

A REMARK ON CONTINUED FRACTIONS

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Very little is known about the continued fraction expansion of algebraic numbers other than quadratic irrationals, though Roth's theorem imposes a limitation on the rapidity with which the partial quotients a_n can increase. In particular, it has never been proved that there exists such a number for which the sequence a_n is unbounded, though this is almost certainly true.

The following much easier question was put to me by Dr. Schinzel: given N , however large, do there exist algebraic numbers other than quadratic irrationals which have infinitely many partial quotients greater than N ? The following simple result answers this question, but does so without throwing any light on the deeper problems.

Let θ be any irrational number, and let P be any large prime. Then one at least of the numbers

$$P^2\theta, \theta, \theta + 1/P, \dots, \theta + (P - 1)/P$$

has $a_n > P - 2$ for infinitely many n .

To prove this, we consider the convergents p_n/q_n to the irrational number $P\theta$. They satisfy the inequality

$$\left| P\theta - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}.$$

If q_n is divisible by P for infinitely many n , we put $q_n = Pq'_n$ and get infinitely many rational approximations p_n/q'_n to $P^2\theta$ satisfying the inequality

$$\left| P^2\theta - \frac{p_n}{q'_n} \right| < \frac{1}{P(q'_n)^2}.$$

This implies that infinitely many of the partial quotients in the continued fraction for $P^2\theta$ are greater than $P - 2$. (This is a consequence of well-known properties of continued fractions; see for example Theorems 184 and 163 in Hardy and Wright's *Introduction to the theory of numbers*.)

In the contrary case, q_n is relatively prime to P for all sufficiently large n . Thus we can determine an integer A_n such that

$$p_n \equiv A_n q_n \pmod{P}, \quad 0 \leq A_n < P.$$

One at least of the integers $0, 1, \dots, P - 1$ must occur infinitely often as A_n . If A is such an integer, we can write

$$p_n = Aq_n + Pr_n$$

for the corresponding infinite set of values n , where r_n is an integer. Now

$$\left| \theta - \frac{A}{P} - \frac{r_n}{q_n} \right| < \frac{1}{Pq_n^2}$$

for infinitely many n , and again this implies that $\theta - A/P$ has infinitely many partial quotients greater than $P - 2$. This proves the result.

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