ALMOST LOCALLY FLAT IMBEDDINGS OF MANIFOLDS

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1. INTRODUCTION

By an *almost locally flat* imbedding of one manifold in another we mean an imbedding that is locally flat except possibly at countably many points. We are concerned with the question as to whether almost locally flat imbeddings of manifolds are actually locally flat at every point, under appropriate dimensional restrictions.

In Theorem 2.4 it is shown that, for an imbedding h of a k-manifold in an n-manifold in the so-called "trivial" range of dimensions $n \geq 2k + 2$, the assumption that h is almost locally flat does imply that h is locally flat at every point. The authors have previously considered the cases $k = 1$ and $k = 2$ of this theorem (see [3], [5], and [8]). The proof of Theorem 2.4 employs J. Stallings' result to the effect that, for $n \geq 5$ and $k \leq n - 3$, a k-sphere that is topologically imbedded in the n-sphere cannot fail to be locally flat at a single point [16] (in other words, cannot be locally flat everywhere with the exception of precisely one point).

It is well-known that a simple closed curve or 2-sphere imbedded in Euclidean 3-space E^3 can fail to be locally flat at a single point (see [9], for instance), whereas Cantrell has proved that an (n - 1)-sphere which is imbedded in the n-sphere S^n cannot fail to be locally flat at a single point if $n > 3$ [4]. In Section 4 we show that, for each $n \geq 3$, S^n contains a topological (n - 2)-sphere which fails to be tame but is locally flat except possibly at a single point. Thus, for imbeddings of S^k in S^n, local flatness except at a single point implies tameness if and only if $n > 3$ and $k \neq n - 2$.

In Section 3 we show that, if M and N are two topological manifolds such that no imbedding of M in N fails to be locally flat at precisely one or two points, then every almost locally flat imbedding of M in N is locally flat at every point. Finally, polyhedral imbeddings of the 4-sphere S^4 in S^n are discussed in Section 5.

If M is a k-dimensional submanifold of the n-manifold N, then M is said to be *locally flat* at the point $p \in M$ if there exists a neighborhood U (in N) of p such that the pair $(U, U \cap M)$ is homeomorphic to the pair (E^n, E^k). An imbedding h of the manifold M into the manifold N is said to be *locally flat* at the point $x \in M$ if h(M) is locally flat at h(x). An imbedding or submanifold is called *locally flat* if it is locally flat at every point. A closed subset P of a complex K is said to be *tame* if there exists a homeomorphism f of K onto itself such that f(P) is a subcomplex of some subdivision of K. A topological k-sphere S (k-cell C) in E^n is *flat* if there exists a homeomorphism g of E^n onto itself such that g(S) is the boundary of a (k + 1)-simplex (such that g(C) is a k-simplex).

By a *combinatorial* n-*manifold* we shall mean one without boundary, in other words, a connected separable metric space that is triangulated as a simplicial complex in which the link of each vertex is a combinatorial (n - 1)-sphere.

Received November 14, 1964.

This research was supported by the National Science Foundation, Grant 23790. The second-named author is an Alfred P. Sloan Fellow.

2. IMBEDDINGS IN THE TRIVIAL RANGE

If f is a piecewise linear mapping of the complex K into the complex L, we define the *singular set* $S(f)$ of f as the closure of the set of points $x \in K$ such that $f^{-1}f(x)$ contains more than one point. We quote first the following well-known general position lemma.

LEMMA 2.1. *Let* K *be a finite* k-*complex,* K_0 *a subcomplex of* K, *and* M *a combinatorial* n-*manifold* ($k \leq n$). *If* f *is a piecewise linear map of* K *into* M *such that* $f \mid K_0$ *is a homeomorphism, and* $\varepsilon > 0$, *then there exists a piecewise linear map* g *of* K *into* M *such that*

a) $d(f(x), g(x)) < \varepsilon$ *for each* $x \in K$,

b) $f(x) = g(x)$ *if* $x \in K_0$,

c) *the dimension of* $S(g)$ *is at most* $2k - n$.

A proof of a generalization of Lemma 2.1 may be found in [12] or in Chapter 6 of [18].

In proving that an almost locally flat imbedding in the trivial range is locally flat, the first step is to show that if such an imbedding fails to be locally piecewise linear at an isolated point, then it is nevertheless locally flat there. This is done (with a slight dimensional improvement) in the following lemma.

LEMMA 2.2. *Let* h *be an imbedding of a combinatorial* k-*ball* B *in* E^n, *where* $k \geq 2$ *and* $n \geq 2k + 1$. *If* h *is locally piecewise linear except at a single point* $p \in \text{Int } B$, *then* h *is locally flat at* p.

Proof. Consider B as imbedded in a combinatorial k-sphere S^k, and denote by S^{k-1} the combinatorial $(k - 1)$-sphere $\text{Bd } B$ and by C the combinatorial k-ball $\text{Cl}(S^k - B)$. Since $n - k \geq 3$, E. C. Zeeman's result on unknotting combinatorial spheres [17] implies that the piecewise linear imbedding $h: S^{k-1} \to E^n$ can be extended to a piecewise linear imbedding h' of C into E^n, and we may assume that $h(p) \notin h'(C)$. Then h and h' together define a map f of S^k into E^n that is locally piecewise linear except at the single point p.

However, f may have singularities in $S^k - p$. To eliminate these, choose combinatorial k-balls A and A_0 in S^k such that $A_0 \cap S(f) = \emptyset$ and $p \in \text{Int } A \subset \text{Int } A_0$, and let S^k be subdivided so as to contain A and A_0 as subcomplexes. Define

$$K = S^k - \text{Int } A \quad \text{and} \quad K_0 = A_0 - \text{Int } A,$$

and choose $\varepsilon > 0$ less than the distance from $f(A)$ to $f(K - K_0)$. Then Lemma 2.1 yields a piecewise linear imbedding g of K into E^n such that

$$g \mid K_0 = f \mid K_0 \quad \text{and} \quad g(K - K_0) \cap f(A) = \emptyset.$$

Hence g and $f \mid A$ together define an imbedding Ψ of S^k into E^n that is locally piecewise linear except at p. It therefore follows from Zeeman's theorem [17] on the unknotting of combinatorial ball pairs that Ψ is locally flat except possibly at p. Stallings' result [16] now implies that Ψ is also locally flat at p. Since the topological k-sphere $\Psi(S^k)$ contains a neighborhood (in $h(B)$) of $h(p)$, it follows that h is locally flat at p.

Using H. Gluck's modification [10] of a theorem of T. Homma [13], Gluck [10] and C. Greathouse [11] have shown that if f is a locally flat imbedding of the closed combinatorial k-manifold M in the combinatorial n-manifold N, where $n \geq 2k + 2$, then, for each $\varepsilon > 0$ and each neighborhood U of f(M), there exists an ε-homeomorphism h of N onto itself such that hf is piecewise linear and $h \mid N - U = 1$. The following local version of this result is proved in [8], with the minor difference that there it is stated in terms of imbedded submanifolds instead of imbeddings of manifolds.

LEMMA 2.3. *Let* h *be an imbedding of the compact combinatorial* k-*manifold* M *with boundary in the combinatorial* n-*manifold* N, *where* $n \geq 2k + 2$, *such that* h *is locally flat at each point of the open subset* G *of* Int M. *Then, given* $\varepsilon > 0$ *and a neighborhood* U *of* h(G) *in* N, *there exists an* ε-*homeomorphism* g *of* N *onto itself such that* gh *is locally piecewise linear at each point of* G *and* $g \mid (N - U) \cup h(M - G)$ *is the identity.*

LEMMA 2.3 now enables us to apply Lemma 2.2 to almost locally flat imbeddings.

THEOREM 2.4. *Let* h *be an imbedding of a topological* k-*manifold (not necessarily triangulated)* M *in a topological* n-*manifold* N, *with* $n \geq 2k + 2$, *and denote by* E *the set of points at which* h *is not locally flat. Then* E *contains no isolated points and is therefore either empty or uncountable.*

Proof. Suppose, to the contrary, that p is an isolated point of E. Let U be a Euclidean neighborhood of h(p) in N, sufficiently small so that $G = h^{-1}(U)$ is contained in a Euclidean neighborhood (in M) of p that contains no other point of E. Then Lemma 2.3 yields a homeomorphism g of N onto itself such that gh is locally piecewise linear (relative to arbitrarily assigned combinatorial triangulations of G and U) at each point of G - p. Since p lies interior to a combinatorial k-ball B such that $B \subset G$ and $gh(B) \subset U$, Lemma 2.2 implies that the imbedding gh, and therefore h, is locally flat at p (actually, Lemma 2.2 applies only if $k \geq 2$; in the case $k = 1$ we use [5]). This contradiction proves that E contains no isolated points. That E must then be either empty or uncountable then follows from the fact that every nonempty closed perfect subset of a manifold is uncountable.

COROLLARY 2.5. *If* f *is an almost locally flat imbedding of the closed combinatorial* k-*manifold* M *in the combinatorial* n-*manifold* N $(n \geq 2k + 2)$, *then, given* $\varepsilon > 0$ *and a neighborhood* U *of* f(M) *in* N, *there exists an* ε-*homeomorphism* h *of* N *onto itself such that* $h \mid N - U = 1$ *and* $hf\colon M \to N$ *is piecewise linear.*

Proof. This follows immediately from Theorem 2.4 above and Theorem 8.1 of [10].

COROLLARY 2.6. *Every almost locally flat imbedding of* S^k *in* S^{2k+2} *is flat.*

This follows immediately from Corollary 2.5 and the fact that any piecewise linear imbedding of S^k in S^{2k+2} is flat.

3. A REDUCTION OF THE PROBLEM

The purpose of this section is to show how the problem of proving that almost locally flat imbeddings are locally flat can be reduced to proving that an imbedding cannot fail to be locally flat at precisely one or two points.

LEMMA 3.1 (Rosen [14]). *If* A *is a countable compact subset of* E^n, *then there is a flat arc in* E^n *that contains* A.

A subset X of an n-manifold M is said to be *cellular* in M if M contains a sequence $\{Q\}_{i=1}^{\infty}$ of closed n-cells such that $Q_{i+1} \subset \operatorname{Int} Q_i$ for each $i \geq 1$ and such that $X = \bigcap_{i=1}^{\infty} Q_i$. The proof of the following lemma is easy and we therefore omit it.

LEMMA 3.2. *If* X *is cellular in the manifold* M, *then the decomposition space* M/X *(obtained by collapsing* X *to a point) is homeomorphic to* M.

Let R denote the closed ray consisting of the points $x \in E^1$ with $x \geq 0$. An arc L in a topological n-manifold M will be said to be *locally flat* at the point $p \in L$ if p has a neighborhood U such that the pair $(U, U \cap L)$ is homeomorphic either to (E^n, R) or to (E^n, E^1), according as p is or is not an endpoint of L.

LEMMA 3.3. *If* L *is a locally flat arc contained in the topological* n-*manifold* M, *then* L *is cellular in* M.

Proof. For a Euclidean space M the lemma is known ([2], [5]). We can therefore establish the lemma by showing that some open n-cell in M contains L.

Let a and b be the endpoints of L, and order the points of L from a to b. Let x be the least upper bound of the set of points $y \in L$ such that L_y, the subarc of L from a to y, is contained in an open n-cell U_y. We have only to show that $x = b$ (clearly $x > a$). If $x < b$, we select an open n-cell neighborhood V of x such that $(V, V \cap L) \simeq (E^n, E^1)$. Let z be a point of $V \cap L$ $(z < x)$, and let U_z be an open n-cell in M such that $L_z \subset U_z$. Let h be a homeomorphism of M onto itself such that

$$h \mid (M - V) = 1, \quad h(L) = L, \quad h(z) = x.$$

If $z_1 \in L$ is such that $z_1 \in U_z$ and $z < z_1 < x$, then $h(U_z)$ is an open n-cell containing $L_{h(z_1)}$, and $h(z_1) > x$. This contradiction implies that $x = b$.

LEMMA 3.4. *Let* L *be an arc contained in a topological* n-*manifold* M $(n > 3)$, *and let* E *be the set of points at which* L *fails to be locally flat. If* E *is not empty, then* E *is uncountable.*

Proof. Suppose E is not empty. We show that E is uncountable by showing that there are no isolated points of E. If x is an isolated point of E, we let U be an open n-cell neighborhood of x such that $E \cap U = x$, and we let L be a subarc of L such that $x \in L$ and $L \subset U$. We then have an arc in a Euclidean space of dimension $n > 3$ that is locally flat except at one point. But this is impossible by [5], and it follows that E is uncountable.

THEOREM 3.5. *Let* M *and* N *be topological manifolds, with* M *compact, such that no imbedding of* M *into* N *fails to be locally flat at precisely one or two points. Then every almost locally flat imbedding of* M *into* N *is locally flat.*

Proof. Let f be an almost locally flat imbedding of M into N, and let E be the set of points at which f(M) fails to be locally flat. If E is not empty, we use the Doyle-Hocking construction (see the proof of Theorem 1 of [7]) to obtain a set $U \subset f(M)$ such that U is topologically equivalent to E^k ($k = \dim M$) and U contains the compact countable set E. We let a be an isolated point of E and use Lemma 3.1 to select an arc L, flat in U, such that $E - a \subset L$ and $a \notin L$. Since L is locally flat in f(M) at each point and f(M) is locally flat in N at each point of $L - E$, it follows that L is locally flat in N at each point of $E - a$. By Lemma 3.4, L is locally flat in N at every point, and by Lemma 3.3 it then follows that L is cellular in N.

If $\phi: M \to M/f^{-1}(L)$ and $\Psi: N \to N/L$ are the natural projection maps, then, by Lemma 3.2, $\phi(M)$ and $\Psi(N)$ are topological copies of M and N, respectively. The imbedding $f: M \to N$ induces an imbedding $g: \phi(M) \to \Psi(N)$ such that $\Psi f = g\phi$, and clearly g is locally flat except possibly at the two points $\phi f^{-1}(a)$ and $\phi f^{-1}(L)$. But then the hypothesis of the theorem implies that g is locally flat at every point. Since ϕ and Ψ restrict to homeomorphisms on neighborhoods of $f^{-1}(a)$ and a, respectively, it follows that f(M) is locally flat at a after all. This contradiction shows that E is empty, so that f is locally flat at every point.

4. IMBEDDINGS OF SPHERES WITH CODIMENSION TWO

The purpose of this section is to demonstrate that, for each $n \geq 3$, there exists a wild (n - 2)-sphere in E^n that is locally flat except possibly at a single point.

LEMMA 4.1. *If* K *is a compact rectilinear complex in* S^n, *then* $\pi_1(S^n - K)$ *is finitely generated.*

Proof. Let R be a regular neighborhood of K in S^n. Then $\pi_1(S^n - K)$ is isomorphic to $\pi_1(S^n - \text{Int } R)$, since $S^n - \text{Int } R$ is a deformation retract of $S^n - K$. But $S^n - \text{Int } R$ is a compact polyhedron; hence its fundamental group is isomorphic to its edge-path group and is therefore finitely generated.

The proof of Theorem 1 of [1] can be applied directly to establish the following lemma.

LEMMA 4.2. *Let* S *be a topological* (n - 3)*-sphere in* E^{n-1}, *and let* T *be the suspension of* S *in* E^n. *Then* $\pi_1(E^n - T)$ *is isomorphic with* $\pi_1(E^{n-1} - S)$.

LEMMA 4.3. *For each* $n \geq 3$ *there exists a topological* (n - 2)*-sphere* S^{n-2} *in* E^n *that is locally flat except possibly at a single point, and such that* $\pi_1(E^n - S^{n-2})$ *is not finitely generated.*

Proof. To begin the inductive proof, consider the wild arc A (the "mildly wild" 2-frame) of Debrunner and Fox [6]. By connecting the endpoints of A with a polygonal arc we obtain a wild 1-sphere S^1 in E^3 such that S^1 is locally flat except at a single point and $\pi_1(E^3 - S^1)$ is not finitely generated.

Now assume that S^{k-2} is a topological (k - 2)-sphere in E^k such that S^{k-2} is locally flat except possibly at the point $p \in S^{k-2}$ and such that $\pi_1(E^k - S^{k-2})$ is not finitely generated. If T is the suspension of S^{k-2} in E^{k+1}, then it follows from Lemma 4.2 that $\pi_1(E^{k+1} - T)$ is not finitely generated. If the arc L is the suspension of the point p, then it is clear that the (k - 1)-sphere T is locally flat at each point of T - L.

Now let ϕ be the projection of E^{k+1} onto E^{k+1}/L. Since the arc L is cellular in both E^{k+1} and T, it follows from Lemma 3.2 that $\phi(E^{k+1})$ is a copy of E^{k+1} and that $S^{k-1} = \phi(T)$ is a topological (k - 1)-sphere. Furthermore, S^{k-1} is locally flat except possibly at the single point $\phi(L)$, and $\pi_1(\phi(E^{k+1}) - S^{k-1})$ is not finitely generated, because $\phi(E^{k+1}) - S^{k-1}$ is homeomorphic to $E^{k+1} - T$. This completes the inductive proof.

The existence of the desired examples now follows immediately from Lemmas 4.1 and 4.3.

THEOREM 4.4. *For each* $n \geq 3$ *there exists a topological* (n - 2)*-sphere* S^{n-2} *in* E^n *that is not tame but is locally flat except possibly at a single point.*

We are unable to determine whether, for $n \geq 4$, S^{n-2} does in fact fail to be locally flat at the exceptional point p. If S^{n-2} were actually locally flat at p, for $n \geq 4$, then S^{n-2} would be a wild locally flat (n - 2)-sphere in E^n.

5. POLYHEDRAL IMBEDDINGS OF THE 4-SPHERE

Throughout this section, let T be a combinatorial triangulation of the n-sphere, and let K denote a subcomplex of T that is homeomorphic to S^4. Rosen has shown in [14] that, if $n = 5$, it follows that K is flat. It is well known that this conclusion does not follow in the case $n = 6$ (for example, if K is the 4-sphere in S^6 obtained by suspending three times the trefoil knot in S^3, then Lemma 4.2 implies that K is not flat). We discuss here the cases $n \geq 7$.

LEMMA 5.1. *If* $n \geq 7$, *then* K *is locally flat except possibly at its vertices.*

Proof. Consider a vertex v of K. Let $X = \text{lk}(v, T)$ (the link of v in T) and $Y = \text{lk}(v, K)$. Then Y is a combinatorial 3-manifold [15, p. 240]. If p is a point of the open star of v in K, other than v, then there exists a unique point q of Y such that the line segment vq contains p. By subdividing, if necessary, we may assume that q is a vertex of T. Let

$$B^{n-1} = \text{St}(q, X) \quad \text{and} \quad B^3 = \text{St}(q, Y).$$

Then (B^{n-1}, B^3) is a combinatorial $(n - 1, 3)$-ball pair. Since $n - 4 \geq 3$, it follows from [17] that B^3 is piecewise linearly unknotted in B^{n-1}. Therefore the combinatorial (n, 4)-ball pair (vB^{n-1}, vB^3) is unknotted. Since $p \in \text{Int}\, vB^3$ and $vB^3 \subset K$, it follows that K is unknotted at p. Since each point of K lies interior to some vertex star, this implies that K is locally flat except possibly at its vertices.

A topological k-sphere in S^n is said to be *weakly flat* if its complement in S^n is homeomorphic to the complement of a standard k-sphere (the boundary of a linear $(k + 1)$-simplex).

THEOREM 5.2. *If* $n \geq 7$, *then* K *is weakly flat in* S^n, *and if* $n \geq 10$, *then* K *is flat in* S^n.

Proof. The second part follows immediately from Lemma 5.1 and Theorem 2.4.

In the case $n \geq 7$, choose an arc $L \subset K$, as in the proof of Theorem 3.5, that contains all the vertices of K and is cellular in both K and S^n. Then K/L is a 4-sphere in the n-sphere S^n/L, locally flat except possibly at a single point. It therefore follows from [16] that K/L is flat in S^n/L. But since $S^n - K$ is homeomorphic to $S^n/L - K/L$, it follows that K is weakly flat in S^n.

The referee has pointed out that the first half of Theorem 5.2 also follows from Theorem 2' of [14], together with Lemma 5.1 above.

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