A MAXIMAL SET WHICH IS NOT COMPLETE

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The definition of maximal set is due to John Myhill. A recursively enumerable set M is called *maximal* if the complement of M is infinite and if for each recursively enumerable set S, either M contains all but finitely many members of S or M contains all but finitely many members of the complement of S. Friedberg [2] was the first to construct a maximal set. A recursively enumerable set K is said to be *complete* if every recursively enumerable set is recursive in K. Post [3] originated the notion of completeness and constructed a complete set. It has been conjectured that every maximal set is complete. This conjecture is strongly false, since we show there exist infinitely many degrees of unsolvability which are degrees of maximal sets. Our main result is: *a set is recursive if and only if it is recursive in every maximal set.*

Our argument is a marriage of Friedberg's construction of a maximal set with the combinatorial principle we introduced in [4]. It is possible to regard Friedberg's construction as a convergent sequence of Post-like constructions of the type found in [3]. Similarly, our construction can be viewed as a convergent sequence of Friedbergian constructions. The convergence is made possible by certain concealed, effective properties of Friedberg's construction. For example, at stage s of Friedberg's procedure, several members of the apparent complement of R_e, the e-th recursively enumerable set, are put into M. Then it is shown that eventually all but finitely many members of the complement of R_e are put into M or all but finitely many members of R_e are put into M. Let S_e be the set of all members of the apparent complement of R_e which are put into M at any stage s. Then S_e is the cumulative effect of Friedberg's procedure applied to R_e. It can be shown that S_e is recursive.

Let C be a nonrecursive set which is recursive in a complete set. Our formal argument is devoted to the construction of a maximal set M such that C is not recursive in M. We preface the formal argument with an intuitive sketch of its key ideas. The requirements we must meet are:

F_e: C is not recursive in M with Gödel number e;

G_e: M contains all but finitely many members of R_e or M contains all but finitely many members of the complement of R_e;

H: the complement of M is infinite.

At stage s we attack G_e for all $e \leq s$. For each $e \leq s$, we put finitely many members of the apparent complement of R_e into M, in order to meet G_e. We exercise restraint in order to satisfy requirements H and F_e for each $e \leq s$. Occasionally we are faced with a conflict between requirements F_i and G_j; if $i \leq j$, we favor F_i; if $i > j$, we favor G_j. Thus if we wish to add members to M to meet requirement G_e, we are unconcerned with possible effects on F_i for any $i > e$, but we may be deterred by the possible effects on F_i for some $i \leq e$.

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Let us consider the interaction between requirements F_0, G_0 and F_1; we ignore H, since it is met by a procedure similar to that of Friedberg. In order to meet F_0, we must take steps to ensure that C is not recursive in M with Gödel number 0. We want

$$c(n) = \{0\}^M(n)$$

to be false or undefined for some n, where c(n) is the representing function of C. Since C is recursive in a complete set, there exists a recursive function c(s, n) (defined below) such that $\lim_s c(s, n)$ exists and is equal to c(n) for all n. In order to meet requirement F_0 at stage s, we compare c(s, n), a recursive approximation of the representing function of C, with U(y(s, n, 0)), a recursive approximation of the partial function $\{0\}^M(n)$. The function m(s, 0) is the length of the longest "acceptable" initial segment of the natural numbers on which c(s, n) and U(y(s, n, 0)) agree. (The definition of "acceptable" is a minor technical matter which we save for the formal argument.) We now consider adding several members to M in order to meet requirement G_0, but we are not permitted to add any members capable of destroying the equality of c(s, n) and U(y(s, n, 0)) for any $n < m(s, 0)$; we are dedicated to preserving the latter equality. Requirement F_1 is treated like F_0 save for the important difference that the equality associated with F_1 is vulnerable to change as a result of an attempt to meet G_0.

The proof that F_0 is eventually met depends only on the fact that C is not recursive. Suppose F_0 is not met. Then $c(n) = \{0\}^M(n)$ for all n. It follows that for each n, $\lim_s c(s, n)$ exists, $\lim_s U(y(s, n, 0))$ exists, and

$$\lim_s c(s, n) = \lim_s U(y(s, n, 0)).$$

Fix n. There must be a t such that $c(t, n) = U(y(t, n, 0))$ and $n < m(t, 0)$; but U(y(t, n, 0)) is not subject to change at any stage following stage t, since F_0 has the highest priority of any requirement. But then we have an effective method for deciding whether or not $n \in C$, since y is recursive. The proof that F_1 is eventually met is similar. The only difference lies in the fact that U(y(s, n, 1)) is vulnerable to attempts to meet G_0. Fortunately, we are able to show that S(0), the set of all numbers added to M for the sake of G_0, is recursive. It follows that when the situation demands it, we can determine $\lim_s U(y(s, n, 1))$ in an effective manner. The recursiveness of S(0) is a consequence of the fact that F_0 is eventually met.

We attempt to meet G_0 by adding members to M from time to time in a fashion similar to Friedberg's [2]. The only difficulty we face consists of our inability to add any number less than m(s, 0) to M at stage s. We know that F_0 is met. Thus there must be an n such that $\lim_s U(y(s, n, 0))$ does not exist or

$$\lim_s c(s, n) \neq \lim_s U(y(s, n, 0)).$$

In either event the set $\{m(s, 0) \mid s \geq 0\}$ is bounded. But then any sufficently large number can be put into M without harming F_0.

Unfortunately, we need an additional argument to show G_1 is met. We know F_0 and F_1 are met. We cannot add any number less than m(s, 0) to M at stage s because of the need to protect F_0. This difficulty, as we have just seen in the case of G_0, eventually vanishes. We cannot add any number less than m(s, 1) to M at stage s, if the purpose of adding that number is to meet G_1. F_0 and F_1 are the only obstacles confronting us whenever we wish to augment M for the sake of G_1. Although F_0 eventually vanishes, F_1 may not. Suppose we augment M infinitely often for the sake of meeting G_0; each such augmentation may destroy an equality associated with F_1. In other words, it may happen infinitely often that for the sake of meeting G_0, we add a number to M at stage s which is less than m(s, 1). It then may not follow that the set $\{m(s, 1) \mid s \geq 0\}$ is bounded. But all is not lost.

We are saved by Lemma 4 below. It follows from Lemma 4 that there exists an n such that (a) or (b) holds:

(a) $\{1\}^M(n)$ is undefined;

(b) $m(s, 1) \leq n$ for infinitely many s.

With the help of either (a) or (b) we can show there are infinitely many opportunities for augmenting M in order to meet G_1. Note that finitely many opportunities for augmenting M for the sake of meeting G_1 may not be enough. The difference between G_0 and G_1 is important: the obstacle to meeting G_0 created by F_0 eventually vanishes completely; the obstacle to meeting G_1 created by F_1 may never vanish completely, but it at least vanishes for a moment infinitely often. Requirement G_e, for each $e > 1$, has the same general character as G_1.

The main combinatorial difference between Friedberg [2] and the present paper may be expressed as follows. In [2] a requirement R is confronted by an obstacle finitely often; eventually a stage is reached when all obstacles to requirement R vanish forever. In the present paper, requirement R may be confronted by an obstacle infinitely often; nonetheless, we can show R is not confronted by any obstacle at infinitely many stages of the construction. It is somewhat surprising that such a severely limited ability to meet requirements is sufficient.

We conclude our paper with a conjecture concerning maximal sets.

THEOREM 1. *A set is recursive if and only if it is recursive in every maximal set.*

Proof. Let C be a nonrecursive set. We define a maximal set M such that C is not recursive in M. Let K be a complete, recursively enumerable set. If C is not recursive in K, then any maximal set will serve for M. Suppose C is recursive in K. Let c, k respectively, be the representing function of C, K respectively; let e be a Gödel number such that $c(n) = \{e\}^k(n)$ for all n. Let g be a recursive function whose range is K. We define two recursive functions:

$$k(s, n) = \begin{cases} 0 \text{ if } (Et)_{t<s}\,(g(t) = n)\,, \\ 1 \text{ otherwise}\,, \end{cases}$$

$$c(s, n) = 1 + U\left(\mu y_{y<s}\, T_1^1\Big(\prod_{i<y} p_i^{k(s,i)},\, e,\, n,\, y\Big)\right).$$

Then for each n, $\lim_s k(s, n)$ exists and equals k(n), and $\lim_s c(s, n)$ exists and equals $1 + c(n)$. For each e and s, we define a finite set R_e^s as follows:

$$n \in R_e^s \leftrightarrow (Ey)_{y<s}\, T_1(e, n, y)\,.$$

If W is a recursively enumerable set, then there exists an e such that

$$W = \bigcup \{R_e^s \mid s \geq 0\}\,.$$

We define seven recursive functions $y(s, n, e)$, $m(s, e)$, $z(s, n, e)$, $h(s, n, e)$, $r(s, n, e)$, $L(s, n, e)$, and $M(s, n)$ simultaneously by induction on s.

Stage $s = 0$. We set $y(0, n, e) = m(0, e) = z(0, n, e) = 0$ and

$$h(0, n, e) = r(0, n, e) = L(0, n, e) = M(0, n) = 1$$

for all n and e.

Stage $s > 0$. We define

$$y(s, n, e) = \begin{cases} \mu y T_1^1\left(\prod_{i<y} p_i^{M(s-1,i)}, e, n, y\right) & \text{if there exists such a } y < s, \\ 0 & \text{otherwise}. \end{cases}$$

The definition of $m(s, e)$ has two cases.

Case m1. There is a $t < m(s - 1, e)$ such that

$$y(s, t, e) \neq y(s - 1, t, e) \ \& \ c(s, t) \neq U(y(s, t, e)).$$

Let $m(s, e)$ be the least such t.

Case m2. Otherwise, let $m(s, e)$ be the least t such that

$$m(s - 1, e) \leq t < 2m(s - 1, e) + s \ \& \ (En)_{n \leq t}(c(s, n) \neq U(y(s, n, e))).$$

Note that the least number operator in Case m2 is bounded.

For each n such that $M(s - 1, n) = 1$, let $z(s, n, e)$ be the sum of the members of

$$\{2^{e-f} \mid f \leq e \ \& \ n \in R_f^s\} \cup \{0\};$$

if $M(s - 1, n) \neq 1$, let $z(s, n, e) = 2^{e+1}$. Note that $z(s, n, e) < 2^{e+1}$, unless $M(s - 1, n) \neq 1$. For each n, let $h(s, n, e)$ be the cardinality of

$$\{m \mid m < n \ \& \ z(s, m, e) = z(s, n, e)\}.$$

We define $r(s, n, e)$ and $L(s, n, e)$ for all n and e by means of a simultaneous induction on e:

$$r(s, n, e) = \begin{cases} 0 & \text{if } (Ei)_{i<e}(Et)_{t \leq n}(Em)(m < y(s, t, e) \ \& \ L(s, m, i) \neq M(s - 1, m)), \\ 1 & \text{otherwise}; \end{cases}$$

$$L(s, n, e) = \begin{cases} M(s - 1, n) & \text{if } (Ei)_{i \leq e}(Et)(t < m(s, i) \ \& \ r(s, t, i) = 1 \ \& \ n < y(s, t, i)), \\ M(s - 1, n) & \text{if } h(s, n, e) < e \vee n \in R_e^s, \\ M(s - 1, n) & \text{if } (t)_{s \geq t > n}(z(s, t, e) \neq 1 + z(s, n, e) \vee z(s, t, e) = 2^{e+1}), \\ 0 & \text{otherwise}. \end{cases}$$

Finally, we set $M(s, n) = 0$ if and only if $(Ee)_{e \leq s}(L(s, n, e) = 0)$.

Since all quantifiers and least number operators occurring above are properly bounded, and since c(s, n) is a recursive function, it is easy to verify that M(s, n) is a recursive function. Observe that

$$1 \geq M(s, n) \geq M(s + 1, n) \geq 0$$

for all s and n. Thus for each n, $\lim_s M(s, n)$ exists and is at most 1; let it equal M*(n). Let M be the recursively enumerable set whose representing function is M*. Then

$$n \in M \leftrightarrow (Es)(M(s, n) = 0).$$

We need two notes:

(N1) $(s)(n)(e)(r(s, n, e) = 0 \rightarrow r(s, n + 1, e) = 0)$;

(N2) $(s)(n)(e)(n < m(s, e) \rightarrow y(s, n, e) > 0)$.

Note (N1) is an immediate consequence of the definition of r. Note (N2) is proved by induction on s. If s = 0, then m(s, e) = 0. Suppose s > 0 and

$$(n)(e)(n < m(s - 1, e) \rightarrow y(s - 1, n, e) > 0).$$

Fix n and e, and suppose n < m(s, e). If case m2 applies to the definition of m(s, e), then y(s, n, e) = y(s - 1, n, e) and n < m(s - 1, e), or

$$0 < c(s, n) = U(y(s, n, e));$$

and consequently, y(s, n, e) > 0, since U(0) = 0. Suppose Case m1 applies. Then n < m(s - 1, e) and y(s - 1, n, e) > 0. Since n < m(s, e), either

$$y(s, n, e) = y(s - 1, n, e)$$

or c(s, n) = U(y(s, n, e)).

We adopt some of the terminology of Friedberg [2]. If $z(s, n, e) = i < 2^{e+1}$, then we say n is in the i-th e-state at stage s. If $z(s, n, e) = 2^{e+1}$, then n has no e-state at stage s. Note that n has no e-state at stage s if and only if M(s - 1, n) = 0. Two useful notes concerning e-states are:

(N3) if s < t and n is in some e-state at stage s and at stage t, then the e-state of n at stage s is less than or equal to the e-state of n at stage t;

(N4) if m and n are in different e-states at stage s, then there exists a unique $c \leq e$ such that the c-states of m and n at stage s differ by one, and such that the lower c-state is even.

We say n is an inhabitant of the i-th e-state if for some s, n is in the i-th e-state at stage s. We say n is a resident of the i-th e-state if there is a t such that for all $s \geq t$, n is in the i-th e-state at stage s. We say the i-th e-state is well-inhabited, well-resided respectively, if it has infinitely many inhabitants, residents respectively. Observe that for each e, the 0-th e-state is well-inhabited, since z(0, n, e) = 0 for all n and e.

LEMMA 1. *The highest well-inhabited* e-*state has at least* e *residents.*

Proof. Let y exceed any number which is an inhabitant of an e-state higher than the i-th, where the i-th e-state is the highest well-inhabited e-state. Let w be so large that

$$(n)_{n<y}(Es)_{s<w}(n \in M \rightarrow M(s, n) = 0).$$

In addition, let w be so large that no number passes from the i-th e-state to a higher e-state at stage s for any $s \geq w$. Let J be the set of all n such that n is in the i-th e-state at stage s for some $s \geq w$. Let k be one of the e smallest members of J. We show k is a resident of the i-th e-state. Suppose it is not. Then $L(s, k, c) = 0$, $M(s, k) = 0$, and $M(s - 1, k) = 1$ for some s and c, where $s \geq w$. First suppose $c \geq e$. Then the definition of $L(s, k, c)$ tells us that $h(s, k, c) \geq c$. This means the set

$$\{m \mid m < k \ \& \ z(s, m, c) = z(s, k, c)\}$$

has at least c members. But then the set

$$\{m \mid m < k \ \& \ z(s, m, e) = z(s, k, e) = i\}$$

has at least c members, since $c \geq e$. This last contradicts the definition of k, since $e \leq c$ and $s \geq w$. Now suppose $c < e$. Then the definition of $L(s, k, c)$ tells us $2^{c+1} > z(s, v, c) = 1 + z(s, k, c)$ for some $v > k$. But then

$$z(s, v, e) > z(s, k, e) = i,$$

since $c < e$ and since the definition of z is based on the fact that

$$1 + 2 + 2^2 + \cdots + 2^n < 2^{n+1}.$$

Then $v \leq y$, since v is in an e-state higher than the i-th at stage s. Now $k < v \leq y$, $k \in M$, $M(s - 1, k) = 1$ and $s \geq w$. This last contradicts the definition of w.

It follows from Lemma 1 that the complement of M is infinite, since a resident of an e-state cannot be a member of M. The proof of Lemma 1 is close to the one given by Friedberg [2].

For each e, we say e is stable if $\lim_s y(s, n, e)$ exists and is positive for all n. If e is not the Gödel number of a system of equations, then e is not stable. Thus there are infinitely many nonstable e. Let $\{e_0 < e_1 < e_2 < \cdots\}$ be the set of all nonstable e. For each j, let n_j be the least n such that $\lim_s y(s, n, e_j)$ does not exist or does equal 0. If

$$i = \sum c_k 2^k \text{ and } j = \sum d_k 2^k,$$

where $0 \leq c_k \leq d_k \leq 1$, then we say j extends i. If $i < 2^{e+1}$ and $i + 1$ extends i, then i is even and $i < 2^{e+1} - 1$. We introduce two predicates.

A(e): if e is stable, then the set $\{m(s, e) \mid s \geq 0\}$ is finite.

B(e): if the i-th e-state is well-resided and $i + 1$ extends i, then the $(i + 1)$-th e-state is not well-inhabited.

It will follow from (e)A(e) that C is not recursive in M. It will follow from (e)B(e) that M is maximal. We prove (e)A(e) and (e)B(e) by means of a simultaneous induction on e.

LEMMA 2. *If* $n < m(s, e)$ *and* $r(s, n, e) = 1$, *then* $y(s, n, e) = y(s + 1, n, e)$ *and* $n < m(s + 1, e)$.

Proof. Since $n < m(s, e)$, we have $y(s, n, e) > 0$ by Note (N2). But then

$$y(s, n, e) = \mu y T_1^1\left(\prod_{j<y} p_j^{M(s-1,j)},\ e, n, y\right).$$

Since $r(s, n, e) = 1$, the definition of r tells us that $L(s, j, i) = M(s - 1, j)$ when $j < y(s, n, e)$ and $i < e$. Since $n < m(s, e)$ and $r(s, n, e) = 1$, the definition of L tells us that $L(s, j, i) = M(s - 1, j)$ when $j < y(s, n, e)$ and $i \geq e$. Thus $M(s, j) = M(s - 1, j)$ for all $j < y(s, n, e)$. But then $y(s, n, e) = y(s + 1, n, e)$.

Notes (N1) and (N2) make clear that the above argument works equally well for each $t < n$. Thus

$$(t)_{t \leq n}(y(s, t, e) = y(s + 1, t, e)).$$

Suppose $m(s + 1, e) \leq n$. Then $m(s + 1, e) < m(s, e)$, and Case m1 of the definition of $m(s + 1, e)$ holds. This means there is a $t \leq n$ (namely, $m(s + 1, e)$) such that $y(s, t, e) \neq y(s + 1, t, e)$.

LEMMA 3. $(i)_{i<e} B(i) \rightarrow A(e)$.

Proof. We need another predicate.

D(i): if the j-th i-state is well-resided, k extends j, and $k < 2^{i+1}$, then the k-th i-state is not well-inhabited.

Fix $i < e$. We prove D(i) with the help of $(i)_{i<e} B(i)$. Suppose D(i) is false. Let j and k be such that the j-th i-state is well-resided, k extends j, $k < 2^{i+1}$, and the k-th i-state is well-inhabited. By Note (N4) and the fact that k extends j, there must be a $c \leq i$ such that any number in the j-th e-state is in the j'-th c-state, any number in the k-th e-state is in the (j' + 1)-th c-state, and j' + 1 extends j. But then the j'-th c-state is well-resided and the (j' + 1)-th c-state is well-inhabited. These last remarks contradict B(c).

Let S(i) denote

$$\{n \mid (Es)_{s \geq i}(L(s, n, i) = 0 \ \&\ M(s \dot{-} 1, n) = 1)\}.$$

Fix $i < e$. We use D(i) to show S(i) is recursive. Since the complement of M is infinite, there exists at least one i-state which is well-resided. We claim there is at most one well-resided i-state. Suppose the k-th i-state and the j-th i-state are both well-resided. By note (N4), there is a $c \leq i$ such that any number in the j-th i-state is in the j'-th c-state, any number in the k-th i-state is in the k'-th c-state, j' and k' differ by 1, and the lower c-state, let us say j', is even. Then j' + 1 extends j', the j'-th c-state is well-resided, and the (j' + 1)-th c-state is well-inhabited. This last contradicts B(c).

Let the j-th i-state be the unique well-resided i-state. Let m be larger than any member of

$$\{n \mid k < 2^{i+1} \ \&\ k \text{ extends } j \ \&\ (Es)(z(s, n, i) = k)\}$$

or

$$\{n \mid n \text{ is a resident of the } k\text{-th } i\text{-state for some } k \neq j\};$$

the existence of m follows from D(i) and the definition of j. Fix $n \geq m$. We give an intuitive, effective procedure for deciding whether or not n is a member of S(i). We fix $w > 0$, and we consider the status of n at stage w. If $M(w - 1, n) = 0$, then we can tell whether $n \in S(i)$ by examining all stages of our construction prior to stage w. Suppose $M(w - 1, n) = 1$. Suppose n is in the k-th i-state at stage w. First let $k \neq j$. Since $n \geq m$, n is not a resident of the k-th i-state. But then there must be an $s \geq w$ such that $M(s, n) = 0$ or such that the i-state of n at stage s extends k. If $M(s, n) = 0$, then we can tell whether $n \in S(i)$ by examining all stages prior to stage s. Now let $k = j$. Since $n \geq m$, n cannot be in an i-state higher than the j-th at any stage $s \geq w$. Thus either n is a resident of the j-th i-state or $M(s, n) = 0$ and $z(s, n, i) = j$ for some $s \geq w$. Suppose there is an s with the latter property. Then $L(s, n, c) = 0$ & $M(s - 1, n) = 1$ for some c. We claim $c \neq i$. Suppose $c = i$. Then the definition of L provides us with a t such that

$$t > n, \quad z(s, t, i) < 2^{i+1}, \quad z(s, t, i) = 1 + z(s, n, i), \quad n \notin R_i^s.$$

Then j is even, since $z(s, n, i) = j$ and $n \notin R_i^s$. Let $j' = z(s, t, i)$. We see that $j' < 2^{i+1}$, j' extends j, and $t > n \geq m$. This last is contrary to the definition of m. Consequently, $c \neq i$ and $n \notin S(i)$.

We recapitulate. At stage $s = 0$, n is in the 0-th i-state. As we pass from one stage to the next, n is put into M, or the i-state of n is extended, or the i-state of n remains the same. If n lands in an i-state other than the j-th, then n must eventually move on to M or to a higher i-state. If n lands in the j-th i-state, then $n \notin S(i)$. If n is put into M, it immediately becomes clear whether or not $n \in S(i)$. Since there are only $2^{i+1} - 1$ i-states, it eventually becomes clear whether or not $n \in S(i)$.

We suppose A(e) is false and show that, contrary to hypothesis, C is recursive. Thus e is stable, and the set $\{m(s, e) \mid s \geq 0\}$ is infinite. Let R(n, s) denote the predicate

$$(i)(m)(t)(m < y(s, t, e) \ \& \ t \leq n \ \& \ i < e \rightarrow m \notin S(i) \vee M(s - 1, m) = 0)$$
$$\& \ n < m(s, e) \ \& \ s > e.$$

Since S(i) is recursive for each $i < e$, R must be recursive. Since the set $\{m(s, e) \mid s \geq 0\}$ is infinite, and since $\lim_s y(s, n, e)$ exists and is positive for each n, it follows that $(n)(Es)R(n, s)$. Let $w(n) = \mu s R(n, s)$. Fix n. We show that $\lim_s y(s, n, e) = y(w(n), n, e)$. Let $s \geq w(n)$ be such that

$$y(w(n), n, e) = y(s, n, e) \ \& \ R(n, s).$$

Since R(n, s) holds, it must be that $n < m(s, e)$ and $r(s, n, e) = 1$. It follows from Lemma 2 and Note (N1) that

$$y(s + 1, n, e) = y(s, n, e) \ \& \ R(n, s + 1).$$

Finally we show that

$$1 + c(n) = \lim_s c(s, n) = U(y(w(n), n, e)).$$

If this last is true, then C is recursive, since w is recursive.

Fix n, and suppose $\lim_s c(s, n) \neq U(y(w(n), n, e))$. Let s^* be so large that

$$c(s, n) = \lim_s c(s, n) \neq U(y(w(n), n, e) = U(y(s, n, e))$$

for all $s \geq s^*$. Fix $s > s^*$, and suppose $m(s - 1, e) \leq m(s^*, e) + n$. If Case m1 of the definition of $m(s, e)$ applies, then $m(s, e) < m(s^*, e) + n$. Suppose Case m2 applies. If $n < 2m(s - 1, e) + s$, then $m(s, e) = m(s - 1, e)$ or $m(s, e) \leq n$; if $n \geq 2m(s - 1, e) + s$, then $m(s, e) \leq n$. Thus $m(s, e) \leq m(s^*, e) + n$ for all $s \geq s^*$. This last is impossible since we have assumed that the set $\{m(s, e) \mid s \geq 0\}$ is infinite.

LEMMA 4. *For each* k *and* v *there is an* $s \geq v$ *such that*

$$(j)_{j<k}(r(s, n_j, e_j) = 0 \vee y(s, n_j, e_j) = 0 \vee m(s, e_j) \leq n_j).$$

Proof. Fix k and v. Suppose there is no s with the desired properties. We define an infinite, descending sequence of natural numbers.

We propose the following system of equations as a definition by induction of two functions, S and Q.

$$S(0) = v;$$

$$Q(t) = \mu j_{j<k}(r(S(t), n_j, e_j) = 1 \;\&\; y(S(t), n_j, e_j) > 0 \;\&\; n_j < m(S(t), e_j));$$

$$S(t + 1) = \mu s(Em)(s \geq S(t) \;\&\; m < y(S(t), n_{Q(t)}, e_{Q(t)}) \;\&\; M(s, m) \neq M(S(t) - 1, m)).$$

Clearly $S(0) \geq v$. Suppose $t \geq 0$, $S(t)$ is well-defined, and $S(t) \geq v$. Then $Q(t) < k$, since we have supposed the lemma to be false. Thus

$$y(S(t), n_{Q(t)}, e_{Q(t)}) > 0,$$

and $\lim_s y(s, n_{Q(t)}, e_{Q(t)})$ does not exist or does equal 0. Consequently, there must be an $s > S(t)$ such that

$$0 \leq y(s, n_{Q(t)}, e_{Q(t)}) \neq y(S(t), n_{Q(t)}, e_{Q(t)});$$

this last can happen only if there is an m such that

$$m < y(S(t), n_{Q(t)}, e_{Q(t)}) \;\&\; M(s - 1, m) \neq M(S(t) - 1, m).$$

But then $S(t + 1)$ is well-defined and $S(t + 1) \geq v$.

For each $t \geq 0$, let

$$u(t + 1) = \mu m(M(S(t + 1), m) \neq M(S(t) - 1, m)).$$

Fix $t > 0$. We show $u(t + 1) < u(t)$. Since we know that

$$u(t + 1) < y(S(t), n_{Q(t)}, e_{Q(t)}),$$

it suffices to show that

$$y(S(t), n_{Q(t)}, e_{Q(t)}) \leq u(t).$$

It follows from the definition of S that

$$M(w, m) = M(S(t-1)-1, m)$$

whenever $S(t) > w \geq S(t-1)$ and $m < y(S(t-1), n_{Q(t-1)}, e_{Q(t-1)})$. Consequently,

$$M(S(t), u(t)) \neq M(S(t)-1, u(t)).$$

There must be an i such that $L(S(t), u(t), i) = 0 \neq M(S(t)-1, u(t))$. Let

$$s = S(t), \quad n = n_{Q(t)}, \quad e = e_{Q(t)}.$$

Suppose $e \leq i$. Then $r(s, n, e) = 1$ and $n < m(s, e)$, since $Q(t) < k$. But then the definition of L tells us that $y(s, n, e) \leq u(t)$, since $L(s, u(t), i) \neq M(s-1, u(t)) = 1$. Now suppose $i < e$. Then again $r(s, n, e) = 1$ and $n < m(s, e)$. But then the definition of $r(s, n, e)$ tells us that $y(s, n, e) \leq u(t)$.

LEMMA 5. $(i)_{i \leq e} A(i)$ & $(i)_{i < e} B(i) \to B(e)$.

Proof (by reductio ad absurdum). Suppose the i-th e-state is well-resided, $i+1$ extends i, and the $(i+1)$-th e-state is well-inhabited. We claim there is a z such that

(a) if $c < e$, the j-th c-state is well-resided, k extends j, $k < 2^{c+1}$, and n is an inhabitant of the k-th c-state, then $n < z$, and

(b) if n is one of the e smallest residents of the i-th e-state, then $n < z$.

In the proof of Lemma 3, we observed that $(i)_{i<e} D(i)$ follows from $(i)_{i<e} B(i)$; but then z exists. Let w be so large that every one of the e smallest residents of the i-th e-state is in the i-th e-state at stage w. Let w have the additional property that some number greater than z is in the $(i+1)$-th e-state at stage w. Since R_c^s is finite for each c and s, and since $i+1 > 0$, it follows that for each s, the $(i+1)$-th e-state is empty or has a greatest member. We define a partial function:

$$x(s) \simeq \text{greatest member of } (i+1)\text{-th e-state at stage } s.$$

We claim that $x(s)$ is defined for each $s \geq w$ and that $x(s) \geq x(s-1)$ for each $s > w$. Clearly $x(w)$ is defined, and $x(w) \geq z$. Fix $s \geq w$. Suppose $x(s)$ is defined and $x(s) \geq z$. To show that $x(s+1)$ is defined and $x(s+1) \geq x(s)$, we consider three alternatives:

(1) at stage $s+1$, the e-state of $x(s)$ is $i+1$;

(2) at stage $s+1$, the e-state of $x(s)$ is greater than $i+1$;

(3) at stage s, $x(s)$ is put in M.

If (1) holds, then $x(s+1)$ is defined and $x(s+1) \geq x(s)$. Suppose (2) holds. Note that $i+1$ is odd, since $i+1$ extends i. Then $x(s) \in R_e^s$, since $z(s, x(s), e) = i+1$. It follows that $x(s) \in R_c^{s+1} - R_c^s$ for some $c < e$, since (2) holds. The c-state of $x(s)$ at stage s is well-resided, because $c < e$ and the i-th e-state is well-resided, and because $z(s, x(s), e) = i+1$. Let the c-state of $x(s)$ at stage s be j and at stage $s+1$ be k. Then k extends j, and by (a), $x(s) < z$. But, fortunately, we supposed $x(s) \geq z$, so (2) cannot hold. Suppose (3) holds. Then

$$L(s, x(s), d) \neq M(s-1, x(s)) = 1$$

for some d. The definition of L tells us that

$$2^{d+1} > z(s, t, d) = 1 + z(s, x(s), d) \ \& \ x(s) \notin R_d^s$$

for some $t > x(s)$. First suppose $d > e$. Then

$$z(s, t, e) = z(s, x(s), e) \ \& \ t > x(s).$$

This last contradicts the definition of $x(s)$. Now suppose $d = e$. Then $x(s) \notin R_e^s$, and so $z(s, x(s), e)$ is even. But $z(s, x(s), e) = i + 1$ is odd, since $i + 1$ extends i. Finally, suppose $d < e$. Let $z(s, x(s), d) = j$ and $j + 1 = k$. The j-th d-state is well-resided, since $d < e$ and since the i-th e-state is well-resided. In addition, k extends j, since $x(s) \notin R_d^s$. We know that t is an inhabitant of the k-th d-state. By (a), $t < z \leq x(s)$. But $t > x(s)$.

Thus $x(s)$ is defined for all $s \geq w$, and $x(s + 1) \geq x(s)$ for all $s \geq w$. We use the function x to obtain the desired absurdity. If $j \leq e$ and j is stable, let $m(j)$ be the greatest member of $\{m(s, j) \mid s \geq 0\}$; $m(j)$ exists by $A(j)$. If $j \leq e$ and j is not stable, let $m(j)$ be n_k, where $j = e_k$. Recall that n_k is the least witness to the fact that e_k is not stable. If $j \leq e$ and $t < m(j)$, then $\lim_s y(s, t, j)$ exists and is positive. Let y be such that

$$(j)_{j \leq e} (t)_{t < m(j)} (y(s, t, j) \leq y).$$

Let $w^* \geq w$ be so large that

$$(m)_{m < y} (s)_{s \geq w^*} (M(s - 1, m) = \lim_s M(s, m)).$$

Let n be a resident of the i-th e-state such that $n \geq z$ and $n \geq y$. It follows from the definitions of n and z that $h(s, n, e) \geq e$ for all sufficiently large s. We know i is even, so $n \notin R_e^s$ for any s. The function x is unbounded, since it is nondecreasing after stage w, and since the $(i + 1)$-th e-state is well-inhabited. Note that the definition of R_e^s implies that $x(s) \leq s$ when $x(s)$ is defined. But then for all sufficiently large s,

$$s \geq x(s) > n \ \& \ 2^{e+1} > z(s, x(s), e) = 1 + z(s, n, e).$$

Let $v \geq w^*$ be so large that for each $s \geq v$,

$$h(s, n, e) \geq e \ \& \ n \notin R_e^s \ \& \ (Et)_{s \geq t > n} (2^{e+1} > z(s, t, e) = 1 + z(s, n, e)).$$

By Lemma 4, there is an $s \geq v + e$ such that for each $k \leq e$,

$$r(s, n_k, e_k) = 0 \vee y(s, n_k, e_k) = 0 \vee m(s, e_k) \leq n_k.$$

We claim $L(s, n, e) = 0$. Since $s \geq v$, it is enough to show

$$(j)_{j \leq e} (t)(r(s, t, j) = 0 \vee n \geq y(s, t, j) \vee m(s, j) \leq t).$$

Fix $j \leq e$ and $t < m(s, j)$. Suppose j is stable. Then $t < m(j)$, since $m(s, j) \leq m(j)$; and consequently, $y(s, t, j) \leq y \leq n$. Suppose j is not stable. Then $j = e_k$ for some $k \leq e$. If $t < m(j)$, then $y(s, t, j) \leq n$. Suppose $t \geq m(j) = n_k$. Then $m(s, j) > n_k$, and so

$$r(s, n_k, e_k) = 0 \vee y(s, n_k, e_k) = 0.$$

If $r(s, n_k, e_k) = 0$, then $r(s, t, j) = 0$, since $n_k \leq t$ and $j = e_k$. It follows from Note (N2) that $y(s, n_k, e_k) > 0$, since $m(s, e_k) > n_k$. Thus $L(s, n, e) = 0$ and $s \geq e$. Hence $n \in M$. But this last is absurd, because n is a resident of the i-th e-state.

Lemmas 4 and 5 constitute a proof of $(e)A(e)$ and $(e)B(e)$. We use $A(e)$ to show C is not recursive in M with Gödel number e. Suppose

$$1 + c(n) = \{e\}^{M}(n)$$

for all n. Then $\lim_s y(s, n, e)$ exists and is positive for each n. By $A(e)$, the set $\{m(s, e) \mid s \geq 0\}$ is finite; let its greatest member be m. Let s be so large that $s > m$ and

$$c(s, n) = 1 + c(n) = \{e\}^{M}(n) = U(y(s, n, e))$$

for all $n \leq m$. Consider the definition of $m(s, e)$. If Case m1 holds, then $m < m(s, e)$, which is impossible by the definition of m. If Case m2 holds, then $m < m(s, e)$, since $m < s$.

We use $(i)_{i \leq e} B(i)$ to show that M contains all but finitely many members of R_e or M contains all but finitely many members of the complement of R_e. Suppose not. Then there exist two different e-states such that each is well-resided. Then, for some $c \leq e$, there exist two c-states, the j-th and the $(j + 1)$-th, such that the j-th is well-resided, the $(j + 1)$-th is well-inhabited, and $j + 1$ extends j.

THEOREM 2. *There exist infinitely many degrees which are degrees of maximal sets.*

Proof. We proceed exactly as in Theorem 1 save for one detail. Let $C_0, C_1, \cdots, C_m$ be a finite sequence of nonrecursive sets of degrees less than or equal to $0'$. Let K be a complete set. For each $i \leq m$, let e_i be such that

$$\{e_i\}^{k}(n) = c_i(n)$$

for all n, where k, c_i respectively, is the representing function of K, C_i respectively. Define $k(s, n)$ as in Theorem 1. For each $i \leq m$, let

$$c(i, s, n) = 1 + U\Big(\mu y_{y < s}\, T_1^1\Big(\prod_{j < y} p_j^{k(s,j)}, e_i, n, y\Big)\Big) .$$

In the definition of $m(s, e)$ at stage $s > 0$, replace e by $(e)_1$ and $c(s, n)$ by $c(m \dot{-} (e)_0, s, n)$. The argument of Theorem 1 is easily repeated. $A(e)$ is now used to prove that $C_{m \dot{-} (e)_0}$ is not recursive in M with Gödel number e.

It is a result of Dekker [1] that each nonzero, recursively enumerable degree is the degree of a hyper-simple set which is not hyper-hyper-simple. In a forthcoming paper, C. E. M. Yates will show that there exists a complete maximal set. In another forthcoming paper, D. A. Martin will show that there exists a nonzero, recursively enumerable degree which is not the degree of any maximal set. We conclude by wondering if there is any simple way of characterizing those recursively enumerable degrees that are degrees of maximal sets.

REFERENCES

1. J. C. E. Dekker, *A theorem on hypersimple sets*, Proc. Amer. Math. Soc. 5 (1954), 791-796.

2. R. M. Friedberg, *Three theorems on recursive enumeration. I. Decomposition. II. Maximal set. III. Enumeration without duplication*, J. Symb. Logic 23 (1958), 309-316.

3. E. L. Post, *Recursively enumerable sets of positive integers and their decision problems*, Bull. Amer. Math. Soc. 50 (1944), 284-316.

4. G. E. Sacks, *Recursive enumerability and the jump operator*, Trans. Amer. Math. Soc. 108 (1963), 223-239.

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