THE RADIUS OF UNIVALENCE OF CERTAIN ANALYTIC FUNCTIONS

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In two recent papers, MacGregor presented the following results [3, pp. 527-519] and [4, pp. 522-523].

Suppose that $f(z) = z + a_2 z^2 + \cdots$ *and* $g(z) = z + b_2 z^2 \cdots$ *are analytic in the unit disc* $D = \{z \mid |z| < 1\}$.

(a) *If* g *is univalent in* D *and if* $\Re[f(z)/g(z)] > 0$ *in* D, *then* f *is univalent in the disc* $|z| < 1/5$.

(b) *If* g *is univalent and star-like in* D *and if* $\Re[f(z)/g(z)] > 0$ *in* D, *then* f *is univalent and star-like in the disk* $|z| < (2 - \sqrt{3})$.

(c) *If* g *is univalent in* D *and if* $|f(z)/g(z) - 1| < 1$ *in* D, *then* f *is univalent in the disc* $|z| < 1/3$.

(d) *If* g *is univalent and star-like in* D *and if* $|f(z)/g(z) - 1| < 1$ *in* D, *then* f *is univalent and star-like in* D.

As MacGregor points out, the radii in (b), (c) and (d) are the best possible, while that in (a) is not known to be the best possible radius.

In this note we present two theorems that improve the results stated above.

THEOREM 1. *Suppose that* $f(z) = z + a_2 z^2 + \cdots$ *and* $g(z) = z + b_2 z^2 + \cdots$ *are analytic in the unit disc* D, *and suppose that* $\Re[f(z)/g(z)] > 0$ *in* D. *If* g *is univalent in* D, *then* f *is univalent and star-like in the disc* $|z| < (2 - \sqrt{3})$. *This result is sharp.*

Our proof of Theorem 1 depends on the following result, which has independent interest.

LEMMA. *Let* $g(z) = z + b_2 z^2 + \cdots$ *be analytic and univalent in the unit disc* D. *Then the inequality*

$$\Re \frac{zg'(z)}{g(z)} \geq \frac{1 - |z|}{1 + |z|} \tag{1}$$

holds for $|z| < \tanh \frac{1}{2} = 0.46212 \cdots$. *The bound* (1) *is sharp for each* z; *it is attained by a rotation of the Koebe function* $k(z) = z/(1 - z)^2$.

Proof. It is known that for each z in D the complex number $u = \log [zg'(z)/g(z)]$ lies in the closed disc $\overline{K}(\rho)$ whose center is at the origin and whose radius is $\rho = \log [(1 + |z|)/(1 - |z|)]$ [2, p. 113]. Now the function $w = e^u$ maps $\overline{K}(\rho)$ onto a convex region whenever the inequality

$$\Re \left(1 + u\frac{w''(u)}{w'(u)}\right) = \Re(1 + u) \geq 0$$

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holds on the boundary of $\overline{K}(\rho)$. Hence the image $\overline{R}$ of $\overline{K}(\rho)$ under the mapping $w = e^u$ is the closure of a simply covered convex domain in the w-plane, whenever

$$1 - \rho = \log \frac{1 + |z|}{1 - |z|} \geq 0,$$

that is, whenever $|z| \leq \tanh \frac{1}{2}$. A simple calculation now shows that $\overline{R}$ lies in the strip

$$\frac{1 - |z|}{1 + |z|} = e^{-\rho} \leq \Re w \leq e^{\rho} = \frac{1 + |z|}{1 - |z|}$$

when $|z| \leq \tanh \frac{1}{2}$. The inequality (1) now follows.

Proof of Theorem 1. If we set $f(z) = g(z)\,p(z)$, then we obtain

$$\frac{zf'(z)}{f(z)} = \frac{zp'(z)}{p(z)} + \frac{zg'(z)}{g(z)}. \tag{2}$$

For the function $p(z)$, whose real part is positive in the unit disc D, we have the known inequality [1, p. 18]

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{2|z|}{1 - |z|^2}, \tag{3}$$

which holds for all z in D. From (1), (2), and (3) we obtain

$$\Re \frac{zf'(z)}{f(z)} \geq \frac{1 - |z|}{1 + |z|} - \frac{2|z|}{1 - |z|^2}, \tag{4}$$

which holds for $|z| \leq \tanh \frac{1}{2}$. It is well known that so long as the right-hand member of (4) is positive, f is star-like. Hence we conclude that $w = f(z)$ maps the disc $\left\{z \mid |z| < (2 - \sqrt{3}) < \tanh \frac{1}{2}\right\}$ onto a domain in the w-plane that is star-like with respect to the origin.

To show that the radius $2 - \sqrt{3}$ cannot be improved, we choose

$$p(z) = (1 + z)/(1 - z), \qquad g(z) = z/(1 - z)^2.$$

Then $f(z) = p(z)\,g(z) = z(1 + z)/(1 - z^3)$ is univalent (and star-like) in the disc $|z| < (2 - \sqrt{3})$, with $f'(z)$ vanishing at the point $z = (\sqrt{3} - 2)$. This completes the proof of Theorem 1.

THEOREM 2. *Suppose that* $f(z) = z + a_2 z^2 + \cdots$ *and* $g(z) = z + b_2 z^2 + \cdots$ *are analytic in the unit disc* D, *and suppose that* $|f(z)/g(z) - 1| < 1$ *in* D. *If* g *is univalent in* D, *then* f *is univalent and star-like in the disc* $|z| < 1/3$. *This result is sharp.*

Proof. We can write $[f(z)/g(z) - 1] = z\phi(z)$, where ϕ is analytic in D and $|\phi(z)| \leq 1$. For such functions ϕ, MacGregor proved that the inequality

$$\left|\frac{\phi(z) + z\phi'(z)}{1 + z\phi(z)}\right| \leq \frac{1}{1 - |z|} \tag{5}$$

holds in D [4, p. 522]. If we apply (1) and (5) to the obvious identity

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + z\,\frac{\phi(z) + z\phi'(z)}{1 + z\phi(z)},$$

we obtain

$$\Re\,\frac{zf'(z)}{f(z)} \geq \frac{1 - 3|z|}{1 - |z|^2}, \tag{6}$$

which is valid for $|z| < \tanh\frac{1}{2}$. Since $\frac{1}{3} < \tanh\frac{1}{2}$, our result now follows from (6).

As MacGregor notes [4, p. 523], the function $f(z) = (z + z^2)/(1 - z)^2$ is an extremal function for this theorem.

We point out that our proof of Theorem 2 above is a mild adaptation of MacGregor's proof of his Theorem 3 [4, p. 523]; we used (1) where he invoked a stronger result.

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