A GENERALIZATION OF THE BIEBERBACH COEFFICIENT PROBLEM FOR UNIVALENT FUNCTIONS

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1. INTRODUCTION

Let S denote the class of functions

$$F(z) = \sum_{n=1}^{\infty} A_n z^n \tag{1.1}$$

that are regular and univalent in the unit disc $E(z: |z| < 1)$. The Bieberbach conjecture is the assertion that the coefficients A_n satisfy the inequality

$$|A_n| \leq n|A_1| \quad (n = 2, 3, \cdots). \tag{1.2}$$

This conjecture is known to be correct for $n = 2, 3$, and 4. Recently, Hayman [2] showed that

$$\big||A_{n+1}| - |A_n|\big| < A|A_1| \quad (n = 2, 3, \cdots), \tag{1.3}$$

for some constant A independent of F(z). About the same time, Pommerenke [7, Theorem 4] established (1.3) with $A \leq 3e^2/4$, for the subclass of S consisting of functions that are close-to-convex in E. (A function is close-to-convex if there exists a starlike function $g(z) = \sum_1^\infty b_n z^n$ such that

$$\Re \frac{z F'(z)}{g(z)} \geq 0 \tag{1.4}$$

in E; see [3], where the definition is given in terms of a convex auxiliary function.) Pommerenke further showed [7, Theorem 4] that if F(z) is close-to-convex, but not convex in one direction [9], then there exists a $\delta = \delta(g) > 0$ such that

$$|A_{n+1}| - |A_n| = O(1/n^\delta). \tag{1.5}$$

In particular, if F(z) is starlike in E and (1.5) is not satisfied, then F(z) must be of the form

$$F(z) = A_1 z(1 - \varepsilon_1 z)^{-1}(1 - \varepsilon_2 z)^{-1}, \tag{1.6}$$

where $|\varepsilon_1| = |\varepsilon_2| = 1$.

An inequality stronger than either of the inequalities (1.2) and (1.3) is the assertion that

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$$\Big|\, n\,|A_n| - m\,|A_m| \,\Big| \le |n^2 - m^2| \cdot |A_1| \tag{1.7}$$

for all nonnegative integers n and m (convention: $A_0 = 0$). If (1.7) were true, then (1.2) would follow from it. We can see this in two ways by putting $m = 0$ or $m = n - 1$. If we set $m = n - 1$, then (1.7) implies the inequality

$$n\,|A_n| \le (2n - 1)\,|A_1| + (n - 1)\,|A_{n-1}|\,. \tag{1.8}$$

Then (1.2) follows from (1.8) by induction. However, (1.8) is obviously a stronger assertion than (1.2).

For $m = n - 1$, (1.7) is equivalent to the assertion

$$\Big|\, |A_n| - |A_{n-1}| + \frac{1}{n}\,|A_{n-1}| \,\Big| \le \left(2 - \frac{1}{n}\right) |A_1|\,, \tag{1.9}$$

so that (1.3) would follow from (1.9) and the known inequality $|A_{n-1}| < e\,|A_1|\,(n - 1)$ (see [4]).

At this time we have hardly enough evidence to justify the conjecture that (1.7) is true for the whole class S, although this interesting possibility at least suggests itself. Support for the suggestion is given in this paper. We show that (1.7) is satisfied if F(z) is convex in one direction, or if F(z) is close-to-convex and $m - n$ is an even integer. Each of the two subclasses of S contains the Koebe function $z(1 - \varepsilon z)^{-2}$ ($|\varepsilon| = 1$), so that (1.7) is sharp. From (1.7) we obtain an improved estimate for the constant A in (1.3), for the subclasses of S under discussion.

2. FUNCTIONS CONVEX IN ONE DIRECTION

Definition. Let f(z) be regular and univalent in $E(z: |z| < 1)$, and let f(z) map E onto a domain G. We call f(z) *convex in one direction* in E if G consists of the union of parallel rectilinear segments with not more than one segment on any one straight line.

THEOREM 1. *Let* $f(z) = \sum_1^\infty a_n z^n$ *be regular and univalent in* $E(z: |z| < 1)$ *and map* E *onto a domain* G *convex in one direction. Let* n *and* m *be nonnegative integers. Then*

$$\Big|\, n\,|a_n| - m\,|a_m| \,\Big| \le |n^2 - m^2| \cdot |a_1|\,,$$

where $a_0 = 0$. *Equality is attained for the function* $a_1 z (1 - \varepsilon z)^{-2}$, *where* $|\varepsilon| = 1$.

Proof. It is sufficient to prove Theorem 1 for the special case $m = n - 1$, and we may assume $a_1 = 1$. For in the general case, with $m < n$, the theorem for the special case then gives the inequalities

$$(n - 1)^2 - n^2 \le n\,|a_n| - (n - 1)\,|a_{n-1}| \le n^2 - (n - 1)^2,$$

$$(n - 2)^2 - (n - 1)^2 \le (n - 1)\,|a_{n-1}| - (n - 2)\,|a_{n-2}| \le (n - 1)^2 - (n - 2)^2,$$

. . . .

$$m^2 - (m + 1)^2 \le (m + 1)\,|a_{m+1}| - m\,|a_m| \le (m + 1)^2 - m^2.$$

By addition, we deduce that

$$\Big| n\,|a_n| - m\,|a_m| \Big| \le n^2 - m^2 .$$

In the proof, we may also assume that $f(z)$ is continuous on $|z| = 1$ and that G is convex in the direction of the imaginary axis. The circle $|z| = 1$ then consists of two arcs E_1 and E_2 such that $\Re f(z)$ is nonincreasing on E_1 and nondecreasing on E_2. The general case, where $f(z)$ is not assumed to be continuous on $|z| = 1$, is taken care of by approximating the domain G by a sequence of domains $G_1 \subset G_2 \subset \cdots$, where $\lim_{n\to\infty} G_n = G$ and $0 \in G_1$, and where the domains G_n are convex in one direction and are bounded by Jordan curves. For a proof, see for example de Bruijn [1].

For each G_n, the corresponding mapping function $w = f_n(z)$ taking E onto G_n may be uniformly approximated by the associated de la Vallée Poussin polynomials $g_m^{(n)}(z)$ of degree m:

$$\lim_{m\to\infty} g_m^{(n)}(z) = f_n(z),$$

where $g_m^{(n)}(z)$ is convex in one direction in E and regular on $|z| = 1$, for all m [6, p. 318]. In the proof of Theorem 1 we may assume then that $f(z)$ is not only continuous, but even analytic on $|z| = 1$. We take $a_1 = 1$, for simplicity.

With this assumption it follows [9, p. 467] that there exist real numbers μ and ν and an analytic function $F_1(z)$, with $\Re F_1(z) \ge 0$ in $|z| \le 1$, such that, with the notation $q(z) = z f'(z)$,

$$q(-i e^{i\mu} z) = \frac{z F_1(z)}{(1 + i e^{i\nu} z)(1 + i e^{-i\nu} z)}.$$

We may then write $q(z)$ in the form

$$q(z) = \frac{i e^{-i\mu} z F_1(i e^{-i\mu} z)}{(1 - e^{i(\nu-\mu)} z)(1 - e^{-i(\nu+\mu)} z)} = \frac{i(\varepsilon_1 \varepsilon_2)^{1/2} z P_1(z)}{(1 - \varepsilon_1 z)(1 - \varepsilon_2 z)},$$

where $\varepsilon_1 = e^{i(\nu-\mu)}$, $\varepsilon_2 = e^{-i(\nu+\mu)}$, $P_1(0) = -i/(\varepsilon_1 \varepsilon_2)^{1/2}$, and $\Re P_1(z) > 0$ in E. $P_1(z)$ is regular on $|z| \le 1$. Thus

$$\Re \left\{ e^{i\gamma} \frac{z f'(z)}{t(z)} \right\} \ge 0 \qquad (|z| < 1)$$

for $e^{i\gamma} = -i/(\varepsilon_1 \varepsilon_2)^{1/2}$, where $t(z)$ is defined as

$$t(z) = \frac{z}{(1 - \varepsilon_1 z)(1 - \varepsilon_2 z)} = z + \sum_2^\infty c_n z^n \qquad \left(c_n = \frac{\varepsilon_2^n - \varepsilon_1^n}{\varepsilon_2 - \varepsilon_1}, \quad c_n - \varepsilon_1 c_{n-1} = \varepsilon_2^{n-1} \right).$$

We note that $|c_n - \varepsilon_1 c_{n-1}| = 1$, since $|\varepsilon_1| = |\varepsilon_2| = 1$. Now let

$$e^{i\gamma} \frac{z f'(z)}{t(z)} = P(z) \cos\gamma + i \sin\gamma,$$

where $P(z)$ is analytic in E, $P(0) = 1$, and $\Re P(z) > 0$ in E. Let $P(z) = \sum_0^\infty p_n z^n$, where $p_0 = 1$. Then $|p_n| \leq 2$. It now follows that

$$na_n = c_n + \cos\gamma\, e^{-i\gamma}(p_1 c_{n-1} + p_2 c_{n-2} + \cdots + p_{n-2} c_2 + p_{n-1}),$$

$$(n-1)a_{n-1} = c_{n-1} + \cos\gamma\, e^{-i\gamma}(p_1 c_{n-2} + p_2 c_{n-3} + \cdots + p_{n-2}),$$

$$|n a_n - \varepsilon_n (n-1) a_{n-1}|$$

$$\leq |c_n - \varepsilon_1 c_{n-1}| + \cos\gamma \left(|p_{n-1}| + \sum_1^{n-2} |p_k|\, |c_{n-k} - \varepsilon_1 c_{n-k-1}| \right)$$

$$\leq 1 + \cos\gamma \sum_1^{n-1} |p_k| \leq 1 + (2n-2)\cos\gamma \leq 2n - 1,$$

$$\Big| n|a_n| - (n-1)|a_{n-1}| \Big| \leq |n a_n - \varepsilon_1 (n-1) a_{n-1}| \leq 2n - 1 = n^2 - (n-1)^2,$$

which was to be proved.

THEOREM 2. *If* $f(z) = \sum_1^\infty a_n z^n$ *is regular and univalent in* $E(z: |z| < 1)$ *and maps* E *onto a domain* G *convex in one direction, then*

$$\left(-3 + \frac{2}{n}\right)|a_1| \leq |a_n| - |a_{n-1}| \leq \left(2 - \frac{1}{n}\right)|a_1| \qquad (n = 2, 3, \cdots).$$

Proof. Let $m = n - 1$ in Theorem 1. Then we have the inequalities

$$-(2n-1)|a_1| \leq n|a_n| - (n-1)|a_{n-1}| \leq (2n-1)|a_1|,$$

$$-(2n-1)|a_1| - |a_{n-1}| \leq n(|a_n| - |a_{n-1}|) \leq (2n-1)|a_1| - |a_{n-1}|.$$

Since $|a_{n-1}| \leq (n-1)|a_1|$ (see [9]) we see that

$$-(3n-2)|a_1| \leq n(|a_n| - |a_{n-1}|) \leq (2n-1)|a_1|,$$

and the desired inequality follows when we divide by n.

3. CLOSE-TO-CONVEX FUNCTIONS

THEOREM 3. *Let* $f(z) = \sum_1^\infty a_n z^n$ *be regular, univalent, and close-to-convex in* $E(z: |z| < 1)$, *and let* m *and* n *be nonnegative integers such that* $n - m$ *is even. Then*

$$\Big| n|a_n| - m|a_m| \Big| \leq |n^2 - m^2| \cdot |a_1|.$$

Equality holds for the functions $a_1 z(1 - \varepsilon z)^{-2}$ $(|\varepsilon| = 1)$.

Proof. We may assume that $a_1 = 1$. Since $f(z)$ is close-to-convex in E, there exist a real number α $(|\alpha| \leq \pi/2)$ and a function $g(z) = z + \sum_2^\infty b_n z^n$, regular, univalent, and starlike in E, such that [3]

$$\Re\left\{e^{i\alpha}\frac{zf'(z)}{g(z)}\right\} \geq 0 \qquad (|z|<1). \tag{3.1}$$

We may assume that f(z) and g(z) are regular for $|z| \leq 1$. Otherwise, for $0 < t < 1$, we define

$$F(z) = \frac{1}{t} f(tz), \qquad G(z) = \frac{1}{t} g(tz).$$

Then G(z) is regular and starlike in $|z| \leq 1$, and

$$\Re\left\{e^{i\alpha}\frac{zF'(z)}{G(z)}\right\} \geq 0 \qquad (z \in E).$$

It follows that F(z) is regular and close-to-convex for $|z| \leq 1$.

Since g(z) is regular and starlike for $|z| \leq 1$, it is in particular starlike in the direction of its diametral line [5], [10]. For a suitable real constant β, we may represent g(z) in the form

$$g(z) = e^{i\beta}h(e^{-i\beta}z) = z + \cdots, \tag{3.2}$$

where h(z) is starlike with respect to the origin and has the real axis as its diametral line. However, in this case there exists a function P(z), analytic for $|z| \leq 1$ with $P(0) = 1$ and $\Re P(z) > 0$ in E, such that

$$h(z) = \frac{zP(z)}{1-z^2} \qquad (z \in E). \tag{3.3}$$

From (3.1) we also deduce that

$$e^{i\alpha}\frac{zf'(z)}{g(z)} = P_1(z)\cos\alpha + i\sin\alpha, \tag{3.4}$$

where $P_1(0) = 1$ and $\Re P_1(z) > 0$ in E. Since $|\alpha| \leq \pi/2$, it follows that $\cos\alpha \geq 0$. From (3.2), (3.3), and (3.4) we obtain the relation

$$f'(z) = \frac{e^{-i\alpha}P(e^{-i\beta}z)(P_1\cos\alpha + i\sin\alpha)}{1-e^{-2i\beta}z^2} = \frac{1+c_1z+c_2z^2+\cdots}{1-e^{-2i\beta}z^2}. \tag{3.5}$$

Let

$$P(e^{-i\beta}z) = 1 + \sum_1^\infty q_n z^n, \qquad e^{-i\alpha}(P_1\cos\alpha + i\sin\alpha) = 1 + \sum_1^\infty r_n z^n.$$

Since $\Re P(z) > 0$ and $\Re P_1(z) > 0$ in E, it follows that $|q_n| \leq 2$, $|r_n| \leq 2$ $(n = 1, 2, \cdots)$. Therefore

$$|c_n| = |r_n + q_1 r_{n-1} + \cdots + q_{n-1}r_1 + q_n| \leq 2 + 4 + \cdots + 4 + 2 = 4n.$$

By (3.5),

$$(1-e^{-2i\beta}z^2)\left(\sum_1^\infty n a_n z^{n-1}\right) = 1 + \sum_1^\infty c_n z^n,$$

$$|(n+1)a_{n+1} - e^{-2i\beta}(n-1)a_{n-1}| = |c_n| \leq 4n,$$

and

$$\big||(n+1)|a_{n+1}| - (n-1)|a_{n-1}|\big| \leq 4n = (n+1)^2 - (n-1)^2 \tag{3.6}$$

$$(n = 1, 2, \cdots; \ a_0 = 0, \ a_1 = 1).$$

If m is any nonnegative integer $(m < n)$ and $m - n$ is even, then

$$\begin{aligned} (n-2)^2 - n^2 &\leq n|a_n| - (n-2)|a_{n-2}| \leq n^2 - (n-2)^2, \\ (n-4)^2 - (n-2)^2 &\leq (n-2)|a_{n-2}| - (n-4)|a_{n-4}| \leq (n-2)^2 - (n-4)^2, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ m^2 - (m+2)^2 &\leq (m+2)|a_{m+2}| - m|a_m| \leq (m+2)^2 - m^2. \end{aligned}$$

By addition we obtain the inequality in the theorem.

THEOREM 4. *Let* $f(z) = \sum_1^\infty a_n z^n$ *be regular, univalent, and close-to-convex in* $E(z: |z| < 1)$. *Then*

$$-\left(6 - \frac{8}{n}\right)|a_1| \leq |a_n| - |a_{n-2}| \leq \left(4 - \frac{4}{n}\right)|a_1| \qquad (n = 3, 4, \cdots).$$

Proof. Taking $m = n - 2$ in Theorem 3, we obtain the inequalities

$$-(4n-4)|a_1| \leq n|a_n| - (n-2)|a_{n-2}| \leq (4n-4)|a_1|,$$

$$-(4n-4)|a_1| - 2|a_{n-2}| \leq n(|a_n| - |a_{n-2}|) \leq (4n-4)|a_1| - 2|a_{n-2}|.$$

Since $|a_{n-2}| \leq (n-2)|a_1|$ (see [8]), Theorem 4 follows.

For close-to-convex functions $f(z)$ in E we have been able to prove the inequality (1.7) only with the assumption that $n - m$ is even. The possibility remains that the inequality (1.7) holds for arbitrary nonnegative integers m and n. In support of this proposition we prove (1.7) for close-to-convex functions in the special case $n = 3$, $m = 2$. (Since $|a_2| \leq 2$, (1.7) is trivially true for the class S, for the case $n = 2$, $m = 1$.)

THEOREM 5. *Let* $f(z) = z + \sum_2^\infty a_n z^n$ *be regular, univalent, and close-to-convex in* $E(z: |z| < 1)$, *relative to the starlike function* $g(z) = z + \sum_2^\infty b_n z^n$. *Then the sharp inequalities*

$$|3a_3 - b_2 a_2| \leq 5 \quad \textit{and} \quad \big|3|a_3| - 2|a_2|\big| \leq 3^2 - 2^2 = 5$$

hold.

Proof. By hypothesis, there exists a real number α such that

$$\Re\{e^{i\alpha} z f'(z)/g(z)\} \geq 0 \qquad (z \in E).$$

Since $g(z)$ is starlike in E,

$$zg'(z)/g(z) = P(z) = 1 + \sum_1^\infty p_n z^n ,$$

where $\Re P(z) > 0$ in E and $P(0) = 1$. It follows that

$$(n - 1)b_n = b_{n-1} p_1 + b_{n-2} p_2 + \cdots + b_1 p_{n-1} \qquad (b_1 = 1).$$

In particular, $b_2 = p_1$ and $2b_3 = b_2 p_1 + p_2$, so that $|b_2| \leq 2$ and

$$\left| b_3 - \frac{b_2^2}{2} \right| = \left| \frac{p_2}{2} \right| \leq 1. \tag{3.7}$$

We may also write

$$e^{i\alpha} z f'(z)/g(z) = P_1(z) \cos \alpha + i \sin \alpha = e^{i\alpha} + \sum_1^\infty \tilde{p}_n z^n ,$$

where $\Re P_1(z) > 0$ in E and $P_1(0) = 1$. Thus

$$3a_3 = b_3 + \cos \alpha \; e^{-i\alpha} (\tilde{p}_1 b_2 + \tilde{p}_2),$$

$$2a_2 = b_2 + \cos \alpha \; e^{-i\alpha} \tilde{p}_1 ,$$

$$3a_3 - b_2 a_2 = \left(b_3 - \frac{b_2^2}{2} \right) + \cos \alpha \; e^{-i\alpha} \left[\tilde{p}_1 \left(b_2 - \frac{b_2}{2} \right) + \tilde{p}_2 \right],$$

$$|3 a_3 - b_2 a_2| \leq \left| b_3 - \frac{b_2^2}{2} \right| + \cos \alpha \, [2(1) + 2] \leq 1 + 4 \cos \alpha \leq 5. \tag{3.8}$$

Since $|b_2/2| \leq 1$, it follows that

$$3 |a_3| - 2 |a_2| \leq \left| 3 a_3 - \left(\frac{b_2}{2} \right) (2a_2) \right| \leq 5.$$

Equality holds for the functions $z(1 - \varepsilon z)^{-2}$ ($|\varepsilon| = 1$). This completes the proof of Theorem 5.

In the special case of Theorem 5 where $f(z)$ is univalent and starlike with respect to the origin in E, we can take $g(z) = f(z)$, $\alpha = 0$, and $b_2 = a_2$, so that

$$|3 a_3 - a_2^2| \leq 5, \qquad \big| 3 |a_3| - 2 |a_2| \big| \leq 5. \tag{3.9}$$

However, the inequalities (3.9) follow more easily from (3.7). For if $f(z)$ is starlike, we can take $b_3 = a_3$ and $b_2 = a_2$ in (3.7), and we obtain the inequalities

$$|3 a_3 - a_2^2| \leq |a_3| + |2 a_3 - a_2^2| \leq 3 + 2 = 5,$$

$$-4 \leq 3 |a_3| - 2 |a_2| \leq |3 a_3 - a_2^2| \leq 5.$$

It also follows, for starlike functions, that

$$|a_3| - |a_2| \leq \left| a_3 - \left(\frac{a_2}{2} \right) a_2 \right| \leq 1. \tag{3.10}$$

We omit the proof of the following theorem, since the method is quite similar to the one used in the proof of Theorem 1.

THEOREM 6. *For some real number* α $(|\alpha| \leq \pi/2)$, *let the function* $f(z) = z + \sum_2^\infty a_n z^n$, *regular in* $E(z: |z| < 1)$, *satisfy the inequality*

$$\Re \left\{ e^{i\alpha} \frac{f'(z)}{g'(z)} \right\} \geq 0 \qquad (z \in E),$$

where $g(z) = z + \sum_2^\infty c_n z^n$ *is regular and convex in one direction in* E. *Then, for all* n *and* m, *the Taylor coefficients satisfy the sharp inequality*

$$\Big| n\,|a_n| - m\,|a_m| \Big| \leq \frac{1}{3} \Big| n(2n^2 + 1) - m(2m^2 + 1) \Big| .$$

In particular, $|a_n| \leq \frac{1}{3}(2n^2 + 1)$ $(n = 2, 3, \cdots)$.

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