ON THE BOUNDARY BEHAVIOR OF CONFORMAL MAPS

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1. INTRODUCTION AND SUMMARY

Let f be schlicht in $|z| < 1$, and suppose that $c = \lim_{x \to 1-0} f(x) \neq \infty$ exists. Then $f(z) \to c$ as $z \to 1$ in any Stolz angle, hence also as $z \to 1$ in some domain G that is tangential at $z = 1$ (by this we mean that G contains every Stolz angle in a sufficiently small neighborhood of $z = 1$). In other words,

$$f(z) \to c \quad \text{as } z \to 1 \ (z \in G), \tag{1}$$

and as a consequence

$$\text{the image } f(G) \text{ has finite area.} \tag{2}$$

Note, however, that G depends on f. George Piranian raised the following question: Does there exist a domain G tangential at $z = 1$ that is *independent* of f and for which (1) or (2) holds. For example: Does (1) or (2) always hold for $G = \{|z - 1/2| < 1/2\}$?

Our answer is negative in both cases.

THEOREM 1. *Let* $\{z_p\}$ *be any sequence with* $|z_p| < 1$ *such that* $z_p \to 1$ *and* $\arg(z_p - 1) \to \pi/2$ $(p \to \infty)$. *Then there exists a function* f *that is bounded and schlicht in* $|z| < 1$, *for which* $\lim_{x \to 1-0} f(x)$ *exists but* $\{f(z_p)\}$ *diverges.*

THEOREM 2. *Let* G *be any subdomain of* $|z| < 1$ *that is tangential at* $z = 1$. *Then there exists a function* f, *schlicht in* $|z| < 1$, *for which* $\lim_{x \to 1-0} f(x) \neq \infty$ *exists but* f(G) *has infinite area.*

2. PRELIMINARY RESULTS

It will be more convenient to work in the half-plane $\Re z > 0$ than in the disk $|z| < 1$. So we assume that $\{z_p\}$ is a sequence with $\Re z_p > 0$ and $z_p \to 0$, $\arg z_p \to \pi/2$ $(p \to \infty)$.

Let $0 < h < 1$, and let D be a simply connected domain in the half-plane $u > h$ of the $w = u + iv$-plane that is symmetric with respect to the real axis, contains the point $w = 1$ and some rectangle $\{h < u < h + a, \ |v| < 1\}$. For $0 < \varepsilon < 1$, let D_ε consist of the rectangle $R = \{0 < u < h, \ -4 < v < 4\}$ and of D, the two domains being connected by an opening of width 2ε; that is, let

$$D_\varepsilon = R \cup (h - i\varepsilon, \ h + i\varepsilon) \cup D.$$

Let the schlicht function $w = f_\varepsilon(z)$, normalized by the conditions $f_\varepsilon(0) = 0$ and $f_\varepsilon(1) = 1$, map the half-plane $\Re z > 0$ onto D_ε.

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LEMMA. *For every* p_0 *there exist* $\varepsilon > 0$ *and* $p > p_0$ *such that*

(3) $$0 < \Re f_\varepsilon(z_p) < h \quad \textit{and} \quad \Im f_\varepsilon(z_p) > 2.$$

Proof. Let the function $b(\varepsilon)$ be determined by the equation $f_\varepsilon(ib(\varepsilon)) = 4i$. We shall first show that

(4) $$b(\varepsilon) \searrow 0 \quad \text{continuously as } \varepsilon \searrow 0.$$

(i) Monotonicity is clear, since $\varepsilon_1 < \varepsilon_2$ implies $D_{\varepsilon_1} \subset D_{\varepsilon_2}$, so that the harmonic measure of the segment $(-4i, 4i)$ with respect to $w = 1$ is smaller in D_{ε_1} than in D_{ε_2}.

(ii) To see the continuity of $b(\varepsilon)$, we first extend f_ε across $(-ib(\varepsilon), ib(\varepsilon))$ by the reflection principle. The z-plane, slit along $z = iy$ $(|y| \geq b(\varepsilon))$, will thus be mapped by f_ε onto $\tilde{D}_\varepsilon$, the union of D_ε with its reflection about $(-4i, 4i)$, plus this open segment. If now $\varepsilon \searrow \varepsilon_0 > 0$, the kernel K_z of the regions in the z-plane with respect to $z = 0$ is the plane slit along $z = iy$ $(|y| \geq \lim_{\varepsilon\searrow\varepsilon_0} b(\varepsilon))$, while in the w-plane the kernel with respect to $w = 0$ is just $K_w = \tilde{D}_{\varepsilon_0}$. By the Carathéodory kernel theorem, the limit function will map K_z onto K_w, where $z = 0$ and $z = 1$ remain fixed and whereby points on $\Re z = 0$ are mapped into points on $\Re w = 0$, so that $z = i \cdot \lim_{\varepsilon\searrow\varepsilon_0} b(\varepsilon)$ corresponds to $w = 4i$. Since this limit function will also map $\Re z > 0$ onto D_{ε_0} with fixed points 0 and 1, it must be the function f_{ε_0}, and it follows that $i \cdot \lim_{\varepsilon\searrow\varepsilon_0} b(\varepsilon) = i \cdot b(\varepsilon_0)$. Likewise we can show that $\lim_{\varepsilon\nearrow\varepsilon_0} b(\varepsilon) = b(\varepsilon_0)$.

(iii) Assume that $b(\varepsilon) \to \alpha > 0$ $(\varepsilon \to 0)$. Then the kernel of the slit z-planes with respect to $z = 0$ would not degenerate for $\varepsilon \to 0$, the functions f_ε would converge uniformly, in a certain neighborhood of $z = 0$, to a function vanishing at $z = 0$, and it would follow that $|f_\varepsilon(\delta)| < h$ for all $\varepsilon < \varepsilon_0$ and some sufficiently small $\delta > 0$, which we now keep fixed. On the other hand, we may consider the maps f_ε from $\Re z > 0$ onto D_ε with 1 as fixed point instead of 0. Since the kernel of the regions D_ε with respect to $w = 1$ is D, we see again by the Carathéodory kernel theorem that $f_\varepsilon(\delta)$ must lie in D for sufficiently small ε, and this contradicts the relation $|f_\varepsilon(\delta)| < h$.

Now that (4) is proved, we attack proposition (3). Let $\zeta = z/b(\varepsilon)$ and

$$\phi_\varepsilon(\zeta) = f_\varepsilon(b(\varepsilon)\zeta).$$

Making an analytic continuation as above, we see that $w = \phi_\varepsilon(\zeta)$ maps P, the ζ-plane slit along $[-i\infty, -i] \cup [i, i\infty]$, onto $\tilde{D}_\varepsilon$. Note that $\phi_\varepsilon(0) = 0$ and $\phi_\varepsilon'(0) > 0$ for all $\varepsilon > 0$. As $\varepsilon \to 0$, the domains $\tilde{D}_\varepsilon$ converge in the Carathéodory sense to the rectangle $\tilde{R} = \{-h < u < h,\ -4 < v < 4\}$. Hence $\phi_\varepsilon(\zeta) \to \phi(\zeta)$ locally uniformly in P, where $\phi(\zeta)$ is the function that maps P onto $\tilde{R}$ with $\phi(0) = 0$, $\phi'(0) > 0$.

Consider the image S of the segment $[3i, h/2 + 3i]$ under ϕ^{-1}. We choose $\varepsilon_0 > 0$ so small that

$$|\phi_\varepsilon(\zeta) - \phi(\zeta)| < h/2 \quad \text{for } \zeta \in S,\ 0 < \varepsilon < \varepsilon_0.$$

Because $\arg z_p \to \pi/2$ $(p \to \infty)$, there exists a point $\zeta_p \in S$ with $\arg \zeta_p = \arg z_p$, for sufficiently large p (say for $p > p_1 \geq p_0$). Using $z_p \to 0$ $(p \to \infty)$ and (4), we see that we can determine $\varepsilon = \varepsilon(p) > 0$ such that $z_p = \zeta_p b(\varepsilon)$ for $p > p_2 \geq p_1$. We have the relation

$$f_\varepsilon(z_p) = \phi_\varepsilon(\zeta_p) \in [3i, h/2 + 3i] \qquad (p \geq p_2),$$

and this proves (3).

3. PROOF OF THEOREM 1

Let $h_0 = 2$, $1 > h_1 > h_2 > \cdots > 0$, and

$$R_k = \{u + iv: h_{k+1} < u < h_k, \ -4 < v < 4\} \qquad (k = 0, 1, \cdots).$$

Let $H_0 = R_0$ and $H_1 = R_0 \cup (h_1 - i\varepsilon_1, h_1 + i\varepsilon_1) \cup R_1$, with $0 < \varepsilon_1 < 1$. Let f_1 be the function that maps $\Re z > 0$ onto H_1 and is real on the positive real axis, with $f_1(1) = 1$. By the lemma, we can choose a small ε_1 and a point z_{p_1} in our given sequence such that

$$(5) \qquad h_2 < \Re f_1(z_{p_1}) < h_1, \qquad \Im f_1(z_{p_1}) > 2.$$

On the other hand, since $f_1(z_p) \to h_2$ $(p \to \infty)$, there exists another point z_{q_1} such that

$$(6) \qquad h_2 < \Re f_1(z_{q_1}) < h_1, \qquad \Im f_1(z_{q_1}) < 1.$$

In the second step, let $H_2 = H_1 \cup (h_2 - i\varepsilon_2, h_2 + i\varepsilon_2) \cup R_2$, with $0 < \varepsilon_2 < 1$. Denote the normalized mapping function by f_2. The choice of ε_2 depends on two conditions. First, the above statements (5) and (6) should remain valid with f_1 being replaced by f_2. This can be achieved by the Carathéodory kernel theorem, since H_2 converges to H_1 as $\varepsilon_2 \to 0$. Furthermore, ε_2 should be chosen so small that there exists a point z_{p_2} $(p_2 > p_1)$ with

$$h_3 < \Re f_2(z_{p_2}) < h_2, \qquad \Im f_2(z_{p_2}) > 2.$$

This can be done by (3). Now ε_2 is fixed, and we pick a point z_{q_2} such that

$$h_3 < \Re f_2(z_{q_2}) < h_2, \qquad \Im f_2(z_{q_2}) < 1.$$

It is clear how the construction continues. The domains H_k converge (with respect to $w = 1$) to a domain H, and the normalized mapping functions f_k and f satisfy the condition $f_k(z) \to f(z)$. The function f maps the positive real axis into itself. In particular, $f(x) \to 0$ as $x \to +0$. By the construction, we have the inequalities

$$\Im f_k(z_{p_j}) > 2, \qquad \Im f_k(z_{q_j}) < 1 \qquad (j = 1, 2, \cdots, k).$$

Letting $k \to \infty$ for fixed j, we obtain the inequalities

$$\Im f(z_{p_j}) \geq 2, \qquad \Im f(z_{q_j}) \leq 1 \qquad (j = 1, 2, \cdots).$$

Hence the sequence $\{f(z_p)\}$ diverges. This proves Theorem 1.

4. PROOF OF THEOREM 2

The construction in the last section has to be slightly modified. Consider infinite strips $T_k = \{h_{k+1} < u < h_k\}$, instead of the rectangles R_k. Also, let S_k be a closed rectangle in T_k. We can make the numbers ε_k so small that $S_j \subset f_k(G)$ for $j = 1, \cdots, k$. Hence $S_j \subset f(G)$ for $j = 1, 2, \cdots$. If $\bigcup S_j$ has infinite area, then $f(G)$ also has infinite area.

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