COVERING THEOREMS FOR UNIVALENT FUNCTIONS

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1. INTRODUCTION

Let M_+ denote the class of univalent meromorphic functions $f(z)$ in the unit disc $|z| < 1$, hereafter called D, such that

(1) $$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \qquad (a_2 \geq 0)$$

in a neighborhood of the origin. Any meromorphic univalent function in D can be transformed into a member of M_+ by a suitable mapping of D onto itself and a normalization. Let U_+ denote the subclass of M_+ containing the functions that are regular in D, and let S_+ and C_+ denote the starlike and convex subclasses, respectively, of U_+. For $f \in M_+$, $\rho(\phi, f)$ represents the distance along a fixed ray $\arg w = \phi$ from $w = 0$ to the nearest boundary point of the map of D by $w = f(z)$. Put $m(\phi) = \inf \rho(\phi, f)$ for $f \in M_+$, and similarly define $u(\phi)$, $s(\phi)$, and $c(\phi)$ for the classes U_+, S_+, and C_+, respectively. Scott [2] has proved that

$$u(\phi) = 1/2 \ (0 \leq |\phi| \leq \pi/2) \qquad \text{and} \qquad u(\pi) = 1/4,$$

and he obtained estimates for $u(\phi)$ in the range $\pi/2 < |\phi| < \pi$. (The class U introduced here is the closure of the class U_+ used in [2], but it is evident that $u(\phi)$ is the same for U and U_+.) In this paper it will be shown that

$$1/2 = m(\phi) = u(\phi) = s(\phi) < c(\phi) < \pi/4 \qquad (0 \leq |\phi| \leq \pi/2),$$

$$0 < m(\phi) < u(\phi) < s(\phi) < c(\phi) \qquad (\pi/2 < |\phi| < \pi),$$

$$0 = m(\pi) < u(\pi) = s(\pi) = 1/4 < c(\pi) = 1/2.$$

Our principal method of proof is subordination. We use the following elementary properties of bounded analytic functions: If, in D,

$$f(z) = b_1 z + b_2 z^2 + \cdots \qquad \text{and} \qquad |f(z)| < 1,$$

then [1, p. 168]

(2) $$|b_2| < 1 - |b_1|^2,$$

and equality holds if and only if for some real α

(3) $$f(z) = z \frac{b_1 + z e^{i\alpha}}{1 + \bar{b}_1 z e^{i\alpha}}.$$

If, in addition, $f(z)$ is univalent in D, then [1, p. 224]

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(4) $$|b_2| \leq 2|b_1|(1 - |b_1|),$$

and equality holds if and only if

(5) $$f(z) = e^{i\alpha} K^{-1}[|b_1| K(ze^{i\beta})],$$

where $\alpha = 2 \arg b_1 - \arg b_2$, $\beta = \arg b_2 - \arg b_1$, and K^{-1} is the inverse of the Koebe function $K(z) = z/(1 - z)^2$.

2. UNIVALENT FUNCTIONS

If $f \in M_+$ and $f(z) \neq \rho e^{i\phi}$ for all z in D, then $\rho \geq m(\phi)$. Thus the following is a covering theorem for M_+.

THEOREM 1. $m(\phi) = 1/2$ *for* $0 \leq |\phi| \leq \pi/2$, *and* $m(\phi) = |\sin \phi|/2$ *for* $\pi/2 < |\phi| \leq \pi$. *For fixed* ϕ $(0 \leq |\phi| < \pi)$, *a function* $f \in M_+$ *omits the value* $m(\phi)e^{i\phi}$ *if and only if*

(6) $$f(z) = \frac{z}{1 + e^{-2i\phi} z^2} \quad (0 \leq |\phi| \leq \pi/2),$$
$$f(z) = \frac{z}{1 - 2|\text{ctn}\, \phi|\, z - z^2} \quad (\pi/2 < |\phi| < \pi).$$

Proof. Suppose $f \in M_+$ and f omits the value $\gamma = \rho e^{i\phi}$. Since (1) holds, the function

(7) $$g(z) = \frac{f(z)}{1 - f(z)/\gamma} = z + \left(a_2 + \frac{1}{\gamma}\right) z^2 + \cdots$$

has a removable singularity at any pole of $f(z)$ in D; that is, it can be defined to be regular and univalent in D. From the coefficient inequality for such functions [1, p. 213],

(8) $$|a_2 + e^{-i\phi}/\rho| \leq 2,$$

where equality holds if and only if $g(z) = z/(1 - e^{i\alpha} z)^2$. Since $a_2 \geq 0$,

$$\rho^{-1} \leq |a_2 + \rho^{-1} e^{-i\phi}| \leq 2 \quad (0 \leq |\phi| \leq \pi/2),$$

$$|\sin \phi|\, \rho^{-1} \leq |\Im(a_2 + \rho^{-1} e^{-i\phi})| \leq 2 \quad (\pi/2 \leq |\phi| \leq \pi).$$

By (7) and (8), equality holds for a fixed ϕ $(0 \leq |\phi| < \pi)$ if and only if $f(z)$ is the function (6). Since, for each $\lambda > 0$, $f(z) = z/(1 - \lambda z)$ is in M_+, we conclude that $m(\pi) = 0$. This completes the proof.

Let M_+' be the subclass of functions $f \in M_+$ for which (1) holds with $a_2 \leq 2$, and let $m'(\phi) = \inf \rho(\phi, f)$ for $f \in M_+'$.

THEOREM 2. $m'(\phi) = 1/2$ *for* $0 \leq |\phi| \leq \pi/2$, $m'(\phi) = |\sin \phi|/2$ *for* $\pi/2 < |\phi| \leq 3\pi/4$, *and* $m'(\phi) = |\sec \phi|/4$ *for* $3\pi/4 < |\phi| \leq \pi$. *A function* $f \in M_+'$ *omits the value* $m'(\phi)e^{i\phi}$ *if and only if* $f(z)$ *is the function* (6) *when* $0 \leq |\phi| \leq 3\pi/4$ *and*

(9) $$f(z) = \frac{z}{1 - 2z + e^{2i\phi} z^2} \qquad \left(\frac{3\pi}{4} < |\phi| \le \pi\right).$$

Proof. Since $M'_+ \subset M_+$, $m'(\phi) \ge m(\phi)$. Furthermore, for fixed ϕ $(0 \le |\phi| \le 3\pi/4)$, the function (6) is in M'_+, which proves that $m'(\phi) = m(\phi)$ in this case. An argument like that in Theorem 1 shows that

$$|2 + \rho^{-1} e^{-i\phi}| \le |a_2 + \rho^{-1} e^{-i\phi}| \le 2$$

whenever $\rho \le -\cos \phi/2$, since $0 \le a_2 \le 2$ for $f \in M'_+$. A simple computation gives $\rho \ge 1/4|\cos \phi|$ when $3\pi/4 < |\phi| \le \pi$, and equality holds if and only if $f(z)$ is the function (9).

Since $U_+ \subset M'_+$, $u(\phi) \ge m'(\phi)$. When $0 \le |\phi| \le \pi/2$, the function (6) is regular in D, and the same is true for the function (9) when $\phi = \pi$, so that $u(\phi) = m'(\phi)$ in these cases. Since the extremal functions (6) and (9) are not regular in D when $\pi/2 < |\phi| < \pi$, and since U_+ is compact, it follows that $u(\phi) > m'(\phi)$ in this case. This completes the proof of the following corollary, which is contained in the results of [2]:

COROLLARY 1. $u(\phi) = m'(\phi)$ *for* $0 \le |\phi| \le \pi/2$, $u(\phi) > m'(\phi)$ *for* $\pi/2 < |\phi| < \pi$, *and* $u(\pi) = m'(\pi) = 1/4$.

3. SLIT AND STARLIKE FUNCTIONS

THEOREM 3. *For fixed real* ϕ $(0 \le |\phi| \le \pi)$ *and* $\rho_0 > 0$, *let*

(10) $$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \qquad (|z| < 1,\ a_2 \ge 0)$$

map the unit disc onto a region that omits $\rho e^{i\phi}$ *for all* $\rho \ge \rho_0$. *Then* $\rho_0 \ge \rho(\phi)$, *where*

(11) $$\rho(\phi) = \begin{cases} \sqrt{3}/4 & (0 \le |\phi| \le \pi/2), \\ \sqrt{1 + 2|\sin \phi|}/4 & (\pi/2 < |\phi| \le \pi). \end{cases}$$

If, in addition, $f(z)$ *is univalent in* D, *then* $\rho_0 \ge r(\phi)$, *where*

(12) $$r(\phi) = \begin{cases} 1/2 & (0 \le |\phi| \le \pi/2), \\ (1 + |\sin \phi|)/4 & (\pi/2 < |\phi| \le \pi). \end{cases}$$

For each ϕ $(0 \le |\phi| \le \pi)$ *there is a unique function* (10) *that omits the values on the slit* $\rho e^{i\phi}$ $(\rho \ge \rho(\phi))$ *and a unique univalent function that omits the values on the slit* $\rho e^{i\phi}$ $(\rho \ge r(\phi))$.

Proof. For $p \ne 0$ and $F(z) = 4pz/(1 + z)^2$, the inverse function

$$F^{-1}(w) = \frac{w}{4p} + \frac{w^2}{8p^2} + \cdots$$

maps the complex plane cut along the radial line segment from p to ∞ onto the unit disc. Thus if $f(z)$ is the function (10) and $p = \rho_0 e^{i\phi}$,

$$F^{-1}[f(z)] = \frac{z e^{-i\phi}}{4\rho_0} + \left(\frac{a_2 e^{-i\phi}}{4\rho_0} + \frac{e^{-2i\phi}}{8\rho_0^2}\right) z^2 + \cdots$$

maps $|z| < 1$ into itself. By (2) and (4) this implies

$$\left| a_2 + \frac{e^{-i\phi}}{2\rho_0} \right| \leq \mu(\rho_0), \tag{13}$$

where $\mu(\rho_0) = 4\rho_0 - 1/4\rho_0$ or, when f(z) is known to be univalent in D, $\mu(\rho_0) = 2 - 1/2\rho_0$. Since $a_2 \geq 0$,

$$\frac{1}{2\rho_0} = \left| 0 + \frac{e^{-i\phi}}{2\rho_0} \right| \leq \left| a_2 + \frac{e^{-i\phi}}{2\rho_0} \right| \leq \mu(\rho_0) \qquad (0 \leq |\phi| \leq \pi/2),$$

$$\frac{|\sin \phi|}{2\rho_0} = \left| \Im\left(a_2 + \frac{e^{-i\phi}}{2\rho_0} \right) \right| \leq \mu(\rho_0) \qquad (\pi/2 < |\phi| \leq \pi).$$

This shows that $\rho_0 \geq \rho(\phi)$, where $\rho(\phi)$ is given by (11) and that when f(z) is univalent in D, $\rho_0 \geq r(\phi)$, where $r(\phi)$ is given by (12).

When the function f(z) of (10) is univalent in D, equality holds in (13) if and only if $F^{-1}[f(z)]$ has the form prescribed in (5). Then

$$f(z) = F\left\{ e^{i\alpha} K^{-1}\left[\frac{1}{4\rho_0} K(e^{i\beta} z) \right] \right\} = z + 2e^{i\beta}\left(1 - \frac{1 + e^{i\alpha}}{4\rho_0}\right) z^2 \cdots, \tag{14}$$

where $\alpha + \beta = -\phi$. For $0 \leq |\phi| \leq \pi/2$ and $\rho_0 = 1/2$, the coefficient of z^2 in (14) is real and nonnegative if and only if

$$\sin \beta - \sin(\alpha + \beta) = \sin \beta + \sin \phi = 0 \qquad \text{and} \qquad \cos \beta - \cos \phi \geq 0,$$

or, equivalently, $\beta = -\phi$ and $\alpha = 0$. Thus for fixed ϕ $(0 \leq |\phi| \leq \pi/2)$ we have shown that $\rho_0 = 1/2$ if and only if

$$f(z) \equiv F\left\{ K^{-1}\left[\frac{1}{2} K(e^{-i\phi} z) \right] \right\} \equiv z/(1 + e^{-2i\phi} z^2). \tag{15}$$

For $\pi/2 < |\phi| \leq \pi$ and $\rho_0 = (1 + |\sin \phi|)/4$, the coefficient of z^2 in (14) is real and nonnegative if and only if

$$|\sin \phi| \sin \beta + \sin \phi = 0 \qquad \text{and} \qquad |\sin \phi| \cos \beta - \cos \phi \geq 0,$$

or equivalently, $\beta = \pm\pi/2$, where the sign is chosen opposite to that of ϕ. Thus $\rho_0 = (1 + |\sin \phi|)/4$ for fixed ϕ $(\pi/2 < |\phi| \leq \pi)$ if and only if

$$f(z) \equiv F\{\pm i e^{-i\phi} K^{-1}[(1 + |\sin \phi|)^{-1} K(\mp iz)]\}, \tag{16}$$

where the upper or lower sign is used according as ϕ is positive or negative.

If univalence is not required for the function f(z) of (10), a similar argument with the functions (3) replacing (5) yields the extremal functions. We find that $\rho_0 = \sqrt{3}/4$ for fixed ϕ $(0 \leq |\phi| \leq \pi/2)$ if and only if

$$f(z) = \frac{z(1+\sqrt{3}\,e^{-i\phi}z)(1+e^{-i\phi}z/\sqrt{3})}{(1+2e^{-i\phi}z/\sqrt{3}+e^{-2i\phi}z^2)^2},$$

and that $\rho_0 = \sqrt{1+2|\sin\phi|}/4 = \mu/4$ for $\pi/2 < |\phi| \leq \pi$ if and only if

$$f(z) = \frac{z(1 \pm i\mu z)(1 \pm iz/\mu)}{[1+(e^{-i\phi} \pm i)z/\mu \pm ie^{-i\phi}z^2]^2},$$

where the upper or lower sign is used according as ϕ is negative or positive.

Since the functions in the second part of Theorem 3 are in U_+, the following is obtained from (12):

COROLLARY 2. *For* $\pi/2 < |\phi| < \pi$, $u(\phi) \leq (1+|\sin\phi|)/4$.

For fixed ϕ $(0 \leq |\phi| \leq \pi/2)$, the extremal function (15) and, for $\phi = \pi$, the extremal function (16) is in S_+. When $\pi/2 < |\phi| < \pi$, however, an examination of the map of D by (16) shows that it is not starlike with respect to the origin. Since S_+ is compact, the lower estimate (with strict inequality) in the following result is a consequence of Theorem 3.

COROLLARY 3. *For* $0 \leq |\phi| \leq \pi/2$, $s(\phi) = 1/2$; *for* $\pi/2 < |\phi| < \pi$,

$$(1+|\sin\phi|)/4 < s(\phi) \leq \lambda^{-\lambda}(1-\lambda)^{\lambda-1}/4 \qquad (\textit{where } \lambda = |\phi|/\pi),$$

and $s(\pi) = 1/4$.

The upper estimate for $s(\phi)$ $(\pi/2 < |\phi| < \pi)$ is obtained from the function

$$f(z;\lambda) = \frac{z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{2\lambda-1} \qquad \left(\frac{\pi}{2} < |\phi| = \pi\lambda \leq \pi\right),$$

which is in S_+ and maps $|z| < 1$ onto the complex plane cut along the rays $\arg w = \pm|\phi|$ from ∞ to the points of modulus $1/4\lambda^{\lambda}(1-\lambda)^{1-\lambda}$.

The last two corollaries show that $u(\phi) < s(\phi)$ for $\pi/2 < |\phi| < \pi$, but Theorem 2 and Corollary 1 show that $u(\phi) = s(\phi)$ for all other values of ϕ.

Since $\overline{f(\bar{z})}$ and $t^{-1}f(tz)$ $(0 < t < 1)$ are in the same class as $f(z)$ for $f(z)$ in U_+ or S_+, the curves $w = u(\phi)e^{i\phi}$ and $w = s(\phi)e^{i\phi}$ bound starlike regions that are symmetric with respect to the real axis. These regions are not convex in a neighborhood of $w = -1/4$.

4. HALF-PLANE AND CONVEX FUNCTIONS

Since $w = T(z) = 2dz/(1+e^{-i\phi}z)$ maps D onto the half-plane $\Re\, e^{-i\phi}w < d$, the method of proof for Theorem 3 can be used to prove the following:

THEOREM 4. *If* $w = f(z) = z + a_2z^2 + \cdots$ $(a_2 \geq 0;\ |z| < 1)$ *maps the unit disc into a half-plane* $\Re\, e^{-i\phi}w < d$, *then* $d \geq d(\phi)$, *where*

$$d(\phi) = \begin{cases} \sqrt{2}/2 & (0 \leq |\phi| \leq \pi/2), \\ \frac{1}{2}\sqrt{1+|\sin\phi|} & (\pi/2 < |\phi| \leq \pi). \end{cases}$$

If, in addition, f(z) *is univalent in* D, *then* $d \geq h(\phi)$, *where*

$$h(\phi) = \begin{cases} 3/4 & (0 \leq |\phi| \leq \pi/2), \\ (2 + |\sin \phi|)/4 & (\pi/2 < |\phi| \leq \pi). \end{cases} \tag{17}$$

For each ϕ $(0 \leq |\phi| \leq \pi)$, *there is a unique analytic function* (1) *that maps* D *into* $\Re\, e^{-i\phi} w < d(\phi)$, *and a unique analytic univalent function* (1) *that maps* D *into* $\Re\, e^{-i\phi} w < h(\phi)$.

The extremal functions in the final statement of the theorem are obtained from the relation $T^{-1}[f(z)] = g(z)$, where $d = d(\phi)$ or $d = h(\phi)$ in T(z), and the form of g(z) is as specified in (3) or (5), respectively. In the univalent case, except for $|\phi| = \pi$, the extremal function is not convex, since it maps the unit disc onto a slit half-plane. However, Theorem 4 applies to all functions of C_+, and since C_+ is compact, it follows that $c(\phi) > h(\phi)$ for $0 \leq |\phi| < \pi$.

THEOREM 5.

$$3/4 < c(\phi) \leq \pi/4 \qquad (0 \leq |\phi| \leq \pi/2),$$

$$\frac{1}{4}\left[|\sin \phi| + \sqrt{3 + \sin^2 \phi}\right] \leq c(\phi) \leq \begin{cases} \dfrac{\pi |\cos \phi|}{2(2|\phi| - \pi)} & \left(\dfrac{\pi}{2} < |\phi| \leq \dfrac{3\pi}{4}\right), \\ \dfrac{1}{2|\cos \phi|} & \left(\dfrac{3\pi}{4} < |\phi| < \pi - \arctan\dfrac{1}{2}\right), \end{cases} \tag{18}$$

$$c(\phi) = \frac{1}{2|\cos \phi|} \qquad \left(\pi - \arctan\frac{1}{2} \leq |\phi| \leq \pi\right).$$

Proof. Computation shows that the lines $\Re\, e^{-i\phi} w = h(\phi)$, where $h(\phi)$ is given by (17), envelope a region E that is symmetric with respect to the real axis. In the upper half-plane, E is bounded

$$\text{by } |w| = 3/4 \quad \text{for } 0 \leq \arg w \leq \pi/2,$$

$$\text{by } |w - i/4| = 1/2 \quad \text{for } \pi/2 < \arg w < \pi - \arctan 1/2,$$

$$\text{and by } \Re\, w = -1/2 \quad \text{for } \pi - \arctan 1/2 \leq \arg w \leq \pi.$$

If $f \in C_+$ and D_f denotes the map of D by f, then each finite boundary point of D_f has a line of support $\Re\, e^{-i\phi} w = d$, and by Theorem 4, $d \geq h(\phi)$. Since these lines of support envelope D_f, whereas the lines $\Re\, e^{-i\phi} w = h(\phi)$ envelope E, it follows that $D_f \supset E$. The lower estimates in (18) are now obtained from the polar form for the boundary of E.

The upper estimates in (18) are obtained from the following functions in C_+:

$$f(z) = e^{-i\phi} \arctan(e^{i\phi} z) \qquad (0 \leq |\phi| \leq \pi/2);$$

$$f(z) = \frac{1}{2(2\lambda - 1)}\left\{\left(\frac{1+z}{1-z}\right)^{2\lambda - 1} - 1\right\} \qquad (\pi/2 < |\phi| = \pi\lambda < 3\pi/4);$$

$$f(z) = z/(1 - z) \qquad (3\pi/4 \leq |\phi| \leq \pi).$$

The region bounded by $w = c(\phi) e^{i\phi}$ is convex, since it is the intersection of a family of convex regions.

Added in proof. A complete determination of $u(\phi)$ has been announced by G. V. Kuz'mina [*Covering theorems for functions which are regular and univalent in the circle* (Russian), Dokl. Akad. Nauk SSSR 160 (1965), 25-28; MR 30, 3204].

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