CIRCLE-PACKINGS AND CIRCLE-COVERINGS ON A CYLINDER

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The problems of the densest circle-packing and the thinnest circle-covering of the plane and the sphere, together with their various ramifications, have a vast literature, which is due both to the intrinsic beauty of these problems and to their applications in many fields of pure and applied mathematics. The corn-ear and similar conformations suggest the analogous problems on a cylinder [1, pp. 170-171], [2].

The perimeter of the orthogonal section of a cylinder is said to be its perimeter. A (cylindrical) circle of radius r is defined as the set of all points of the cylinder whose geodesic distance from a fixed point of the cylinder is at most r. (This definition obviously renders the shape of the cylinder irrelevant.) The questions we want to treat concern the cylinders for which the most economical packing and covering with unit circles is the worst.

Our results are contained in the following theorems.

THEOREM 1. *The greatest packing density of unit circles on a cylinder is at least* $\pi/4$, *and it is* $\pi/4$ *only for a cylinder of perimeter* $2\sqrt{2}$.

THEOREM 2. *The smallest covering density of unit circles on a cylinder is at most* $\pi/2$, *and it is* $\pi/2$ *only for a cylinder of perimeter* 2.

On a cylinder of perimeter $2\sqrt{2}$ a densest packing (with density $\pi/4$) is obtained by the arrangement in which each circle is touched by two others in the vertices of a square (Figure 1). If the perimeter equals 2, we obtain a thinnest covering (with density $\pi/2$) by placing the circles in such a way that each circle is intersected by two others in the vertices of a square (Figure 2). Of course, these extremal arrangements are not unique. For the removal of a circle from the packing or the addition of a circle to the covering does not change the density. As for the density, we can use the same definition as in the Euclidean plane [3].

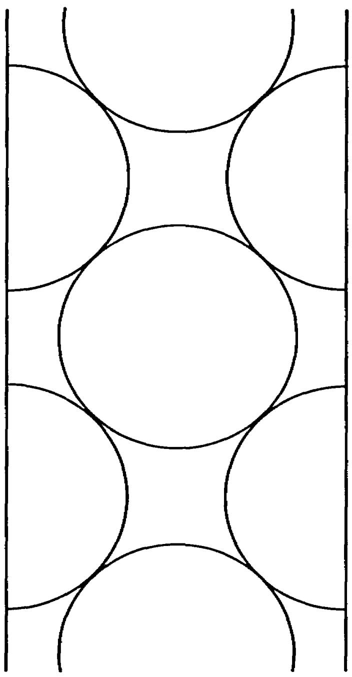

Figure 1.

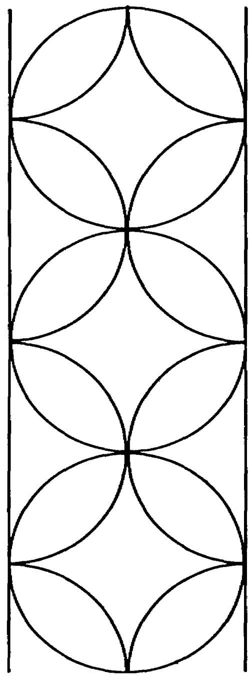

Figure 2.

Proof of Theorem 1. The densest packing of a cylinder with perimeter at most 2 is illustrated by Figure 3. Obviously, the density is at least $\pi/\sqrt{12}$. Thus we can restrict ourselves to the case of a cylinder of perimeter $p > 2$.

Suppose that

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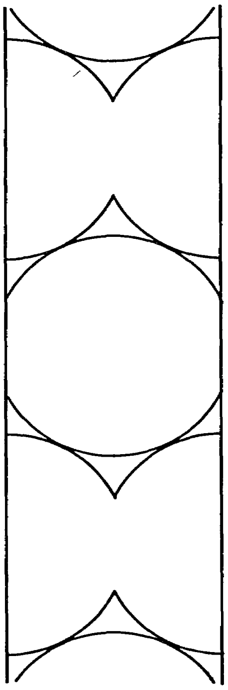

Figure 3.

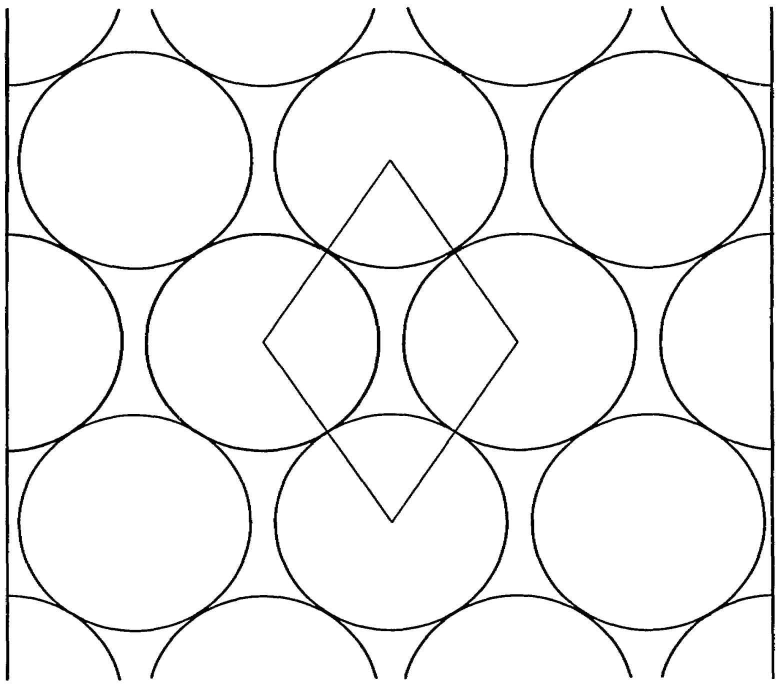

Figure 4.

$$n \leq p/2 \leq \sqrt{3}n,$$

where n is a positive integer. Consider a "cycle" of n unit circles equally spaced at the same height of the cylinder. The packing consisting of such cycles placed in such a way that all neighbouring cycles have exactly 2n points in common has density at least $\pi/4$, for this packing is generated by a rhombus with sides of length 2 (Figure 4). Therefore the area of the rhombus is at most 4, and it is 4 only for a square, that is, for $p = 2\sqrt{2}n$.

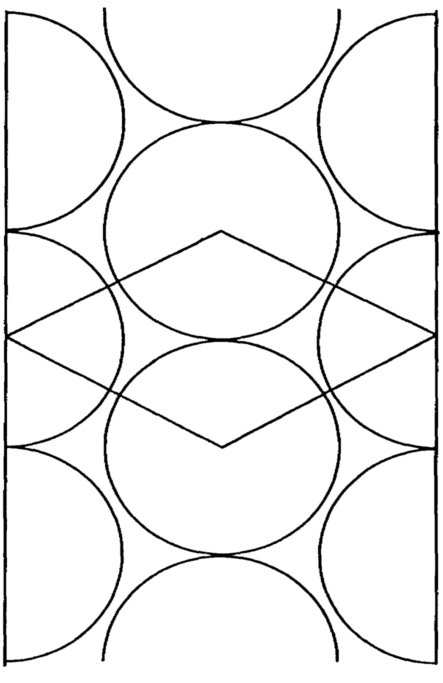

Figure 5.

The intervals $[n, \sqrt{3}n]$ cover all values of $p/2 > 1$ except the values with $\sqrt{3} < p/2 < 2$. As to the values $p/2 = \sqrt{2}n$, observe that for $n \geq 3$,

$$n + 1 < \sqrt{2}n < \sqrt{2}(n + 1).$$

Thus, except for the value $p/2 = \sqrt{2}$, which lies in $(1, \sqrt{3})$, there is only one critical value, namely $p/2 = 2\sqrt{2}$.

We proceed to construct a packing of density greater than $\pi/4$ for all values p such that $\sqrt{3} \leq p/2 < 2$. This packing is generated by a rhombus having a "vertical" diagonal of length 2 and a "horizontal" diagonal of length p (Figure 5). The area of this rhombus being p, the packing-density π/p is indeed less than $\pi/4$.

In the case $p/2 = \sqrt{8}$, a packing of density greater than $\pi/4$ can be obtained as follows. We start with the observation that among the numbers of the form

$$\sqrt{a^2 + ab + b^2} \qquad (a, b = 0, 1, 2, \cdots; a + b > 0)$$

giving the half-length of a lattice-vector in the densest lattice-packing of the plane by unit circles [2], there is one, namely

$$\sqrt{2^2 + 2 \cdot 1 + 1^2} = \sqrt{7},$$

which is only a little less than $\sqrt{8}$. Thus "expanding" the closest lattice-packing in the direction of the vector under consideration in the ratio $\sqrt{8}$: $\sqrt{7}$, we obtain a lattice-packing of density

$$\frac{\pi}{\sqrt{12}}\sqrt{\frac{7}{8}} = \frac{\pi}{4}\sqrt{\frac{7}{6}} > \frac{\pi}{4};$$

this gives rise to a packing with the same density on a cylinder of perimeter $4\sqrt{2}$.

We must still show that on a cylinder of perimeter $2\sqrt{2}$ the density of an arbitrary packing of unit circles is at most $\pi/4$. This follows from the fact that the level-difference between the centers of two consecutive circles is at least $\sqrt{2}$. Therefore the area of the portion of the cylinder determined by two such centers is at least $2\sqrt{2} \cdot \sqrt{2} = 4$, and the circle-density in such a portion is at most $\pi/4$. It follows that the circle-density on the whole cylinder is also at most $\pi/4$.

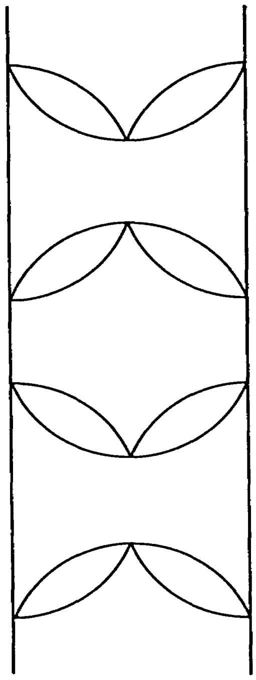

Figure 6.

Proof of Theorem 2. If $p < 2$, the covering displayed in Figure 6 has a density less than $\pi/2$. Thus we can suppose that $p \geq 2$.

We start with the case

$$n \leq p/2 \leq 9n/5,$$

where n is a positive integer. Again we consider cycles each consisting of n equally spaced circles, and we cover the cylinder with them in such a way that the circles of two neighboring cycles intersect at the two center-lines of the cycles. This covering is generated by a rhombus with diagonals of lengths

$$2x = p/n \quad \text{and} \quad 2y = 2\sqrt{1 - (1 - x)^2} = 2\sqrt{2x - x^2}$$

(Figure 7). Thus the area A of the rhombus is given by

$$A = 2xy.$$

We consider the function

$$f(x) = A^2/4 = 2x^3 - x^4,$$

and observe that $f(1) = 1$, $f(9/5) = 9^3/5^4 > 1$, and $f''(x) = 12x(1 - x) < 0$ for $x > 1$. It follows that for $1 \leq x \leq 9/5$, $f(x) \geq 1$; that is, $A \geq 2$, with equality only for $x = 1$. Consequently, the covering-density π/A is at most $\pi/2$, and equality holds only if $p/2 = n$. Since $n - 1 < n < 9(n - 1)/5$ for $n \geq 3$, the only values of $p/2 > 1$ not covered by the intervals $(n, 9n/5]$ are those lying in $(9/5, 2]$. Thus, we need only consider the case $9/5 < p/2 \leq 2$.

Note that if $x = 9/5$, then $y = 3/5$. Therefore in the case

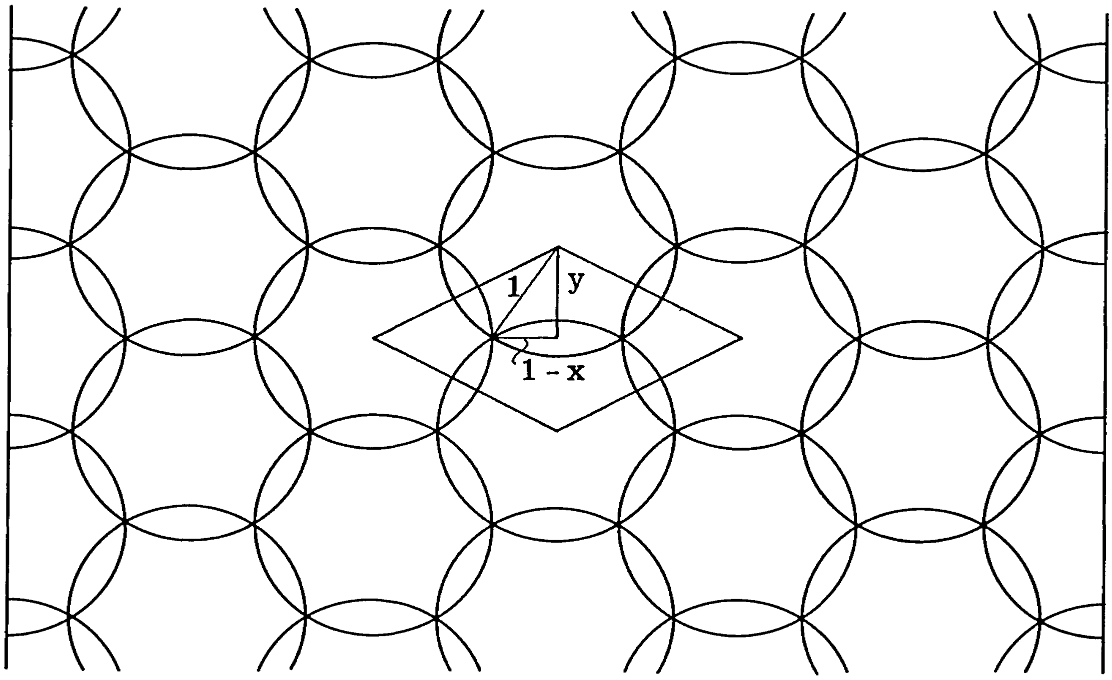

Figure 7.

$$3n/5 \leq p/2 < n \qquad (n = 2, 3, \cdots)$$

the rhombus obtained from that above by interchanging the role of the horizontal and vertical diagonals generates a covering with a density less than $\pi/2$. Choosing $n = 3$, we have settled the problem for all values of p different from 2.

We order the unit circles covering a cylinder of perimeter 2 according to the height of their centers. Let h_{-1}, h_0, h_1 be the heights of three consecutive circles c_{-1}, c_0, c_1. Obviously, $h_1 - h_{-1} \leq 2$, because otherwise the circles different from c_0 would not cover a cylindrical zone of positive height $h_1 - h_{-1} - 2$. But such a zone cannot be covered by the single circle c_0. Therefore the density of the circles in the zone determined by the centers of c_{-1} and c_1 is at least $2\pi/2 \cdot 2 = \pi/2$. Since the cylinder can be decomposed into such zones, the covering-density on the whole cylinder must be at least $\pi/2$.

Looking back upon these proofs, we see that we have established a little more than is stated in the theorems: On a cylinder there always exist a "lattice-packing" and a "lattice-covering" by unit circles of densities at least $\pi/4$ and at most $\pi/2$, respectively (in the sense that the points obtained from the centers by rolling the cylinder along the plane will form a lattice). On the other hand, A. Beck noticed that for certain values of the perimeter the densest packing of unit circles does not constitute a lattice. An analogous statement seems to hold for the covering.

We conclude with the remark that instead of our definition of a circle on a cylinder we can also consider a circular label stuck on the cylinder. In other words, if the circle overlaps itself, we can consider the multiply covered parts with the corresponding multiplicity, so that the area of the circle does not depend on the perimeter of the cylinder. If we define a packing as an arrangement in which different circles do not overlap, then Theorem 1 continues to hold for such circles, without any change. On the other hand, with this definition of a circle, Theorem 2 is valid only for $p > 2x = 1.08 \cdots$, where x is defined by the conditions $0 < x < 1$, $x^4 - 2x + 1 = 0$. For $p = 2x$ the smallest covering-density is $\pi/2$, and for $p < 2x$ it is greater than $\pi/2$.

REFERENCES

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3. ———, *Regular figures*, Pergamon Press, Oxford, 1964.

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