VECTOR FIELDS AND CHARACTERISTIC NUMBERS

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Dedicated to R. L. Wilder on his seventieth birthday.

1. INTRODUCTION

A well-known theorem of Heinz Hopf asserts that on a compact manifold the properly counted number of zeros of a vector field equals the Euler number of the manifold. The purpose of this paper is to show that when a vector field satisfies certain differential equations, then there are other relations between the characteristic numbers of the manifold and local invariants of the vector field near its zeros.

The two cases of greatest interest are (i) *where the vector field is holomorphic and* (ii) *where it defines an infinitesimal motion of a Riemannian manifold.* In the first case all the characteristic numbers will be seen to be determined by the zeros of the vector field. In the second case the Pontrajagin numbers are determined—and of course the Euler numbers—but the Stiefel-Whitney numbers are not. In short, the local behaviour at the zeros of the vector field determines all the rational characteristic numbers in both cases.

To describe these relations explicitly, when the vector field X behaves generically at its zeros, we recall first that the Lie derivative in the direction of X is a well-defined differential operator $\mathfrak{L}(X)$ on all the tensor fields over M, and that it *has order* 0 *at the zeros of* X.

Thus, in particular, $\mathfrak{L}(X)$ induces a linear map L(X) of the tangent space TM to M restricted to the set zero(X):

$$L_p(X) = \mathfrak{L}(X) \mid T_p M \qquad (p \in \text{zero}(X)).$$

The vector field X will be called *nondegenerate* if L(X) is nonsingular at all the zeros of X, and the eigenvalues of L(X) will be referred to as the *characteristic roots* of X at its zeros.

Second, we recall that a complex structure on M endows the real tangent bundle $T_R M$ of M with a complex structure. The Chern classes of this C-bundle are therefore well-determined elements $c_i(M) \in H^{2i}(M, Z)$. Suppose now that $\Phi(c) = \Phi(c_i, \cdots, c_m)$ $(m = \dim_C M)$ is a polynomial in the indeterminates c_i with complex coefficients. By replacing c_i with $c_i(M)$, we obtain a cohomology class $\Phi\{c(M)\}$ in $H^*(M; C)$ whose value on the orientation class [M] will be denoted by $\Phi(M)$. Of course, we define the value of a class $u \in H^*(M; C)$ on [M] to be zero if u contains no elements of degree equal to the dimension of M. Hence $\Phi(M) = 0$ unless Φ involves monomials of "*weight* m" in the c_i, that is, expressions of the form

$$w_a = c_1^{a_1} c_2^{a_2} \cdots c_n^{a_n} \qquad (a_1 + 2a_2 + \cdots + n a_n = m).$$

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The values $w_a(M)$ are called the *Chern numbers* of M.

For our purpose it is convenient also to define the *Chern classes* of an endomorphism $A: V \to V$ of a finite-dimensional complex vector space. By definition, $c_i(A)$ is the ith coefficient of the characteristic polynomial of A; that is,

$$\sum \lambda^i c_i(A) = \det(1 + \lambda A). \tag{1.1}$$

Once this is done, we define the value of Φ on such an endomorphism simply by

$$\Phi(A) = \Phi\{c_1(A), \cdots, c_m(A)\}.$$

With these conventions out of the way, I may state the main theorem.

THEOREM 1. *Let* X *be a nondegenerate vector field that preserves a complex structure on the compact connected manifold* M. *Then for every polynomial* $\Phi(c_1, \cdots, c_m)$ $(m = \dim_C M)$ *of weight not greater than* m,

$$\sum_P \Phi(L)/c_m(L) = \Phi(M). \tag{1.2}$$

Here P *ranges over the zeros of* X, *and* L *denotes the* C*-endomorphism induced by* L(X) *on the complex tangent space to* M *at* P.

We state two immediate corollaries.

COROLLARY 1. *A nonvanishing vector field* X *can preserve a holomorphic structure on* M *only if all the Chern numbers relative to that structure vanish.*

COROLLARY 2. *If* M *admits a nonvanishing holomorphic vector field, then* M *bounds.*

Note finally that when $\Phi = c_m$, (1.2) gives the Hopf formula—which in this case simply asserts that $c_m(M)$ is the number of zeros of X.

COROLLARY 3. *When* Φ *has weight less than* m, *then the right-hand side vanishes.*

For example,

$$\sum_P 1/c_m(L) = 0.$$

Theorem 1 is really a byproduct of M. Atiyah's and my work on the generalized fixed point theorem [1], and its history is as follows. A formula of this type was first conjectured to me by V. Guillemin, who derived some special cases of it from our fixed-point formula and the Riemann-Roch formula of Hirzebruch. Next, M. Atiyah pointed out that there really were sufficiently many such special cases to prove the theorem in general as a consequence of our fixed-point formula.

My aim here has been to present a more direct proof in the framework of the theory of "Real Characteristic Classes" as developed by Chern, Pontrajagin, and Weil; this is done in Section 3, after some preliminaries in Section 2. Thereafter I deal with the Riemannian case in Section 3, while Section 4 describes the Guillemin argument, which one may now use in reverse to get at the Riemann-Roch formula for a manifold that admits a nondegenerate holomorphic vector field. The amusing thing about this very special case is that the Todd genus arises in it quite naturally: out of our fixed-point formula.

2. REVIEW OF CURVATURE AND CHARACTERISTIC CLASSES

I shall use the Chern-Weil theory of connections and characteristic classes in its low-brow form as described in [2], for example, and I shall for the most part adhere to the notation of that paper. To review matters briefly, let E be a complex vector bundle over the manifold M, and let $T^*_C M$ be the *complexified cotangent bundle of* M. A *connection on* E then is a differential operator

$$D: \Gamma(E) \to \Gamma(T^* \otimes E) \qquad (T^* = T^*_C M),$$

which, relative to functions, satisfies the derivation law

$$D(fs) = df \otimes s + f Ds \qquad (s \in \Gamma).$$

(Γ denotes C^∞ sections, throughout; all manifolds, functions, and so forth will be assumed to be C^∞, unless it is otherwise indicated.)

If one extends D as an antiderivation to the whole exterior complex of forms on M with values in E, then D^2 is easily seen to be *linear over the functions on* M, whence D^2 can be realized by multiplication with a 2-form K with values in the endomorphism bundle of E. That is, for all $s \in \Gamma(E)$,

$$D^2 s = Ks, \tag{2.1}$$

where $K \in \Gamma(\wedge^2\{T^*_C M\} \otimes \text{End } E)$ and $\wedge^2$ denotes the second exterior power.

The form K is called the *curvature of the connection* D. If $s = \{s_\alpha\}$ is a frame for E over U, then K determines a matrix $K(s) = \|K_{\alpha\beta}(s)\|$ of two-forms on U; it is given by the formula

$$K \cdot s_\alpha = \sum K_{\alpha\beta}(s) s_\beta ,$$

and if $s^1 = As$ is another such frame, then

$$K(s^1) = A^{-1} K(s) A . \tag{2.2}$$

The *Chern forms* of K are now defined in analogy to (1.1) as the coefficients of the characteristic equation of K:

$$\sum_i \lambda^i c_i(K) = \det(1 + \lambda\kappa^{-1} K) \qquad (\kappa = 2\pi\sqrt{-1}). \tag{2.3}$$

This determinant makes good sense, because it may be interpreted locally as the determinant of the matrix of *even forms* $1 + \lambda\kappa K(s)$, and in view of the invariance of the determinant under inner automorphisms, the transformation law (2.2) implies that the resulting form is independent of the frame s. Thus the $c_i(K)$ are well-determined 2i-forms on M, and it is a basic fact in the Chern-Weil theory that $c_i(K)$ *is closed and represents the* i*th Chern class of* E *independently of the connection underlying the construction of* K.

So much for a brief review of C^∞ complex bundles and their characteristic classes. When both E and M have a complex analytic structure, more is true. One may then choose a Hermitian structure (,) on E and exploit the basic incompatibility of this structure with the complex structure to induce a connection D on E whose curvature form K is d″-closed, in the sense that relative to any *holomorphic* frame $s = \{s_\alpha\}$

(2.4) $d''K(s) = 0$ *and* $Ds = \theta(s) \otimes s$ *with* $\theta(s)$ *of type* (1, 0).

Here d'' is, of course, the usual $\partial/\partial\bar{z}$-part of the differential operator d:

$$d = d' + d''.$$

Indeed, define D in terms of the holomorphic frame s by the formula

$$Ds = (d'N \cdot N^{-1})s,$$

where $N = N(s)$ is the matrix of inner products (s_i, s_j). Then $\theta(s) = d'N \cdot N^{-1}$ is clearly of type (1, 0), and $D^2 s = d'' \cdot \theta(s) \cdot s$, whence $K(s) = d'' \theta(s)$.

Finally, a remark on the complexification $T_C M$ of the tangent bundle $T_R M$ when M has a complex structure: Here $T_C M = T_R M \otimes C$. On the other hand, because $T_R M$ has a complex structure of its own, we may single out the subspaces $T' = T'_C M$ and $T'' = T''_C M$ on which these structures agree and anti-agree. Thus

$$\sqrt{-1} \otimes 1 = 1 \otimes \sqrt{-1} \text{ on } T' \quad \text{and} \quad \sqrt{-1} \otimes 1 = -1 \otimes \sqrt{-1} \text{ on } T''.$$

Then

(2.5) $$T_C M = T'_C M \oplus T''_C M,$$

and T' is in duality with the forms of type (1, 0) while T'' is in duality with the forms of type (0, 1).

If $\{z_i\}$ is a local coordinate system over U, say, then the $\partial/\partial z_i$ and $\partial/\partial \bar{z}_i$ span T' and T'', respectively. Hence any real vector field X is locally of the form

$$X \mid U = \sum a_i \frac{\partial}{\partial z_i} + \bar{a}_i \frac{\partial}{\partial \bar{z}_i},$$

and X is holomorphic if and only if its projection X' on T' is holomorphic, that is, if the a_i are holomorphic.

The inner product of a form θ by a vector field is denoted by $i(X)\theta$. Thus the action of $\mathfrak{L}(X)$ on forms is given by

$$\mathfrak{L}(X) = i(X)d + d\,i(X).$$

A holomorphic vector field preserves the type of a form, whence

(2.6) $$i(X')d'' + d''i(X') = 0,$$

for such vector fields.

3. THE PROOF OF THEOREM 1

Suppose now that X is a nontrivial *holomorphic* vector field on the *complex* manifold M. We choose a Hermitian structure on the holomorphic tangent bundle $T'_C M$ to M, and we thus obtain a curvature form K on M, so that the characteristic classes of M can be computed in terms of the Chern forms $c_r(K)$.

Now, if X generated a fibering, it would be easy to construct a metric on $T'_C M$ such that the resulting forms $c_r(K)$ are *basic relative to the fibering*, in other words, such that $i(X)c_r(K) = 0$. Our first aim is to show that in any case *on the complement of the zero set of* X *the* d''*-cohomology class of* $c_r(K)$ *contains a form* $\hat{c}_r(K)$ *such that*

$$i(X')\hat{c}_r(K) = 0 . \tag{3.1}$$

Here, as throughout, the dash denotes projection on $T'_C M$.

Note that once this is done, Theorem 1 holds for nonvanishing X and forms Φ of *top* weight. Indeed, because top forms in the image of d'' have zero integrals, we may then compute the numbers $\Phi(M)$ by integrating the forms $\hat{\Phi} = \Phi(\hat{c}_1, \hat{c}_2, \cdots, \hat{c}_m)$ on [M], where the $\hat{c}_r$ are chosen to satisfy (3.1) on all of M. But then $i(X')\hat{\Phi} = 0$, because $i(X')$ is an antiderivation. On the other hand, in the top dimension $i(X')$ is injective at all points where $X \neq 0$. Hence $\hat{\Phi} = 0$ on M, and so $\Phi(M) = 0$, as Theorem 1 asserts.

We begin with the following lemma.

LEMMA 1. *Let* K *be the curvature of a Hermitian connection on* $T'_C M$, *and let* X *be a holomorphic vector field on* M. *Then there is a global section* $L' \in \Gamma\, \mathrm{End}(T'_C M)$ *that agrees with* $L(X) \mid T'_C M$ *on the zero set of* X:

$$L' \mid \mathrm{zero}(X) = L(X) \mid T'_C M , \tag{3.2}$$

and such that

$$d'' L' = i(X')K . \tag{3.3}$$

Proof. We may interpret a section of $\mathrm{End}(T'_C M)$ as a differential operator of *degree zero*, that is, one which is linear over the functions, from $\Gamma(T'_C M)$ to $\Gamma(T'_C M)$. With this interpretation, let L' be defined by the relation

$$L' \cdot s = \mathfrak{L}(X) \cdot s - i(X) \cdot D \cdot s . \tag{3.4}$$

Then we must verify first of all that L' maps $T'_C M$ into itself, and that it is linear over the functions. To see this linearity, we use the derivation property of D:

$$L' \cdot fs = (Xf) \otimes s + f \cdot \mathfrak{L}(X) \cdot s - i(X)\{df \otimes s + f Ds\} = fL' \cdot s .$$

Now the stability of $T'_C M$ under L' follows from the fact that holomorphic sections s are preserved by $\mathfrak{L}(X)$.

The property (3.2) is self-evident from (3.4), because $i(X)$ is the zero operator on the zero set of X.

There remains (3.3). By the foregoing, we may verify this relation for a local holomorphic frame s. Then $d'' \cdot \mathfrak{L}(X) \cdot s = 0$, so that

$$d'' \cdot L' \cdot s = -d'' i(X) Ds .$$

However, Ds is of type (1, 0), whence $i(X)$ may be replaced by $i(X')$. Now, using (2.1) and (2.5), one obtains the relations

$$d'' \cdot L' \cdot s = i(X') \cdot d'' D \cdot s = i(X') \cdot K \cdot s . \tag{3.5}$$

Next suppose that

$$\phi: \bigotimes_1^k \mathfrak{gl}(m, C) \to C$$

is a symmetric, multilinear, C-valued function of k arguments from the Lie algebra of $m \times m$ matrices $\mathfrak{gl}(m, C)$ into C, and that it is invariant under the adjoint section of $\mathfrak{gl}(m, C)$ on itself. Thus

$$\sum_{i=1}^{k} \phi(x_1, \cdots, [x_i, y], \cdots, x_k) = 0 \qquad (x_i, y \in \mathfrak{gl}(m, C)).$$

Such a function can naturally be extended to arguments x_i that are forms on M with values in the endomorphism bundle of a bundle E over M, and it then takes its values in the forms on M.

Indeed, relative to a local frame s over V, such x_i have matrix representatives $x_i(s)$ on which ϕ may be evaluated to yield forms on V, and by the invariance hypothesis (3.5) this form is seen to be independent of s.

In our situation we may therefore meaningfully speak of the forms

$$(3.6)\qquad \begin{cases} \phi_K = \phi(K, K, \cdots, K), \\ \phi_K^{(1)} = \binom{k}{1} \phi(L', K, \cdots, K), \\ \phi_K^{(2)} = \binom{k}{2} \phi(L', L', K, \cdots, K), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \phi_K^{(k)} = \phi(L', L', \cdots, L') \end{cases}$$

Note also that all the *forms* $c_i(K)$*—and therefore all the forms generated by these—can be realized as* ϕ_K *for a suitable* ϕ. This follows directly if we polarize and symmetrize the determinant polynomial. Hence (3.1) will be established once we construct for each ϕ a form η_ϕ, defined on $\hat{M} = M$ - zero(X), such that

$$(3.7)\qquad i(X')\,[\phi_K + d''\eta_\phi] = 0 \quad \text{on } \hat{M}.$$

We shall now give a recipe for η_ϕ in terms of the forms $\phi_K^{(r)}$ constructed above. *For this purpose, let* π *be the* 1*-form of type* (1, 0) *determined by the orthogonal projective on* X' *in* T' —that is, the form given on the vector field Y by the formula

$$\pi(Y) = (Y', X')/|X'|^2.$$

My claim is that then the form

$$\eta_\phi = \pi\{\phi_K^{(1)} + \phi_K^{(2)} d''\pi + \cdots + \phi_K^{(k)} (d''\pi)^{k-1}\}$$

has the desired property (3.7).

First we derive the basic identities

(3.8) $$i(X')\phi_K^{(r)} = d''\phi_K^{(r+1)} \qquad (r = 0, \cdots, k-1;\ \phi_K^{(0)} = \phi_K),$$

(3.9) $$i(X')d''\pi = 0.$$

To establish (3.8), we use the antiderivative property of $i(X')$ and the symmetry of ϕ to obtain the relation

$$i(X')\phi_K^{(r)} = \binom{k}{r}(k-r)\phi(\underbrace{L, L, \cdots, L}_{r}; i(X')K; K, \cdots, K).$$

Next, using the same property of d'' and (2.4), we see that

$$d''\phi_K^{(r+1)} = \binom{k}{r+1}\cdot r\,\phi(\underbrace{L, \cdots, L}_{r+1}; d''L; K, \cdots, K),$$

and this then yields (3.8). The other relation follows directly from (2.6), whence $i(X')d''\pi = -d''i(X')\pi = -d''\cdot 1 = 0$.

Let us next define Λ as the sum

$$\Lambda = \phi_K^{(1)} + \phi_K^{(2)}(d''\pi) + \cdots + \phi_K^{(k)}(d''\pi)^{k-1}.$$

Then, as a consequence of (3.8), we get the equation

$$d''\Lambda = i(X')\{\phi_K + (d''\pi)\Lambda\},$$

from which it follows, in particular, that $i(X')d''\Lambda = 0$.

Hence, finally, one proves (3.7) by the reduction

$$\begin{aligned} i(X')\{\phi_K + d''\eta\} &= i(X')(\phi_K + d''[\pi\cdot\Lambda]) \\ &= i(X')\{\phi_K + (d''\pi)\Lambda - \pi d''\Lambda\} \\ &= d''\Lambda - d''\Lambda + \pi i(X')d''\Lambda = 0. \end{aligned}$$

At this stage, it is also clear how the rest of the proof of Theorem 1 must go, at least for forms of weight m. Given a $\Phi(c_1, \cdots, c_m)$ of weight m, we may construct the multilinear ϕ associated with Φ so that

$$\Phi(M) = \phi_K[M].$$

Next, let M_ε be the complement of an ε-neighborhood of the set zero(X), relative to some Riemannian structure on M. Then, because ϕ_K is a smooth form on all of M, and zero(X) is at least of codimension 2, we may write

(3.10) $$\Phi(M) = \lim_{\varepsilon\to 0}\phi_K[M_\varepsilon].$$

Now, since ϕ_K has top dimension, η_ϕ is of type $(m, m-1)$, whence

$$d''\eta_\phi = d\eta_\phi.$$

Applying Stokes' theorem, we therefore obtain the equation

$$\Phi(M) = \lim_{\varepsilon \to 0} (\phi_K + d''\eta_\phi)[M_\varepsilon] - \lim_{\varepsilon \to 0} \eta_\phi[\partial M_\varepsilon].$$

The first term now drops out, by our previous argument, so that we are finally left with

$$\Phi(M) = -\lim_{\varepsilon \to 0} \eta_\phi[\partial M_\varepsilon]. \tag{3.11}$$

In this way, then, the characteristic numbers $\Phi(M)$ *are determined by the local behavior of* X *along its zero set.*

To derive the formula (1.2), it is therefore merely necessary to evaluate the limit on the right-hand side of (3.11) when zero(X) consists of isolated points at which X vanishes nondegenerately. The boundary ∂M_ε then simply consists of ε-spheres S_ε about the exceptional points.

Let P be one of these, and let $\lambda_1, \cdots, \lambda_m$ be the eigenvalues of $L_P^{\prime}$. For simplicity we shall first assume these to be distinct. We shall study the behavior of η_ϕ near P in detail, and for this purpose it is convenient to introduce good coordinates $\{z_i\}$ and good metrics on M and T'(M) near P. Precisely, we first choose the z_i so that, on a coordinate patch U containing P,

$$X' \mid U \sim \sum \lambda_i z_i \frac{\partial}{\partial z_i},$$

the $\sim$ denoting equality modulo terms that vanish to at least the second order at P.

Next we choose a Hermitian structure on T'(M) so that

$$\left(\sum |z_\alpha|^2\right)\pi \mid U \sim \sum \lambda_i^{-1} \bar{z}_i \, dz_i .$$

Setting $d''\pi = w$, we easily see that

$$\left(\sum |z_\alpha|^2\right)^m \pi \cdot w^{m-1} \sim \frac{(m-1)!}{\prod \lambda_j} \left\{ \sum_{i=1}^{m} dz_1 d\bar{z}_1 \cdots (\bar{z}_i^{(i)} dz_i) \cdots dz_n d\bar{z}_n \right\} \tag{3.12}$$

while

$$\left(\sum |z_\alpha|^2\right)^k \pi \cdot w^{k-1} \sim 0 \qquad \text{for } k < m-1 . \tag{3.13}$$

It follows that if the metric on M near P is chosen so that $S_\varepsilon = \left\{ z \,\middle|\, \sum |z_\alpha|^2 = \varepsilon \right\}$, then the limit $\eta_\phi[S_\varepsilon]$ as $\varepsilon \to 0$ depends only on the last term in the expansion of η_ϕ. Hence, by (3.12) and (3.13), we see that

$$\lim_{\varepsilon \to 0} \eta_\phi(S_\varepsilon) = \frac{\rho_m}{\pi \lambda_i} \cdot \phi(L', L', \cdots, L') \qquad (L' = L_P^{\prime}), \tag{3.14}$$

where ρ_m is a universal constant depending only on m, given by the integral

$$\rho_m = (m-1)! \int_{\sum |z_\alpha|^2 = 2} \sum_{i=1}^{n} |dz_1|^2 \cdots (\bar{z}_i \, dz_i) \cdots |dz_n|^2 .$$

Let us now apply this result to the computation of $\Phi(M)$, where Φ is a polynomial of weight m. Let ϕ be its polarization, so that

$$\phi(A, \cdots, A) = \Phi(c_1(A), \cdots, c_m(A)).$$

Then it is clear by (2.3) that $\Phi(M) = \kappa^{-m}\phi(K, K, \cdots, K)\cdot[M]$. Applying (3.10) and (3.14), we see that

$$\Phi(M) = -\kappa^m \rho_m \sum_P \frac{\Phi(L'_P)}{c_m(L'_P)},$$

which agrees with Theorem 1 up to a universal constant. This constant can be evaluated either by a direct computation of ρ_m or by an example. Indeed, if we grant the Hopf formula, which corresponds to the case $\Phi = c_m$ on complex projective space, then we see that $\kappa^m p_m = -1$, as was to be shown.

There remains the case where ϕ has degree $k < m$. To deal with it, we consider the top-dimensional form

$$\psi = (d''\pi)^r \cdot \phi_K \qquad (r = m - k).$$

It is easily verified that $i(X')\{\psi + (d''\pi)^{m-k} d''\eta_\phi\} = 0$ on $\hat{M}$. Hence, by our previous argument,

$$\psi[M] = \sum \frac{\phi(L'_P)}{c_m(L'_P)}.$$

On the other hand, $\psi = d\{\pi(d''\pi)^{r-1}\phi_K\}$, so that the left-hand side vanishes.

When the eigenvalues of L_P are not distinct (so that $X \mid U$ need not be in diagonal form), the argument is entirely similar once one observes the identity

$$\left(\sum \lambda_{ij}\, dz_i\, d\bar{z}_j\right)^m = \det \|\lambda_{ij}\| \left(\sum dz_i\, d\bar{z}_j\right)^m.$$

4. THE RIEMANNIAN CASE

In this section, we consider the case of an oriented compact Riemann manifold M that admits an infinitesimal isometry X, and this time our aim is to relate the zeros of X to the Pontrajagin numbers of M. Recall therefore that these numbers may be derived from the Chern classes of the complexified tangent bundle $T_C M = T_R M \otimes C$ in much the same way as before. If dim M = m, we start with a polynomial $\Phi(c)$ in m variables, and we again *define the value of* Φ *on* M *as the integral of the form* $\Phi(c_1(T_C M), c_2(T_C M), \cdots)$ *on* [M]. *Note, however, that this time only the terms of weight* m/2 *contribute to* $\Phi(M)$. Thus $\Phi[M] = 0$ if dim M is odd. Actually, $\Phi(M) = 0$ unless the dimension of M is divisible by 4. Indeed, if E is obtained from a real bundle E_R by complexification, $E = E_R \otimes C$, then we can choose a connection on E_R that preserves a Riemannian structure on E_R. As a consequence, the induced connection on E will have a skew symmetric curvature tensor K(s) relative to each orthonormal frame s. But then all odd terms in the expansion of $\det(1 + \lambda\kappa^{-1}K(s))$ vanish, whence $c_i(K) = 0$ for odd i.

To state our analogue of Theorem 1, we need one additional well-known fact. The vector field X is an *infinitesimal isometry* on M if and only if the Lie derivative

$\mathfrak{L}(X)$ acts *skew-symmetrically* on the vector fields on M relative to the Riemannian norm (,)—that is, if and only if

$$(\mathfrak{L}(X)Y, Z) + (Y, \mathfrak{L}(X)Z) = X(Y, Z) \qquad \text{for } X, Y, Z \in \Gamma(T_R M).$$

It follows immediately that the endomorphism L(X) determined by $\mathfrak{L}(X)$ at the zeros of X is also skew-symmetric. Hence det L(X) = 0 if the dimension is odd, and it is a "square" when the dimension is even. We need a particular square root of this determinant in the even case, namely, the one that is determined by the orientation of M in the following manner:

Let M have dimension m = 2k, and let $e_1, e_2, \cdots, e_m$ be an orthonormal base for $T_P M$ such that

$$L_P e_{2i-1} = \lambda_i e_{2i} \qquad (\lambda_i \in R),$$

and such that, in addition, $e_1 \wedge \cdots \wedge e_m$ is in the orientation of [M].

The product $\lambda_1 \cdots \lambda_m$ is then the square root of $\det L_P(X)$ we seek, and we shall denote it by $\det^{1/2}(L_P)$ or $c_m^{1/2}(L_P)$. With this understood, we can formulate our second theorem as follows.

THEOREM 2. *Let X be an infinitesimal motion of a compact, oriented, even-dimensional Riemann manifold M whose zeros are nondegenerate. If $\Phi(c_1, \cdots, c_m)$ is any polynomial of weight not greater than m/2, then*

$$\sum_P \Phi(L)/\det^{1/2}(L) = \Phi(M) \qquad (L = L_P(X)),$$

where P ranges over the zeros of X.

The proof parallels our earlier argument. We begin by endowing $T_R M$ with the *unique* connection D that preserves the Riemann structure on M and has zero torsion. Next we extend D to all the tensor powers of $T_R M$ as a derivation, and extend it further as an antiderivation to the forms on M with values in $T_R^{(p)}(M)$. Since our Riemann structure identifies $T_R(M)$ with its dual, we thus also have an induced connection on $T_R^*(M)$ and on $\text{End}\{T_R^*(M)\}$, and so forth.

The proper analogue of Lemma 1 is now the following:

LEMMA 2. *In the situation under discussion, the differential operator*

$$L = \mathfrak{L}(X) - i(X) \cdot D$$

is of degree zero and so determines a section L of the endomorphism bundle $\text{End}(T_R M)$; *this section agrees with L(X) on the zero set of X:*

$$L \mid \text{zero}(X) = L(X), \tag{4.1}$$

and its covariant derivative is given by the equation

$$DL = i(X)K. \tag{4.2}$$

Proof. The f-linearity of D follows immediately from the derivation properties of $\mathfrak{L}(X)$ and D. The relation (4.1) follows directly from the definitions. To see (4.2) most conceptually, one can argue as follows. The value of DL on a section s of $T_R M$ is given by

$$\begin{aligned} \mathrm{DL}\cdot s &= \mathrm{D}\cdot \mathrm{L}\cdot s - \mathrm{L}\cdot \mathrm{D}\cdot s \\ &= \mathrm{D}\cdot \mathfrak{L}(\mathrm{X})\cdot s - \mathrm{D}\cdot i(\mathrm{X})\cdot \mathrm{D}\cdot s - \mathfrak{L}(\mathrm{X})\cdot \mathrm{D}\cdot s + i(\mathrm{X})\mathrm{D}^2\cdot s. \end{aligned} \tag{4.3}$$

Now the connection D is canonically associated with the Riemannian structure on M. Hence motions—and therefore also infinitesimal ones—preserve it. Thus $\mathfrak{L}(X)\cdot D = D\cdot \mathfrak{L}(X)$. Finally, since both terms of (4.2) are tensor fields, it is sufficient to verify this relation relative to *one* local frame, and such a frame may always be chosen so that it is parallel to X, in other words, so that $i(X)\cdot D\cdot s = 0$. But then (4.3) reduces to $DL\cdot s = i(X)D^2\cdot s$, and here the right member equals $i(X)K\cdot s$.

Our next lemma is the analogue of the identities (3.8) and (3.9), and it reads as follows.

LEMMA 3. *Let* ϕ *be as in Lemma* 2, *and define forms* $\phi_K^{(r)}$ *by*

$$\phi_K^{(r)} = \binom{k}{r}\phi(\underbrace{L, \cdots, L}_{r}; K, \cdots, K).$$

Also, let π *be the one-form associated with the orthogonal projection of* $T_R M$ *onto* X. *Then, on* M - zero(X),

$$i(X)\phi_K^{(r)} = d\phi_K^{(r+1)}, \tag{4.4}$$

$$i(X)\,d\pi = 0. \tag{4.5}$$

The proof is again quite similar. One needs the basic fact that generally

$$d\phi(X_1, \cdots, X_k) = \sum_i \phi(X_1, \cdots, DX_i, \cdots, X_k)$$

when the X_i are *even*, end-valued forms, and finally, that DK = 0. Hence, in particular,

$$d\phi_K^{(r+1)} = \binom{k}{r+1}(r+1)\phi(\underbrace{L, \cdots, L}_{r}; DL; K, \cdots, K),$$

which by virtue of (4.2) immediately yields (4.4).

To see (4.5), we merely observe that the motion generated by X must preserve π. Hence $\mathfrak{L}(X)\pi = 0$, which implies $i(X)d\pi = 0$, since $i(X)\pi = 1$.

Just as in Section 3, this lemma leads to the following proposition.

Let $w = d\pi$, *and set*

$$\eta_\phi = \pi\{\phi_K^{(1)} + \phi_K^{(2)}w + \cdots + \phi_K^{(k)}w^{k-1}\}.$$

Then $i(X)(\phi_K + d\eta_\phi) = 0$ *on* M - zero(X).

From here on, a procedure parallel to the one described in Section 3 leads to Theorem 2. We shall leave these details to the reader, with only this comment. In the Riemannian situation, ϕ need not be defined on all of $\mathfrak{gl}(m, C)$. If we refer back to orthonormal frames, all the constructions already make sense when ϕ is an invariant form on the Lie algebra $SO(m, R) \subset \mathfrak{gl}(m, C)$. In this way one gets one *new* form, which corresponds to the Euler class, and which one would need if one wanted to use the Hopf formula to evaluate the universal constant that arises in the evaluation of $\lim_{\varepsilon \to 0} \eta_\phi(S_\varepsilon)$.

Let me also remark that one can extend Theorem 2, somewhat artificially, by simply postulating the formulae (4.2) and (4.5) on which the proof rests. Thus one postulates a form L and a connection D on $T_R(M)$ such that

$$i(X)K = DL, \tag{4.6}$$

as well as a one-form π such that $\pi(X) = 1$ and $\mathfrak{L}(X)\pi = 0$. A vector field X for which a π with these properties can be found might be called projectible. Now the property (4.6) is certainly valid *if* X *preserves the connection* D. Thus for *projectible vector fields that preserve a connection, Theorem* 2 *is valid.*

Quite possibly, the projectibility hypothesis is superfluous, here.

5. CONCLUDING REMARKS

We conclude by explaining briefly the relation between Theorem 1 and the fixed-point theorem of Atiyah and myself.

First of all, recall our formula in the instance of a holomorphic endomorphism $f: M \to M$ that is *transversal*—in other words, whose graph meets the diagonal transversally.

Let $H^i(M; \Omega)$ be the cohomology of M in the sheaf of germs of holomorphic functions, and denote the homomorphism induced by f on $H^i(M; \Omega)$ by $H^i(f)$. Finally, at a fixed point of f, let df denote the *holomorphic differential* of f at P:

$$df: T'_P M \to T'_P M .$$

With this understood, our formula reads

$$\sum (-1)^i \operatorname{trace} H^i(f) = \sum_P \frac{1}{\det(1 - df_P)}, \tag{5.1}$$

where the sum is taken over the fixed points of f.

Now, following Guillemin, consider (5.1) when f is replaced by the family of maps

$$f_t = e^{tX},$$

obtained by integrating a holomorphic vector field X. First one observes that at the zeros $\{P\}$ of X

$$df_t|_P = e^{tL_P},$$

so that if X is nondegenerate in our sense, then f_t is transversal and the fixed points of f_t correspond precisely to the zeros of X, at least for small values of $|t|$. Hence (5.1) takes the form

(5.2) $$\sum (-1)^i \operatorname{trace} H^i(e^{tX}) = \sum_P \frac{1}{\det(1 - e^{tL})}.$$

Now the left member of (5.2) is easily seen to define a real analytic function of t, for t near 0. The right member, on the other hand, is a finite sum of Laurent series in t. *Hence the singular parts of these series must cancel out as we sum over* P, *and the constant terms must add up to the left member at* $t = 0$. But for $t = 0$, $e^{tL} = 1$, so that the left member reduces to

$$\chi(M; \Omega) = \sum (-1)^i \dim H^i(M; \Omega),$$

which is usually called the arithmetic genus of M. To understand the right-hand side, define the functions ϕ_k on $\mathfrak{gl}(m, C)$ by

$$\frac{c_m(y)}{\det(1 - e^{ty})} = t^{-m}\phi_0(y) + t^{-m+1}\phi_1(y) + \cdots \qquad (y \in \mathfrak{gl}(m, C))$$

—that is, let ϕ_k be the coefficient of t^{-m+k} in the Laurent expansion of $c_m(y)\det(1 - e^{ty})^{-1}$. Then ϕ_k is homogeneous of degree k in $y \in \mathfrak{gl}(m, C)$ —and also *invariant* because the determinant is invariant. Hence there exist polynomials $T_k(c_1, \cdots, c_m)$ of weight k such that

$$\phi_k(y) = T_k(c_1(y), \cdots, c_m(y)) \qquad (k \leq m),$$

and these are essentially, by definition, the *Todd polynomials*.

Equating the constant terms in (5.5), one gets the formula

$$\chi(M; \Omega) = \sum_P \frac{T_m(L)}{c_m(L)},$$

where T_m is the mth Todd polynomial. Now, if we apply our Theorem 1, we obtain the Riemann-Roch formula of Hirzebruch:

(5.5) $$\chi(M; \Omega) = T_m[M];$$

or alternatively, if we assume (5.5), we get the special case of Theorem 1 corresponding to $\Phi = T_m$.

Applying this procedure to sufficiently many of the sheaves

$$\Omega_1^{a_1} \otimes \Omega_2^{a_2} \otimes \cdots \otimes \Omega_m^{a_m},$$

(where Ω_r denotes the sheaf of germs of holomorphic r-forms on X) and using some standard K-theory, one can derive sufficiently many verifications of Theorem 1 to prove it in general.

A similar argument can be devised to prove Theorem 2 from the fixed-point formula. Actually, one gets more in that way than we obtained in the present treatment. Using the invariance of the de Rham cohomology groups under deformations, one then finds in addition that under the hypotheses of Theorem 2 the identity

$$\sum_{P} \frac{\Phi(L)}{c_m^{1/2}(L)} = 0,$$

is valid for certain forms Φ of degree *greater than* $m/2$.

It would be interesting to explain this phenomenon (which occurs for the L-polynomials) directly in our framework, as indeed it would be interesting to explain all of the other results about the characteristic classes of the tangent bundle, such as the divisibility theorems, in a more directly differential-geometric way.

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