characteristic is of the form* (0, 0, q) *if* $c_1 \neq 0$, (0, q, 0) *if* $c_2 \neq 0$, *and* (q, 0, 0) *if* $c_3 \neq 0$. *In each case,* q *depends only on the equivalence class of the given element.*

Proof. A sufficient condition that there exist values for $\mu_1, \cdots, \nu_3$ such that $u_1' \cdot u_2' = u_1' \cdot u_3' = 0$ (see proof of (12.2)) is that the first two rows of (12.3) be linearly independent, which is the case if $c_1 \neq 0$. A similar argument applies in the other two cases. This proves the first half. To prove the second half, we suppose, for example, that $c_1 \neq 0$, and we consider two equivalent elements (J, Q), (J, Q') of ρ with characteristics (0, 0, q) and (0, 0, q'). We are to prove that q = q'. By (11.3) we may assume that passage from (J, Q) to (J, Q') is effected by an operation $e_W' \in E_W'$ followed by operations $e \in E$. Since dim W = 1, e_W' is given by

$$X_i \to X_i + \lambda_i C_i, \qquad Y_i \to Y_i + \tau_i C_i.$$

Let u_1', u_2', u_3' be the vector whose component matrix is $e_W'(J, Q)$. We find that

$$u_1' \cdot u_2' = \tau_1 c_2 - \tau_2 c_1,$$

$$u_1' \cdot u_3' = \tau_1 c_3 - \tau_3 c_1 + \lambda_2 c_1 q,$$

$$u_2' \cdot u_3' = (1 + \lambda_2 c_2) q + \tau_2 c_3 - \tau_3 c_2.$$

Now these quantities equal 0, 0, q', respectively. Hence, if we multiply them by $c_3, -c_2, c_1$, respectively, and add, we obtain the equation

$$-c_2 \lambda_2 c_1 q + c_1(1 + \lambda_2 c_2) q = c_1 q',$$

and since $c_1 \neq 0$, this implies that q = q'.

13. For $W \in \Omega_p^3$, $\dim W = 1$, let $i(W)$ be the smallest index i such that W contains a vector $C = {}^t(c_1, c_2, c_3)$ with $c_i \neq 0$. Let $(X, Y) \in \Psi(n, p, 3)$ $(n \geq 3)$. We assign to (X, Y) the "normal form" (J, Q) with characteristic

$$(0, 0, q),\ (0, q, 0),\ (q, 0, 0) \quad \text{according as } i(W) \text{ is } 1, 2, \text{ or } 3.$$

Suppose for example, that $i(W) = 2$. Then, under $E \cup E'_W$, each (X, Y) is equivalent to (J, Q), where the characteristic of (J, Q) is $(0, q, 0)$ and q is uniquely determined by the equivalence class of (X, Y) [see (12.4)]. Hence the number of equivalence classes equals the number of possible values of q, namely p. Therefore we see that

(13.1) $$\eta_W(n, p, 3) = p \quad \text{when } \dim W = 1,\ n \geq 3.$$

We also see from (12.2) that

(13.2) $$\eta_W(n, p, 3) = 1 \quad \text{when } \dim W \geq 2,\ n \geq 3.$$

It follows from (12.1), (13.1), (13.2) that if $n \geq 3$, then $\eta_W(n, p, 3)$ depends only on $\dim W$. This is also true of $\zeta_W(m, p, r)$, as can be shown directly. Hence we may let

$$\eta_i(n, p, 3) = \eta_W(n, p, 3) \quad (\dim W = i,\ n \geq 3),$$

$$\zeta_i(m, p, 3) = \eta_W(m, p, 3) \quad (\dim W = i).$$

From (12.1), (13.1), (13.2) we see that for $n \geq 3$,

$$\eta_0(n, p, 3) = p^3, \quad \eta_1(n, p, 3) = p, \quad \eta_k(n, p, 3) = 1 \quad \text{when } k \geq 2.$$

14. It is trivial that

(14.1) $$\zeta_0(0, p, r) = 0,$$

(14.2) $$\zeta_0(m, p, r) = 1 \text{ if } m > 0.$$

Let $Z \in \Upsilon_W(m, p, r)$. Since Z is null, $\dim W = \dim W(Z) < m$. Hence $\Upsilon_W = \emptyset$ if $\dim W \geq m$, and so

(14.3) $$\zeta_i(m, p, r) = 0 \quad \text{if } i \geq m.$$

Further values of ζ_i are

(14.4) $$\zeta_1(2, p, r) = (p - 1)/2,$$

(14.5) $$\zeta_1(3, 3, r) = 3,$$

(14.6) $$\zeta_2(3, 3, r) = 9.$$

To prove (14.4), take $W = W^1$. Then the elements of Υ_W have first rows of the form $(x, -x)$ $(x \neq 0)$, the remaining rows consisting of zeros. A complete set of nonequivalent elements is represented by $(1, -1), (2, -2), \cdots, (\ell, -\ell)$, where $\ell = (p - 1)/2$. For (14.5), take $W = W^1$. Then the elements of Υ_W consist of zeros, except that the first rows are of the form (x, y, z) with $x + y + z = 0$ and x, y, z not all zero. A complete set of nonequivalent elements is represented by

$$(1, 1, 1), \quad (1, 2, 0), \quad (2, 2, 2). \tag{14.7}$$

For (14.6), take $W = W^{12}$. The elements of Υ_W consist of zeros, except that the first two rows are

$$\begin{matrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{matrix}$$

with $\sum x_i = \sum y_i = 0$, the two rows being linearly independent. A complete set of nonequivalent elements is represented by

$$\begin{matrix} 1 & 1 & 1 \\ 1 & 2 & 0, \end{matrix} \quad \begin{matrix} 2 & 2 & 2 \\ 1 & 2 & 0, \end{matrix} \quad \begin{matrix} 1 & 2 & 0 \\ 1 & 1 & 1, \end{matrix} \quad \begin{matrix} 1 & 2 & 0 \\ 2 & 2 & 2, \end{matrix}$$

(14.8)

$$\begin{matrix} 1 & 2 & 0 \\ 0 & 1 & 2, \end{matrix} \quad \begin{matrix} 1 & 2 & 0 \\ 0 & 2 & 1, \end{matrix} \quad \begin{matrix} 1 & 2 & 0 \\ 1 & 0 & 2, \end{matrix} \quad \begin{matrix} 1 & 2 & 0 \\ 2 & 1 & 0, \end{matrix} \quad \begin{matrix} 1 & 2 & 0 \\ 2 & 0 & 1. \end{matrix}$$

15. We shall now determine some values of ξ. From (10.1) and (10.2) we see that

$$\xi(n, m, p, r) = 0 \quad \text{if } r \geq 2n + m,$$

$$\xi(n, 0, p, r) = 0 \quad \text{if } r > 2n,$$

and (12.1) implies that

$$\xi(n, 0, p, r) = \eta(n, p, r) = p^{r(r-1)/2} \quad \text{when } n \geq r.$$

By (11.3) we have for $m > 0$, $n > 0$ the relations

$$\xi(n, m, p, r) = \sum_W \eta_W \zeta_W = \sum_i d_i \eta_i \zeta_i,$$

where $\eta_i = \eta_i(n, p, r)$, $\zeta_i = \zeta_i(m, p, r)$, and d_i is the number of elements of dimension i in Ω_p^r. If $r = 3$, then

$$d_0 = 1, \quad d_1 = d_2 = 1 + p + p^2, \quad d_3 = 1.$$

Using (12.1) and (14.2), we find for $n \geq r$ that

$$\xi(n, 1, p, r) = \eta_0(n, p, r)\zeta_0(1, p, r) = p^{r(r-1)/2}.$$

For $n \geq 3$, (14.3) implies that

$$\xi(n, 3, p, 3) = \eta_0 \xi_0 + (1 + p + p^2)(\eta_1 \zeta_1 + \eta_2 \zeta_2),$$

where $\eta_i = \eta_i(n, p, 3)$, $\zeta_i = \zeta_i(3, p, 3)$. Using the values of η, ζ listed above, we obtained for $n \geq 3$ the values

$$\xi(n, 2, 3, 3) = 1 \cdot 3^3 + (1 + 3 + 3^2)(3 \cdot 1 + 1 \cdot 0) = 66,$$

$$\xi(n, 3, 3, 3) = 1 \cdot 3^3 + (1 + 3 + 3^2)(3 \cdot 3 + 1 \cdot 9) = 261.$$

EFFECTIVE ACTIONS ON SURFACES

16. Let M be a closed oriented 2-manifold, and G a finite group. Denote by $A_e(M, G)$ the totality of effective orientation-preserving actions $a = (G, \mathfrak{M})$ with orbit space M such that $\mathfrak{M}$ is itself a closed oriented 2-manifold. For $a = (G, \mathfrak{M})$, denote by S(a) the totality of points S of M such that ϕ_S^{-1} has fewer than [G: 1] points, and let $B(a) = \phi^{-1} S(a) \subset \mathfrak{M}$. The points of S will be called the *singular points* of a, those of B the *branch points*. A point $\mathfrak{s}$ is a branch point if and only if the stability group $G_{\mathfrak{s}}$ is nontrivial. An equivalence between a and a' induces a one-to-one correspondence between S(a) and S(a') and between B(a) and B(a'). The sets B and S are known to be finite.

Let $a \in A_e(M, G)$. For $\mathfrak{s} \in B(a)$, let $D_{\mathfrak{s}}$ be a topological disc on $\mu(a)$ which is a neighborhood of $\mathfrak{s}$. The discs $D_{\mathfrak{s}}$ can be chosen so that

(1) they are disjoint,

(2) $D_{g\mathfrak{s}} = gD_{\mathfrak{s}}$ $(g \in G, \mathfrak{s} \in B)$,

(3) $\phi D_{\mathfrak{s}}$ is a disc in M which is a neighborhood of $\phi\mathfrak{s}$.

Call $\{D_{\mathfrak{s}}\}$ a *freeing system* of discs if it satisfies (1), (2), (3). Let $\{D_{\mathfrak{s}}\}$ be a freeing system for a, and let

$$\mathfrak{M}^\circ = \mu(a) - \bigcup \text{Int } D_{\mathfrak{s}}, \qquad M^\circ = \mathfrak{M}^\circ - \bigcup \text{Int } \phi D_{\mathfrak{s}}.$$

$\mathfrak{M}^\circ$ and M° are compact oriented 2-manifolds, and a induces a free action $a^\circ \in A_f(M^\circ, G)$ such that $\mu(a^\circ) = \mathfrak{M}^\circ$. We shall say that a° is obtained by *freeing* a.

The proof of the following proposition about $A_e(M, G)$ is a straight-forward exercise in surface topology, and we shall omit it. (General reference: [3, pp. 223-230].)

(16.1) *Let* a_1°, a_2° *be obtained by freeing* a_1, a_2. *Then* $a_1^\circ \sim a_2^\circ$ *if and only if* $a_1 \sim a_2$.

Let a° come from freeing $a \in A_e(M^\circ, G)$ by means of a freeing system $\{D_{\mathfrak{s}}\}$. If γ is a boundary curve of $\mu(a^\circ)$, it is also a boundary curve of one of the discs, say $D_{\mathfrak{s}}$. Obviously $G_\gamma = G_{\mathfrak{s}}$, hence G_γ is not trivial. Conversely, if $a^\circ \in A_f(M^\circ, G)$ and if the stability group of each boundary curve of $\mu(a^\circ)$ is nontrivial, then a° comes from freeing some $a \in A_e(M, G)$. For let $\mathfrak{M}$ be formed from $\mu(a^\circ)$ by identifying each boundary curve of $\mu(a^\circ)$ to a point, and let M be formed from M° in similar manner. There is an obvious induced action $a \in A_e(M, G)$ with $\mu(a) = \mathfrak{M}$, and the images of the boundary curves of $\mu(a^\circ)$ are simply the branch points of a. One sees that a° is equivalent to the actions obtained by freeing a.

Let $a^\circ \in A_f(M^\circ, Z_p^r)$, and let h be an epimorphism $H_1(M, Z_p) \to Z_p^r$ characterizing a° (Section 8). *A necessary and sufficient condition that* a° *comes from freeing some* $a \in A_e(M, Z_p^r)$ *is that*

(16.2) $$h(c_i) \neq 0 \qquad (i = 1, \cdots, m),$$

where $c_1, \cdots, c_m$ *are the elements of* $H_1(M, Z_p)$ *represented by the boundary curves of* $\mu(a^o)$. If (X, Y, Z) is the matrix of h (Section 10), *then* (16.2) *is equivalent to the condition that* Z *contain no column of zeros.*

For given m, n, p, r, let $\mathfrak{T}'$ consist of the elements of $\mathfrak{T}$ that have no columns of zeros, and let Ξ' consist of the elements (X, Y, Z) of Ξ such that $Z \in \mathfrak{T}'$. Then $\mathfrak{T}'$ and Ξ' are unions of equivalence classes (under $E \cup E' \cup E''$). Let the numbers of these classes be ξ', ζ'. In view of (10.2), (16.1), (16.2), and the definition of "freeing", we have the following result.

(16.3) THEOREM. *Let* G *be a finite group, and let* $A_f'(M^o, G)$ *consist of those elements* a *of* $A_f(M^o, G)$ *such that the stability groups of the boundary curves of* a *are nontrivial. Let* n *be the genus of* M^o, *and* m *the number of boundary curves of* M^o. *Let* M *be a closed oriented manifold of genus* n, *and let* $A_e'(G, M)$ *consist of the elements* a *of* $A_e(G, M)$ *that have* m *singular points. The process of freeing establishes a one-to-one correspondence between the equivalence classes of* $A_f'(M^o, G)$ *and* $A_e'(M, G)$. *If* a^o *and* a *are representatives of corresponding equivalence classes, then the number of branch points of* a *equals the number of boundary curves of* a^o. *If* $G = Z_p^r$, *then the number of equivalence classes in* $A_e'(M, G)$ *is* $\xi'(n, m, p, r)$.

If we now define ζ_i' just as ζ_i was defined, we have for $m \geq 1$ and $n \geq 1$ [see (6.1)] the formula

$$\xi' = \sum d_i \eta_i \zeta_i'.$$

Some values of ζ' are

$$\zeta_0'(m, p, r) = 0,$$

$$\zeta_k'(m, p, r) = 0 \quad \text{if } m \leq k,$$

$$\zeta_1'(2, p, r) = \zeta_1(2, p, r) = (r - 1)/2,$$

$$\zeta_2'(3, 3, r) = 2,$$

$$\zeta_2'(3, 3, 4) = 8.$$

To verify the last two values, look at (14.7) and (14.8) and delete each element that has a column of zeros.

By (10.1) and (10.2), we see that

$$\xi'(n, m, p, r) = 0 \quad \text{when } r \geq 2n + m, \tag{16.4}$$

$$\xi'(n, 0, p, r) = 0 \quad \text{when } r > 2n. \tag{16.5}$$

For $n \geq 3$,

$$\xi'(n, m, p, 3) = \eta_0 \zeta_0' + (1 + p + p^2)(\eta_1 \zeta_1' + \eta_2 \zeta_2'),$$

where $\eta_i = \eta_i(n, p, 3)$, $\zeta_i' = \zeta_i'(m, p, 3)$. Hence, for $n \geq 3$,

$$\begin{aligned} \xi'(n, 2, 3, 3) &= 3^3 \cdot 0 + (1 + 3 + 3^2)(3 \cdot 1 + 1 \cdot 0) = 39, \\ \xi'(n, 3, 3, 3) &= 3^3 \cdot 0 + (1 + 3 + 3^2)(3 \cdot 2 + 1 \cdot 8) = 182. \end{aligned} \tag{16.6}$$

17. It is possible to obtain some information about effective actions of Z_p^r on a closed surface of given genus.

Let $a \in A_e(M, Z_p^r)$, and free a to obtain $a^\circ \in A_f(M^\circ, Z_p^r)$. Let $n = \text{genus } M^\circ$, let $\mathfrak{m}$ be the number of boundary curves of $\mathfrak{M} = \mu(a^\circ)$, and let m be the same for M°. We assert that

$$\mathfrak{m} = p^{r-1} m .$$

Indeed, let δ be a component of $\phi^{-1}\gamma$, and γ a boundary curve of M°. The stability group G_δ of δ is nontrivial, and since the induced action (G_δ, δ) is free, G_δ must be cyclic, hence of order p. Hence $\phi^{-1}\gamma$ has $p^r/p = p^{r-1}$ components, and M has mp^{r-1} boundary curves.

Let k and $\mathfrak{k}$ be the Euler characteristics of M° and $\mathfrak{M}^\circ$. Then

$$\mathfrak{k} = 2 - 2\mathfrak{n} - \mathfrak{m} = 2 - 2\mathfrak{n} - p^{r-1}m ,$$

$$k = 2 - 2n - m .$$

Since $\mathfrak{k} = p^r k$, we see that

$$\mathfrak{n} = 1 + (n-1)p^r + (m/2)(p^r - p^{r-1}) .$$

Assume now that $p = 3$, $r = 3$. Then

$$\mathfrak{n} = 1 + 27(n-1) + 9m . \tag{17.1}$$

(Since $3 \leq 2n + m$ (10.1), (n, m) can not be $(0, 2)$, and so $\mathfrak{n}$ will not be negative.)

Z_3^3 can not act effectively on a closed oriented surface of genus $\mathfrak{n}$ unless $\mathfrak{n}$ is given by (17.1) for some n, m with $2n + m \geq 3$ (10.1).

The only values of n, m $(2n + m \geq 3)$ that give $\mathfrak{n} = 1$ are $n = 0$, $m = 3$. Since $\xi'(0, 3, 3, 3) = 0$, there are no effective actions of Z_3^3 on a torus.

The only values of n, m $(2n + m \geq 3)$ giving $\mathfrak{n} = 73$ are $n = 3$, $m = 2$ and $n = 2$, $m = 5$. Hence, on a closed oriented surface of genus 73, there are just $\xi'(3, 2, 3, 3) = 39$ effective actions of Z_3^3 with 18 branch points, and $\xi'(2, 5, 3, 3)$ actions with 45 branch points. We do not have the value of $\xi'(2, 5, 3, 3)$, since it does not lie in the range $n \geq r$ under consideration.

REFERENCES

1. L. E. Dickson, *Linear groups: With an exposition of the Galois field theory*, Dover Publications, New York, 1958.
2. S.-T. Hu, *Homotopy theory*, Academic Press, New York, 1959.
3. B. von Kerékjártó, *Vorlesungen über Topologie, I. Flächentopologie*, Springer, Berlin, 1923.
4. J. Nielsen, *Die Struktur periodischer Transformationen von Flächen*, Danske Vid. Selsk. Mat.-Fys. Medd. 15 (1937), 1-77.

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