ON AUTOMORPHISMS OF A LOCALLY COMPACT GROUP

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In this note, which is a continuation of the paper [5], we show that a continuous automorphism T of a locally compact, noncompact, connected group G is not ergodic. We thus answer in part a question raised by Halmos on page 29 of his book [1]. The result has also been announced by T. S. Wu [6].

We follow [1] for concepts in ergodic theory, and [2] for results on Lie groups and Lie algebras. If G is a connected Lie group, we denote by L its Lie algebra. The elements of the Lie algebra L are denoted by $\dot{x}, \dot{y}, \cdots$. If T is a continuous automorphism of a connected Lie group G, then the induced automorphism of L is denoted by $\dot{x} \to \dot{x}^t$ or by t.

LEMMA. *Let* G *be a connected Lie group, and let* T *be a continuous automorphism of* G. *Suppose* T *is ergodic. Then* 1 *is the unique characteristic root of* $\mathrm{Ad}\, g$, *for each element* g *of* G. *Consequently, the associated Lie algebra* L *of* G *is nilpotent.*

Proof. From the identity $(Tg)h(Tg^{-1}) = T(g(T^{-1}h)g^{-1})$ we see that $\mathrm{Ad}\,(Tg) = t(\mathrm{Ad}\, g)t^{-1}$ for all g in G. So, for every complex number λ, the function $\det(\lambda I - \mathrm{Ad}\, g)$ is continuous and T-invariant. Hence it is constant for each λ. So $\lambda = 1$ is its only zero. Therefore $\mathrm{Ad}\, \dot{x}$ is nilpotent for each $\dot{x}$ belonging to L, and L is nilpotent.

THEOREM. *Let* G *be a connected, locally compact group, let* T *be a continuous automorphism of* G, *and let* T *be ergodic. Then* G *is compact.*

Proof. Suppose that G is a Lie group of dimension 1. Then it is abelian, and hence compact, by Theorem 3 of [5]. Now suppose that the theorem is true for all Lie groups of dimension less than n. Let G be a Lie group of dimension n. By the lemma, G is nilpotent. Therefore its centre Z has positive dimension and is invariant under T. Therefore T determines a continuous automorphism T^* of G/Z, in the standard way.

We claim that T^* is ergodic. To prove this, let us take a T^*-invariant Borel subset H^* of G/Z. Let H be the union of cosets in H^*. Then H is a T-invariant Borel subset of G, and by hypothesis, either H or $G \sim H$ has measure 0. The same is true for either H^* or $G/Z \sim H^*$, by the formula for integration on coset spaces [3, p. 131]. Since G/Z is a Lie group of dimension less than n, it is compact, by the induction hypothesis.

Now the group $\mathrm{Ad}\, G$ is completely reducible, since G/Z is compact. Therefore, by the lemma, $\mathrm{Ad}\, G = \{I\}$. Therefore G is abelian, and hence G is compact, by Theorem 3 of [5]. Thus we have proved the theorem in case G is a Lie group.

Now suppose G is merely known to be a locally compact, connected group. Then it contains a maximal compact, normal subgroup N such that G/N is a Lie group [4, p. 172]. Clearly, N is invariant under T. Hence, as in the proof of the case of a Lie group above, T determines an ergodic automorphism T^* of G/N. Therefore G/N is compact, by what was shown above; hence, G is compact.

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Remarks. If $H \subset G$ is a measurable subset that differs from $T(H)$ by a set of measure zero, then the set $H = \bigcup_{i=-\infty}^{\infty} T^i(H)$ differs from H by a set of measure zero and is invariant in the strict sense. Thus the special notion of ergodicity used in the proof is equivalent to the more general notion, because of the special properties of T and of Haar measure.

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