A REFINEMENT OF TWO THEOREMS OF KRONECKER

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Kronecker [1] proved in 1857 that if an algebraic integer α different from zero is not a root of unity, then at least one of its conjugates has absolute value greater than 1. He proved also that if α is a totally real algebraic integer and $\alpha \neq 2\cos\rho\pi$ (ρ rational), then at least one of its conjugates has absolute value greater than 2. The aim of this paper is to refine the above statements as follows.

THEOREM 1. *If an algebraic integer* $\alpha \neq 0$ *is not a root of unity, and if* 2s *among its conjugates* α_i $(i = 1, \cdots, n)$ *are complex, then*

$$\max_{1 \leq i \leq n} |\alpha_i| > 1 + 4^{-s-2}. \tag{1}$$

THEOREM 2. *If a totally real algebraic integer* β *is different from* $2\cos\rho\pi$ (ρ *rational*), *and* $\{\beta_i\}$ $(i = 1, \cdots, n)$ *is the set of its conjugates, then*

$$\max_{1 \leq i \leq n} |\beta_i| > 2 + 4^{-2n-3}. \tag{2}$$

It would be possible to improve 4^{-s-2} and 4^{-2n-3} on the right-hand side of inequalities (1) and (2) by constant factors. This, however, seems of no interest, since probably the order of magnitude of

$$\max_{1 \leq i \leq n} |\alpha_i| - 1 \quad \text{and} \quad \max_{1 \leq i \leq n} |\beta_i| - 2$$

is much greater than that given by (1) and (2), respectively. In fact, for α satisfying the assumptions of Theorem 1, we cannot disprove the inequality

$$\max_{1 \leq i \leq n} |\alpha_i| > 1 + \frac{c}{n}, \tag{3}$$

where $c > 0$ is an absolute constant.

Such a disproof would give an affirmative answer to a question of D. H. Lehmer [2, p. 476], open since 1933, namely, whether to every $\varepsilon > 0$ there corresponds an algebraic integer α such that

$$1 < \prod_{i=1}^{n} \max(1, |\alpha_i|) < 1 + \varepsilon.$$

Inequality (3), if true, is the best possible, as the example $\alpha = 2^{1/n}$ shows. Concerning inequality (2), we observe that there exist totally real algebraic integers, not of the form $2\cos\rho\pi$ (ρ rational), for which

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$$\max_{1 \leq i \leq n} \{|\beta_i| - 2\}$$

is arbitrarily small. This follows from a theorem of R. M. Robinson [3], according to which there are infinitely many systems of conjugate totally real algebraic integers in every interval of length greater than 4, in particular, in $[-2+\varepsilon, 2+2\varepsilon]$, for every $\varepsilon > 0$.

In the subsequent proof of Theorem 1 we make frequent use of the following inequalities, valid for all positive integers m:

$$(4) \qquad (1+x)^m < \frac{1}{1-mx} \quad \left(0 < x < \frac{1}{m}\right),$$

$$(5) \qquad \left(1+\frac{1}{y}\right)^{1/m} > 1 + \frac{1}{m(y+1)} \quad (0 < y).$$

Proof of Theorem 1. Let α_i be real for $i = 1, \cdots, r$ and complex for $i = r+1, \cdots, r+2s = n$ with $\alpha_i = \overline{\alpha_{i+s}}$ $(i = r+1, \cdots, r+s)$. Let

$$|\alpha_\mu| = \max_{1 \leq i \leq n} |\alpha_i| \geq 1.$$

Suppose first that $\mu \leq r$. If $|\alpha_i^2 - 1| \geq 1$ for some $i \leq r$, then $\alpha_i^2 \geq 2$, hence

$$|\alpha_\mu| \geq |\alpha_i| \geq \sqrt{2} \geq 1 + 4^{-s-2}.$$

If $|\alpha_i^2 - 1| < 1$, then, noting that $|\alpha_\mu^2 - 1| = |\alpha_\mu|^2 - 1$ and $|\alpha_i^2 - 1| \leq |\alpha_\mu|^2 + 1$ $(r < i \leq r+2s)$, we deduce from (4) that either

$$\prod_{i=1}^{n} |\alpha_2^2 - 1| \leq (|\alpha_\mu|^2 - 1)(|\alpha_\mu|^2 + 1)^{2s}$$

$$\leq (|\alpha_\mu|^2 - 1)2^{2s}\left(1 + \frac{|\alpha_\mu|^2 - 1}{2}\right)^{2s}$$

$$\leq 2^{2s}\frac{|\alpha_\mu|^2 - 1}{1 - s(|\alpha_\mu|^2 - 1)}$$

or

$$s(|\alpha_\mu|^2 - 1) \geq 1.$$

In the second case, (5) implies that

$$|\alpha_\mu| \geq \left(1 + \frac{1}{s}\right)^{1/2} > 1 + \frac{1}{2(s+1)} > 1 + 4^{-s-2}.$$

Since no α_i is a root of unity, $\Pi_1^n |\alpha_i^2 - 1|$ is a positive integer. Thus in the first case $s(|\alpha_\mu|^2 - 1) \leq 1$, and so

$$1 - s(|\alpha_\mu|^2 - 1) \leq 2^{2s}(|\alpha_\mu|^2 - 1).$$

But then by (5)

$$|\alpha_\mu| \geq \left(1 + \frac{1}{s + 2^{2s}}\right)^{1/2} > 1 + \frac{1}{2(s + 2^{2s} + 1)} > 1 + 4^{-s-2}.$$

Next, suppose $r < \mu \leq r + s$. Let $\alpha_\mu = |\alpha_\mu| e^{2\pi i\theta}$. By Dirichlet's approximation theorem, there exist integers p and q such that

$$(6) \qquad |2\theta q - p| < \frac{1}{9 \cdot 2^{s-1}} \quad \text{and} \quad 1 \leq q \leq 9 \cdot 2^{s-1}.$$

Hence

$$|4q\theta\pi - 2\pi p| < \frac{2\pi}{9 \cdot 2^{s-1}} < 2^{-s+1/2}$$

and

$$\cos 4q\theta\pi > \cos 2^{-s+1/2} > 1 - \frac{1}{2}(2^{-s+1/2})^2 = 1 - 2^{-2s}.$$

This gives the estimate

$$(7) \qquad \begin{cases} |(\alpha_\mu^{2q} - 1)(\alpha_{\mu+s}^{2q} - 1)| = |\alpha_\mu|^{4q} - (\alpha_\mu^{2q} + \bar{\alpha}_\mu^{2q}) + 1 \\ \qquad = |\alpha_\mu|^{4q} - 2|\alpha_\mu|^{2q} \cos 4q\theta\pi + 1 \\ \qquad \leq |\alpha_\mu|^{4q} - 2|\alpha_\mu|^{2q}(1 - 2^{-2s}) + 1. \end{cases}$$

If $|\alpha_i^{2q} - 1| \geq 1$, for some $i \leq r$, then $|\alpha_\mu|^{2q} \geq |\alpha_i|^{2q} \geq 2$. Hence, by (5) and (6),

$$|\alpha_\mu| \geq 2^{1/2q} \geq 2^{-9 \cdot 2^s} \geq 1 + 9^{-1} 2^{-s-1} > 1 + 4^{-s-2}.$$

If $|\alpha_i^{2q} - 1| < 1$, for all $i \leq r$, we use (4) and (7) and obtain the inequality

$$\prod_{i=1}^{n} |\alpha_i^{2q} - 1| \leq \{|\alpha_\mu|^{4q} - 2|\alpha_\mu|^{2q}(1 - 2^{-2s}) + 1\}(|\alpha_\mu|^{2q} + 1)^{2s-2}$$
$$\leq \{|\alpha_\mu|^{4q} - 2(1 - 2^{-2s})|\alpha_\mu|^{2q} + 1\} \frac{2^{2s-2}}{1 - (s-1)(|\alpha_\mu|^{2q} - 1)},$$

or

$$(s - 1)(|\alpha_\mu|^{2q} - 1) \geq 1.$$

In the second case, using (5) and (6), we obtain the estimate

$$|\alpha_\mu| \geq 1 + \frac{1}{2qs} \geq 1 + \frac{1}{9s \cdot 2^s} > 1 + 4^{-s-2}.$$

If the second case does not occur, then $1 - (s - 1)(|\alpha_\mu|^{2q} - 1) > 0$. Since no α_i is a root of unity, $\Pi_1^n|\alpha_i^{2q} - 1|$ is a positive integer. Thus

$$2^{2s-2}\{|\alpha_\mu|^{4q} - 2(1 - 2^{-2s})|\alpha_\mu|^{2q} + 1\} \geq 1 - (s - 1)(|\alpha_\mu|^{2q} - 1),$$

hence

$$|\alpha_\mu^{2q}|^2 - |\alpha_\mu|^{2q}\left(2 - \frac{2s-1}{2^{2s-1}}\right) + \left(1 - \frac{s}{2^{2s-2}}\right) \geq 0.$$

Since $|\alpha_\mu^{2q}| \geq 1$ and

$$1 - \left(2 - \frac{2s-1}{2^{2s-1}}\right) + \left(1 - \frac{s}{2^{2s-2}}\right) = -2^{-2s+1} < 0,$$

it follows that

$$|\alpha_\mu|^{2q} \geq 1 - (2s - 1)2^{-2s} + \sqrt{\{1 - (2s - 1)2^{-2s}\}^2 - (1 - s\cdot 2^{-2s+2})}$$

$$= 1 + \frac{1}{s - \frac{1}{2} + \sqrt{2^{2s-1} + \left(s - \frac{1}{2}\right)^2}}.$$

Now (6) implies that

$$2q\left(s + \frac{1}{2} + \sqrt{2^{2s-1} + \left(s - \frac{1}{2}\right)^2}\right) \leq 9\cdot 2^s\left(s + \frac{1}{2} + \sqrt{2^{2s-1} + \left(s - \frac{1}{2}\right)^2}\right) < 4^{s+2}.$$

It follows from (5) and the last two inequalities that

$$|\alpha_\mu| \geq 1 + \frac{1}{2q\left(s + \frac{1}{2} + \sqrt{2^{2s-1} + \left(s - \frac{1}{2}\right)^2}\right)} > 1 + 4^{-s-2}.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Let β be a totally real algebraic integer satisfying the assumptions of the theorem, and put

$$\alpha = \beta/2 + \sqrt{(\beta/2)^2 - 1}.$$

Then α is an algebraic integer and $\alpha^2 - \beta\alpha + 1 = 0$. All the conjugates of α are zeros of polynomials $g_i(x) = x^2 - \beta_i x + 1$ $(i = 1, 2, \cdots, n)$. At most $2n - 2$ of them are complex, since otherwise $|\beta_i| \leq 2$, contrary to the original theorem of Kronecker. Thus α is not a root of unity, and by Theorem 1,

$$\max_{1 \leq j \leq 2n} |\alpha^{(j)}| \geq 1 + 4^{-n-1}.$$

The complex conjugates of α have absolute value 1. It follows that for some $i \leq n$,

$$|\beta_i|/2 + \sqrt{|\beta_i/2|^2 - 1} > 1 + 4^{-n-1} > |\beta_i|/2 - \sqrt{|\beta_i/2|^2 - 1},$$

hence $g_i(\operatorname{sgn} \beta_i (1 + 4^{-n-1})) < 0$. But then

$$|\beta_i| > (1 + 4^{-n-1}) + \frac{1}{1 + 4^{-n-1}} > 1 + 4^{-n-1} + 1 - 4^{-n-1} + 4^{-2n-3} = 2 + 4^{-2n-3}.$$

This completes the proof.

REFERENCES

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