# ON POLYNOMIAL RINGS

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One approach to Serre's conjecture on the freedom of finitely generated projective modules over polynomial rings is to determine which of the sequences $(P_0, \cdots, P_n)$ of polynomials may be written as the sequence of $n \times n$ determinants of an $n \times (n+1)$ matrix. The assertion proved here is that the Serre conjecture is equivalent to the statement that the sequence $(P_0, \cdots, P_n)$ can be written in this form if the ideal generated by the $P_i$ has homological dimension less than or equal to 1.

Throughout this paper we shall use the following conventions: k denotes an algebraically closed field, and $A_m$ denotes the polynomial ring $k[x_1, \cdots, x_m]$ in the indeterminates $x_1, \cdots, x_m$. If $R = (R_{ij})$ is an $n \times (n+1)$ matrix, $\triangle_j(R)$ denotes the determinant of the $n \times n$ matrix derived from R by deletion of the jth column. If M is a module over $A_m$, rank(M) denotes the dimension of the vector space $Q(A_m) \otimes M$, where $Q(A_m)$ is the field of fractions of $A_m$. By $A_m^{n+1}$ we denote the free $A_m$-module of $(n+1)$-tuples of elements of $A_m$.

*Definition.* If $(P_0, \cdots, P_n)$ is an element of $A_m^{n+1}$, we say that the $P_i$ form a *determinantal sequence* if there exists an $n \times (n+1)$ matrix $R = (R_{ij})$ such that $\triangle_i(R) = P_i$ for each i.

LEMMA 1. *Suppose* $(P_0, \cdots, P_n)$ *and* $(Q_0, \cdots, Q_n)$ *are elements of* $A_m^{n+1}$, *and assume that* T *is an elementary transformation of* $A_m^{n+1}$ *that carries* $(P_0, \cdots, P_n)$ *into* $(Q_0, \cdots, Q_n)$. *The sequence* $(P_0, \cdots, P_n)$ *is determinantal if and only if* $(Q_0, \cdots, Q_n)$ *is determinantal. If* $\varepsilon_i = \pm 1$ *and* $(P_0, \cdots, P_n)$ *is determinantal, then so is* $(\varepsilon_0 P_0, \cdots, \varepsilon_n P_n)$.

*Proof.* Suppose that $R = (R_{ij})$ is an $n \times (n+1)$ matrix with $\triangle_i(R) = P_i$. If T multiplies $(P_0, \cdots, P_n)$ by an element of $A_m$ or if T interchanges the ith and jth entries of $(P_0, \cdots, P_n)$, then $(Q_0, \cdots, Q_n)$ is determinantal. If T multiplies $P_i$ by S and adds it to $P_j$, the assertion is equally clear, since we need only replace the jth column of R by the sum of the jth column and S times the ith column. For the last assertion, it will suffice to consider the case where $\varepsilon_i = 1$ for $i > 1$ and $\varepsilon_0 = -1$. Denote by $e_i$ the basis element $(0, \cdots, 1, \cdots, 0)$ of $A_m^{n+1}$, where 1 occurs only in the ith position, and note that if we replace the row vector $\sum R_{ij} e_j$ in R by $(-w) R_{i0} e_0 + \sum w R_{ij} e_j$, then the assertion follows if we choose w so that $w^n = 1$.

THEOREM. *The following two properties are equivalent:*

(1) *every finitely generated projective module of rank at least* n *over* $A_m$ *is free;*

(2) *if* $(P_0, \cdots, P_k)$ *is an element of* $A_m^{k+1}$ *with* $k \geq n$ *such that the ideal generated by the* $P_i$ *has homological dimension less than* 2, *then* $(P_0, \cdots, P_k)$ *is a determinantal sequence.*

*Proof.* Assume (1) and suppose $(P_0, \cdots, P_k)$ is an ideal of homological dimension less than or equal to 1. If $P_i = G Q_i$ for each i and some G in $A_m$, then

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$(Q_0, \cdots, Q_k)$ has homological dimension less than or equal to 1, and it will suffice to prove that the sequence $(Q_0, \cdots, Q_k)$ is determinantal under the assumption that the $Q_i$ have no common nonconstant divisors. We shall first show that we may assume $Q_0$ and $Q_1$ to be relatively prime. If $Q_0 = 0$, then we either have nothing to prove, or we may apply an elementary transformation to move some nonzero entry to the zeroth position. Assume $Q_0 = F_1 \cdots F_u$, where the $F_i$ are the irreducible factors of $Q_0$. Since the $Q_i$ have no common nonconstant divisors, either $Q_0$ is an element of k and all the $Q_i$ are zero for $i > 0$, or there is a $Q_j$ such that $F_1$ does not divide $Q_j$. If j is not 1, then we may interchange $Q_j$ and $Q_1$ by applying an elementary transformation. We now proceed by induction and suppose that $Q_1$ is not divisible by $F_1, \cdots, F_v$ $(v < u)$; we shall show that by means of an elementary transformation leaving $Q_0, Q_2, \cdots, Q_k$ fixed, we can replace $Q_1$ by a polynomial $Q_1'$ that is not divisible by any of the factors $F_1, \cdots, F_{v+1}$. For some t, the factor $F_{v+1}$ does not divide $Q_t$, and since we have nothing to prove if $F_{v+1}$ does not divide $Q_1$, we may suppose it divides $Q_1$ and not $Q_t$, for some $t \neq 1$. If we replace $Q_1$ by $Q_1 + F_1 \cdots F_v Q_j$, the assertion is clear.

Denote by F a free module over $A_m$ with a basis $f_0, \cdots, f_k$, and define a module K by the exact sequence

$$0 \to K \xrightarrow{\phi} F \xrightarrow{\theta} I \to 0,$$

where I is the ideal generated by the elements $Q_0, \cdots, Q_k$, and where $\theta(f_j) = Q_j$. Since

$$\text{rank } F = k + 1 \quad \text{and} \quad \text{rank } I = 1,$$

the rank of K is at least n. The homological dimension of I is at most 1 by assumption, and thus K is free. Suppose that $\phi(K)$ has a basis of elements $k_i = \sum_j R_{ij} f_j$. Since rank K = t, we see that $\Delta_j(R) \neq 0$ for some j, and we form the equations

$$\sum_{w \neq j} R_{iw} Y_w = -R_{ij} Q_j \quad (1 \leq i \leq t).$$

This system of equations has a unique solution, and we may solve it by Cramer's rule to derive the relations

$$(1) \qquad Q_v = (\varepsilon_v / \Delta_j(R)) \cdot Q_j \Delta_v(R) \quad \text{for } \varepsilon_v = \pm 1 \text{ and } v \neq j.$$

Since $Q_0 \neq 0$, we see that $\Delta_0(R) \neq 0$, and we derive the relations

$$Q_v \Delta_0(R) = \varepsilon_v Q_0 \Delta_v(R).$$

The assumption that $Q_0$ and $Q_1$ are relatively prime implies that $\Delta_0(R) = H P_0$ for some H in $A_m$, and hence $Q_i H = \varepsilon_i \Delta_i(R)$ for each i. By [2], a maximal ideal p of $A_m$ contains the ideal generated by the $\Delta_j(R)$ if and only if the ideal $(A_m)_p \cdot I$ is not projective, where $(A_m)_p$ denotes the local ring of the prime p. Thus if p contains all the $\Delta_j(R)$, it must also contain all the $Q_i$, and hence the radical of the ideal generated by the $\Delta_j(R)$ can have no primes of dimension m - 1; thus H must be an element of k. If we assume (2) and if M is a projective module of rank at least n, then by [3] there exists a free module G such that the direct sum $M \oplus G$ is free.

The freedom of M will then follow by induction if we show that a projective module N is free provided its rank is at least n and there exists an exact sequence

$$0 \to L \xrightarrow{\phi} F \xrightarrow{\theta} N \to 0,$$

where L is free on one generator. Suppose that F has a basis $f_0, \cdots, f_s$ and that $\phi(L)$ is generated by $\sum Q_j f_j$. By [2] or [3], there exist polynomials $P_i$ such that $\sum P_i Q_i = 1$. The ideal generated by the $P_i$ is thus projective; hence, the sequence $(P_0, \cdots, P_s)$ is determinantal, and thus there exists an $(s+1) \times (s+1)$ matrix with first row $(Q_0, \cdots, Q_s)$ whose determinant is 1. This implies that N is free.

The following conclusion now follows from theorems of Bass [1, p. 58], and Seshadri [4].

COROLLARY. *Suppose* $k[x_1, \cdots, x_m]$ *is a polynomial ring over the field* k. *If* $m \le 2$, *then for each sequence* $(P_0, \cdots, P_n)$ *there exists an* $n \times (n+1)$ *matrix* R *such that* $\Delta_i(R) = P_i$ $(0 \le i \le n)$. *If* $m \ge 3$ *and* $n \ge m+1$, *then there exists an* $n \times (n+1)$ *matrix* R *such that* $\Delta_j(R) = P_j$ *for each* j, *if the homological dimension of the ideal generated by the* $P_i$ *is less than* 2.

## REFERENCES


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