SEMIGROUPS OF NONLINEAR TRANSFORMATIONS

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Suppose X is a B-space, let B(X) denote the B-space of bounded linear transformations from X into X, and let I denote the identity operator on X. A family $[T(t),\ 0 \le t < \infty]$ of operators in B(X) is said to be a *uniformly continuous semigroup* if and only if

i) $T(0) = I$ and $T(t)T(s) = T(t+s)$ for $s, t \geq 0$, and

ii) $\|T(s) - T(t)\| \to 0$ as $s \to t$, for $t \geq 0$.

A necessary and sufficient condition that a family $[T(t),\ 0 \le t < \infty]$ of operators in B(X) be a uniformly continuous semigroup is that there exist an operator A in B(X) such that $T(t) = e^{tA}$ for all $t \geq 0$, or equivalently, such that $T(0) = I$, and $(d/dt)T(t)x = AT(t)x$ for x in X and $t \geq 0$ (see Hille and Phillips [3, Section 9.6, p. 287] or Dunford and Schwartz [2, Chapter VIII, p. 613]). Under the additional requirement that X be a *complex* B-space, we extend this theorem to families of nonlinear transformations.

Suppose S is a connected open set in X. If M is a connected open set in $[0, \infty) \times S$ such that (0, x) is in M for each x in S, and f is a function from M into S, then f is said to be a *uniformly continuous semigroup* if and only if it satisfies the following three conditions.

a) f is locally uniformly continuous; that is, for each (t, x) in M, there exists a neighborhood of (t, x) on which f is uniformly continuous.

b) The second-place partial derivative $f_2(t, x)$ exists (as a Fréchet derivative; see Dieudonné [1, pp. 143, 167] or Hille [3, Chapter III, p. 110]) for all (t, x) in M, and f_2 is a locally uniformly continuous function from M into B(X).

c) $f(0, x) = x$ for all x in S, and

$$f(s, f(t, x)) = f(s+t, x) \qquad \text{for } (s, f(t, x)) \text{ and } (s+t, x) \text{ in } M.$$

The three conditions are satisfied, for a suitable set M, if $f(t, x) = T(t)x$ for some uniformly continuous semigroup $[T(t),\ 0 \le t < \infty]$ of operators in B(X), in which case $f_2(t, x) = T(t)$. The main theorems to be proved are as follows.

THEOREM 1. *If* M *is a connected open set in* $[0, \infty) \times S$ *that contains* (0, x) *for each* x *in* S, *and the function* f *from* M *into* S *is a uniformly continuous semigroup, then there exists a continuously (Fréchet) differentiable function* F *from* S *into* X *such that*

$$f_1(t, x) = F(f(t, x)) \qquad \textit{for all } (t, x) \textit{ in } M,$$

where f_1 *denotes the first-place partial derivative (in the ordinary sense) of* f.

THEOREM 2. *If* G *is a continuously (Fréchet) differentiable function from* S *into* X, *then there exist an open set* M *in* $[0, \infty) \times S$ *that contains* (0, x) *for each* x *in* S, *and a function* g *from* M *into* S *such that*

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$$g(0, x) = x \textit{ for } x \textit{ in } S \quad \textit{and} \quad g_1(t, x) = G(g(t, x)) \textit{ for } (t, x) \textit{ in } M.$$

Moreover, this function g *is a uniformly continuous semigroup.*

Proof of Theorem 1. First, we state a definition and establish two lemmas. For each (v, x) in M with $v > 0$, let $F_v(x) = [f(v, x) - x]/v$.

LEMMA. *For* (s, x) *in* M *and sufficiently small* v,

$$\| F_v(f(s, x)) - f_2(s, x) F_v(x) \| \le \| F_v(x) \| \sup \| f_2(s, x) - f_2(s, y) \| \;\; (y \text{ in } [x, f(v, x)]),$$

where $[x, f(v, x)]$ *denotes the closed line interval from* x *to* $f(v, x)$.

Proof.

$$v F_v(f(s, x)) = f(v, f(s, x)) - f(s, x) = f(s + v, x) - f(s, x) = f(s, f(v, x)) - f(s, x),$$

and

$$\| f(s, f(v, x)) - f(s, x) - f_2(s, x)[f(v, x) - x] \|$$
$$\le \| f(v, x) - x \| \sup \| f_2(s, x) - f_2(s, y) \| \;\; (y \text{ in } [x, f(v, x)])$$

by [1, Theorem 8.6.2, p. 156].

LEMMA. *If* z *is in* S, *then there exists an open ball* W *lying in* S *and having center* z *such that* $\lim F_v(x)$ $(v \to 0)$ *exists and is approached uniformly for all* x *in* W.

Proof. Let U denote an open set in M, containing $(0, z)$, such that f and f_2 are uniformly continuous on U. Choose $\alpha > 0$ and $t > 0$ so that (s, x) is in U and

$$\| I - f_2(s, x) \| < 1/2 \quad \text{for } \| x - z \| < \alpha \text{ and } 0 \le s \le t,$$

and so that $\| f(t, x) - x \| < 1$ for $\| x - z \| < \alpha$. Let W denote the open ball with center z and radius α. For x in W, let

$$T_x = (1/t) \int_0^t f_2(s, x)\, ds.$$

Then $\| I - T_x \| < 1/2$ and T_x is invertible for x in W. Also, $\| T_x y \| \ge \| y \|/2$ for y in X and x in W. Choose $\delta > 0$ so that $(v, f(s, x))$, $(s, f(v, x))$, and $(s + v, x)$ are in M for $0 \le s \le t$, x in W, and $0 \le v \le \delta$. For $0 < v < \delta$ and x in W, let

$$H_v(x) = (1/t) \int_0^t F_v(f(s, x))\, ds,$$

$$P_v(x) = H_v(x) - T_x \cdot F_v(x),$$

$$Q_v(x) = \begin{cases} P_v(x)/\| F_v(x) \| & \text{for } F_v(x) \ne 0, \\ 0 & \text{for } F_v(x) = 0. \end{cases}$$

Then

$$\begin{aligned} tv H_v(x) &= \int_0^t [f(s+v, x) - f(s, x)]\, ds \\ &= \int_v^{t+v} f(s, x)\, ds - \int_0^v f(s, x)\, ds - \int_v^t f(s, x)\, ds \\ &= \int_t^{t+v} f(s, x)\, ds - \int_0^v f(s, x)\, ds \\ &= \int_0^v [f(s+t, x) - f(s, x)]\, ds = \int_0^v [f(t, f(s, x)) - f(s, x)]\, ds, \end{aligned}$$

so that

$$H_v(x) = (1/v) \int_0^v F_t(f(s, x))\, ds,$$

and $H_v(x) \to F_t(x)$ as $v \to 0$, uniformly for x in W. Moreover,

$$\|P_v(x)\| \leq (1/t) \int_0^t \|F_v(f(s, x)) - f_2(s, x) F_v(x)\|\, ds,$$

so that

$$\|Q_v(x)\| \leq \sup \|f_2(s, x) - f_2(s, y)\|,$$

where the supremum is taken for s in $[0, t]$ and y in $[x, f(v, x)]$. Therefore, $Q_v(x) \to 0$ as $v \to 0$, uniformly for x in W. Thus, for sufficiently small v, we have the inequalities

$$\|H_v(x)\| < \|F_t(x)\| + 1 < (t+1)/t,$$

$$\|T_x F_v(x)\| - \|F_v(x)\| \cdot \|Q_v(x)\| < (t+1)/t,$$

$$\|T_x F_v(x)\| \geq \|F_v(x)\|/2, \qquad \|Q_v(x)\| < 1/4, \qquad \|F_v(x)\|/4 < (t+1)/t$$

for all x in W. Therefore, $P_v(x) \to 0$ uniformly for x in W, so that $T_x F_v(x) \to F_t(x)$ and $F_v(x) \to T_x^{-1} F_t(x)$ uniformly for x in W. This completes the proof of the lemma.

To complete the proof of Theorem 1, let

$$F(x) = \lim F_v(x) \qquad (v \to 0)$$

for all x in S. It follows from [3, Theorem 3.18.1, p. 113] that F is continuously (Fréchet) differentiable on S. If (s, x) is in M, then $(v, f(s, x))$ and $(s+v, x)$ are in M for sufficiently small v, and

$$\frac{f(s+v, x) - f(s, x)}{v} = F_v(f(s, x)) \to F(f(s, x)).$$

This completes the proof of Theorem 1.

Proof of Theorem 2. By [1, Theorem 10.8.1, p. 299], there exist, for each z in S, a positive number r and an open ball V lying in S and having center z, such that the system

$$g(0, x) = x, \qquad g_1(t, x) = G(g(t, x))$$

has a unique solution on $[0, r) \times V$ that is uniformly continuous. We may take r small enough so that the solution g takes $[0, r) \times V$ into S. Let M be the union of such sets $[0, r) \times V$. Since the solutions are unique on each set $[0, r) \times V$, there is a single solution (we shall denote it by g) that is defined on all of M, and this solution is a locally uniformly continuous function from M into S. By [1, Theorem 10.8.2, p. 300], g has a continuous second-place partial (Fréchet) derivative g_2. Since g is also a solution of the integral equation

$$g(t, x) = x + \int_0^t G(g(s, x))\,ds,$$

we may differentiate to obtain the relation

$$g_2(t, x) = I + \int_0^t G'(g(s, x))\,g_2(s, x)\,ds.$$

This differentiation makes use of the rule for differentiation under the integral sign [1, Theorem 8.11.2, p. 172] and the chain rule for derivatives [1, Theorem 8.2.1, p. 145]. Now, [1, Theorem 10.8.1, p. 299] applies again and yields the local uniform continuity of g_2. If x is in S and (t, x) is in M, let

$$\alpha(s) = g(s + t, x) \qquad \text{for } (s + t, x) \text{ in } M,$$

$$\beta(s) = g(s, g(t, x)) \qquad \text{for } (s, g(t, x)) \text{ in } M.$$

Then

$$\alpha'(s) = g_1(s + t, x) = G(\alpha(s)), \qquad \alpha(0) = g(t, x),$$

$$\beta'(s) = g_1(s, g(t, x)) = G(\beta(s)), \qquad \beta(0) = g(0, g(t, x)) = g(t, x),$$

so that $\alpha(s) = \beta(s)$ for all s common to the domains of α and β, by [1, Theorem 10.8.1, p. 299]. This completes the proof of Theorem 2.

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