EQUIVALENCE OF EMBEDDINGS OF k-COMPLEXES IN E^n FOR $n \leq 2k+1$

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In 1953 V. K. A. M. Gugenheim showed that if a k-complex K is piecewise linearly embedded in E^n ($n \geq 2k+2$) and $h: K \to E^n$ is a piecewise linear homeomorphism, then there exists a piecewise linear isotopy $H_t: E^n \to E^n$ such that H_0 is the identity and $H_1 \mid K = h$ [3, Theorem 5]. Later, R. H. Bing and J. M. Kister [1] proved a slightly sharper result for the same dimensions: If h, as above, is an ε-homeomorphism, then $H_t: E^n \to E^n$ can be chosen to be an ε-isotopy and to move things only on a compact subset of E^n.

This paper concerns the cases where $n < 2k+2$. First it should be remarked that there is no hope of proving as general a theorem as is stated above. It is always possible to find two nonequivalent embeddings in E^n ($n \leq 2k+1$) of the complex consisting of disjoint copies of a k-sphere and an (n - k - 1)-sphere; namely, they can link in one instance and not link in the other instance. Theorem 1 gives a sufficient condition on k-complexes for any two embeddings in E^{2k+1} to be equivalent. It is easy to construct examples of very nice k-complexes in E^n ($n \leq 2k+1$) where it seems that though the homeomorphism h moves things very slightly, the isotopy might have to move things quite far.

Before proceeding with the proofs, we give a few basic definitions. If K is an abstract complex, we use the letter K for its geometric realization and also for the point set in E^n associated with some embedding. If K and L are complexes and $f: K \to L$ is a homeomorphism, then f is called *piecewise linear* if K and L have triangulations T and T', respectively, such that f takes simplexes of T linearly onto simplexes of T'. If K is a complex and I denotes the unit interval, then a *piecewise linear isotopy* of K onto itself is a piecewise linear homeomorphism of $K \times I$ onto $K \times I$, triangulated in the natural way, that takes $K \times \{t\}$ onto $K \times \{t\}$ ($0 \leq t \leq 1$). If H is such a homeomorphism, then we let $H_t(x) = H(x, t)$ for each $x \in K$ and $0 \leq t \leq 1$.

Let K and L be complexes. Let A be a simplex of K. Suppose $A = B * v$, that is, A is the join of a face B of A and a vertex v of A. Then K *collapses to* L *in a simple collapse* if $K = L \cup A$ and $L \cap A = v * \dot{B}$ (we use $\dot{C}$ to denote the boundary of a cell C). The complex K is *collapsible* if in a finite number of simple collapses it collapses to a point. If K is a subcomplex of E^n, a *regular neighborhood* of K is a combinatorial n-manifold with boundary, as defined in [5], that collapses to K in a finite number of simple collapses. It is known [5] that any two regular neighborhoods of a particular embedding of a complex are piecewise linearly homeomorphic.

Finally, the term *linking* is used in the sense of homotopy linking. That is, if R is an i-sphere and T is a k-sphere, each embedded in an n-sphere S, and if $R \cap T = \emptyset$, then R links T if R bounds no singular (i + 1)-cell in S - T. Otherwise, R does not link T.

The following Lemma, used later in the paper, is easily established, and we omit its proof.

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LEMMA 1. *Let* D *be a combinatorial* (k + 1)*-cell in* E^n (k < n). *Suppose that* E *and* E' *are combinatorial* k*-cells such that*

$$\dot{D} = E \cup E' \quad \textit{and} \quad E \cap E' = \dot{E} = \dot{E}'.$$

Suppose further that K *is a* k*-complex in* E^n, $D \cap K = E$, *and* $E \cap \overline{(K - E)} = \dot{E}$. *Then there exists a piecewise linear isotopy* H *of* E^n *onto itself such that*

1) H_0 *is the identity,*
2) $H_t \mid (K - E)$ *is the identity for* $0 \leq t \leq 1$,
3) $H_1(E) = E'$, *and*
4) *there exists a compact set* $Z \subseteq E^n$ *such that* $H_t \mid (E^n - Z)$ *is the identity for* $0 \leq t \leq 1$.

THEOREM 1. *Let* K *be a finite* k*-complex* ($k \geq 2$) *in* E^n ($n = 2k + 1$). *Suppose that* K' *is also a* k*-complex in* E^n *and that* f: K → K' *is a piecewise linear homeomorphism. Then if either*

1) $H^k(K, Z) = 0$ *or*
2) $H_k(K, Z) = 0$ *and* $T_{k-1}(K, Z) = 0$,

there exists a piecewise linear isotopy H_t: $E^n \to E^n$ ($0 \leq t \leq 1$) *such that* H_0 *is the identity and* $H_1 \mid K = f$. *Furthermore, there exists a compact set* $X \subseteq E^n$ *such that* $H_t \mid (E^n - X)$ *is the identity for* $0 \leq t \leq 1$.

Proof. First we assume that the homeomorphism f is simplicial. Let T and T' be triangulations of K and K', respectively, that are isomorphic under f. If $\sigma \in T$, let $\sigma' = f(\sigma) \in T'$. It will be convenient to consider T and T' as triangulations of the abstract complexes K and K'. The simplexes of T and T' will not be linearly embedded in E^n at all times. The i-skeleton of K (of K') will always refer to the i-skeleton with respect to T (to T').

Since $2(k - 1) + 2 \leq n$, it follows from the theorem of Bing and Kister (mentioned at the beginning of this paper) that there exists a piecewise linear isotopy G of E^n onto E^n such that G_0 is the identity and $G_1 \mid K^{k-1} = f \mid K^{k-1}$, where K^{k-1} denotes the (k - 1)-skeleton of K with respect to T. Let σ be a k-simplex of T. Then $G_1(\sigma)$ is a combinatorial k-cell in E^n. Because we can move the vertices of the interior of $G_1(\sigma)$ so that they are in general position with respect to the vertices of K', we may assume that $G_1(\sigma) \cap K' = G_1(\dot{\sigma}) = \dot{\sigma}'$. Since this can be done for each k-simplex of T, and since it can be accomplished by a suitable isotopy, we may assume that G is a piecewise linear isotopy of E^n onto itself with the following properties:

1) G_0 is the identity,
2) $G_1 \mid K^{k-1} = f \mid K^{k-1}$,
3) $G_1(K) \cap K' = G_1(K^{k-1})$ is the (k - 1)-skeleton of K', and
4) there exists a compact set $Y \subseteq E^n$ such that $G_t \mid (E^n - Y)$ is the identity for $0 \leq t \leq 1$.

For simplicity of notation, we assume that K and K' had these properties in the first place. That is, we assume that K and K' have the same (k - 1)-skeleton with respect to T and T'. Furthermore, we assume that $K \cap K'$ is exactly this common (k - 1)-skeleton.

The proof will be complete when we have moved the k-cells of T onto the corresponding cells of T'.

Let σ be a k-cell in T, and let σ' be its corresponding cell in T'. Our first objective is to find a $(k+1)$-annulus A such that $S = \sigma \cup \sigma'$ is one boundary component of A and such that $A \cap (K \cup \sigma') = S$. To get such an annulus, note that S bounds a combinatorial $(k+1)$-cell F in E^n (this was proved by E. C. Zeeman in [6]). Now, by adjusting the vertices of the interior of F to be in general position with respect to the vertices of $K \cup \sigma'$, we find that

$$\dim(F \cap (K \cup \sigma') - (\sigma \cup \sigma)) = (k+1) + k - (2k+1) = 0.$$

Thus we can pick an annular neighborhood in F of $\dot{F}$ that intersects $K \cup \sigma'$ only in S.

Let S' be the other boundary component of A. Then S' is a k-sphere in $E^n - K$. Furthermore, $\pi_i(E^n - K) = 0$ for $i \leq k$. For $i \leq k-1$, this follows from general position arguments. To show that $\pi_k(E^n - K) = 0$, note that $\pi_k(E^n - K) = H_k(E^n - K)$ by the Hurewicz Theorem [2, Chapter 15, Theorem 1.12]. By Alexander Duality [2, Chapter 12, Theorem 8.2] it follows that $H_k(E^n - K) = H^k(K)$. Thus if condition 1) of the statement of the theorem is satisfied, $\pi_k(E^n - K) = 0$. If condition 2) is satisfied, it follows from the Universal Coefficient Theorem [2, Chapter 10, Theorem 5.10] that condition 1) is also satisfied. We may now apply the Engulfing Theorem of J. Stallings [4] to obtain a combinatorial n-cell C such that C is contained in $E^n - K$ and S' is contained in the interior of C. Again using Zeeman's unknotting theorems [6], we find that S' is unknotted in the interior of C and hence bounds a combinatorial $(k+1)$-cell D in the interior of C.

Now we adjust the vertices of the interior of D so that they are in general position with respect to the vertices of A, maintaining, of course, the property that D does not intersect K. When this is done,

$$\dim(D \cap (A - S')) \leq (k+1) + (k+1) - (2k+1) = 1.$$

Since $k+1 \geq 3$, the set $D \cap (A - S')$ does not separate S from S' in A.

Let α be an arc in A such that

1) α has one endpoint in the interior of σ and the other endpoint on S',

2) int $\alpha \subseteq$ int A, and

3) $\alpha \cap D = \alpha \cap S'$ = one endpoint.

Now $A \cup D$ can be divided into two combinatorial $(k+1)$-cells, so that σ can be moved to σ' in two cellular moves across $A \cup D$. One cell, D_1, is to consist of D plus a small tubular neighborhood in A of α. The other cell, D_2, is to consist of A minus that tubular neighborhood of α. Let $E = D_1 \cap S$. Let $E' = \dot{D}_1 - E$. We assume that D_1 was chosen so that E and E' are combinatorial k-cells with E contained in the interior of σ. First we move E across D_1 to E'. Next we move $(\sigma - E) \cup E'$ across D_2 to σ'. Then two applications of Lemma 1 give rise to an isotopy H' of E^n onto itself that takes σ onto σ' and is the identity on $\overline{K - \sigma}$. Thus far, though, the final stage of the isotopy might not agree with $f \mid \sigma$. Notice that $f \circ (H_1')^{-1} \mid \sigma'$ is a piecewise linear homeomorphism of σ' onto itself that is the identity on the boundary of σ'. Hence, by a theorem of Gugenheim [3], there exists a piecewise linear isotopy H" of σ' onto itself such that H_0'' is the identity and $H_1'' = f \circ (H_1')^{-1} \mid \sigma'$. Since σ' is a simplex in E^n, H" can easily be extended to an isotopy of E^n onto itself that moves no part of $H_1'(K)$ except the set

$$H_1'(\text{int } \sigma) = \text{int } \sigma' .$$

We let H'' denote the extended isotopy also.

The isotopy of the conclusion of the theorem, restricted to σ, can be written as the composition $H = H'' \circ H' \circ G$. By applying the above techniques to another k-simplex of T and using H(K) in place of K, we can easily see how to continue to move K onto K' in the prescribed manner.

It has been announced at various times that the regular neighborhood of a contractible 2-complex piecewise linearly embedded in E^5 is a 5-cell, but the status of the proofs is in doubt. The following partial result in this direction is an immediate consequence of Theorem 1.

COROLLARY. *If* K *and* K' *are two piecewise linear embeddings of a contractible* 2*-complex in* E^5 *and if* N *and* N' *are regular neighborhoods of* K *and* K', *respectively, then* N *and* N' *are piecewise linearly homeomorphic.*

THEOREM 2. *Let* k *be an integer* $(k \geq 2)$. *For each integer* n $(k \leq n \leq 2k)$ *there exist a collapsible* k*-complex* K(k, n) *and two piecewise linear embeddings* K' *and* K'' *of* K(k, n) *in* E^n *such that no homeomorphism of* E^n *onto itself takes* K' *onto* K''.

Proof. Let Σ be the boundary of an n-simplex in E^n. Let S be a (k - 1)-sphere, and let T be an (n - k - 1)-sphere disjoint from S. We consider $S \cup T$ as an abstract complex, not as a subset of E^n. Let K(k, n) be the cone over $S \cup T$. Let S' and T' be piecewise linear embeddings of S and T in Σ such that S' and T' link each other in Σ. Let S'' and T'' be piecewise linear embeddings of S and T in Σ such that S'' does not link T'' in Σ. Let v be a point in the interior of Σ. We define K' and K'' as follows: K' is the embedding that takes K(k, n) onto $v * (S' \cup T')$ in the natural way, and K'' is the embedding that takes K(k, n) onto $v * (S'' \cup T'')$ in the natural way. We let K' and K'' denote the complexes as well as the embeddings. Clearly, K' and K'' are not piecewise linearly equivalently embedded in E^n, because

$$\text{lk}(v, K') = S' \cup T' \quad \text{and} \quad \text{lk}(v, K'') = S'' \cup T'' .$$

$S' \cup T'$ and $S'' \cup T''$ are not equivalently embedded in $\text{lk}(v, E^n) = \Sigma$.

To show that no space homeomorphism, piecewise linear or not, takes K' onto K'' is slightly more difficult, and we present only a sketch of the proof. First we parametrize $v * S'$ in the natural way, so that S_t' is a (k - 1)-sphere $(0 \leq t < 1)$ with $S' = S_0'$ and $S_1' = v$. We define T_t', S_t'', and T_t'' in a similar fashion. Each S_t'' bounds a singular k-cell D_t that intersects K'' only in S_t''. Suppose f is a homeomorphism of E^n onto itself that takes K'' onto K'. Then f(v) = v, and we can assume that $f(S'') = S'$. There exists a t $(0 \leq t < 1)$ such that $f(D_t)$ is contained in the interior of Σ. By projecting $f(D_t)$ from v, we can construct a singular k-cell lying in $\Sigma - T'$ and having S' as its boundary. This leads to the contradiction that S' can be shrunk to a point in $\Sigma - T'$, that is, S' does not link T' in Σ.

Question. Present information suggests that the obstruction to obtaining space isotopies in E^{2k+1} that take one embedding of a k-complex onto another embedding is a difficulty with linking rather than knotting. Hence one might ask whether such an isotopy always exists if the complex has at most one nontrivial cycle.

REFERENCES

1. R. H. Bing and J. M. Kister, *Taming complexes in hyperplanes*, Duke Math. J. 31 (1964), 491-511.
2. D. G. Bourgin, *Modern algebraic topology*, Macmillan, New York, 1963.
3. V. K. A. M. Gugenheim, *Piecewise linear isotopy and embedding of elements and spheres (I)*, Proc. London Math. Soc. (3) 3 (1953), 29-53.
4. J. Stallings, *The piecewise-linear structure of Euclidean space*, Proc. Cambridge Philos. Soc. 58, (1962), 481-488.
5. J. H. C. Whitehead, *Simplicial spaces, nuclei and* m-*groups*, Proc. London Math. Soc. (2) 45 (1939), 243-327.
6. E. C. Zeeman, *Unknotting combinatorial balls*, Ann. of Math. (2) 78 (1963), 501-526.

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