LIE DERIVATIONS OF PRIMITIVE RINGS

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1. INTRODUCTION

Let R be a subring of a ring S. Then a *Lie derivation* of R into S is a mapping of R into S such that

(1) $$L(x + y) = L(x) + L(y),$$

(2) $$L([xy]) = [L(x), y] + [x, L(y)]$$

for all $x, y \in R$, where $[xy] = xy - yx$. In this paper we study Lie derivations of a primitive ring R into itself, where we assume that the characteristic of R is unequal to 2 and that R *contains a nontrivial idempotent*. Such mappings will be shown to be of the form $D + T$, where D is an ordinary derivation of R into a primitive ring $\overline{R}$ containing R and T is an additive mapping of R into the center of $\overline{R}$ which maps commutators into zero. Our result falls far short of providing a general solution to a conjecture of Herstein mentioned in [1; p. 529], but it generalizes an unpublished result of Kaplansky.

2. PRELIMINARIES

Throughout this paper we shall suppose that R is a primitive ring of characteristic not 2 and containing an idempotent e ($e \neq 0$, $e \neq 1$). (R need not have an identity.) Furthermore we shall assume that there is a Lie derivation L of R into itself. The ring R will be viewed as a dense subring of the ring $\overline{R}$ of all linear transformations of a vector space over a division ring. Setting $e_1 = e$ and $e_2 = 1 - e$, we let $R_{ij} = e_i R e_j$, $\overline{R}_{ij} = e_i \overline{R} e_j$, and we note that

$$R = \sum \oplus R_{ij} \quad \text{and} \quad \overline{R} = \sum \oplus \overline{R}_{ij} \quad (i, j = 1, 2).$$

Z will denote the center of R, Z' the center of $\overline{R}$, and it is clear that $Z \subset Z'$. (The symbol $\subset$ denotes inclusion in the wide sense.) We now state three lemmas which we shall need later on:

LEMMA 1. *If* $a \in \overline{R}_{ij}$ *and* $ax = 0$ *for all* $x \in R_{jk}$, *then* $a = 0$.

LEMMA 2. *If* $a \in R_{ii}$ *and* $[ax] = 0$ *for all* $x \in R_{ii}$, *then* a *is an element of the center of* $\overline{R}_{ii}$.

LEMMA 3. *The center of* $\overline{R}_{ii}$ *is* $e_i Z'$.

The proofs rest on well-known properties of primitive rings and will be omitted.

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3. THE IDEMPOTENT e UNDER LIE DERIVATION

LEMMA 4. *For all* $x \in R$,

$$x\{e L(e) + L(e)e + e L(e)e - L(e)\} - \{e L(e) + L(e)e + e L(e)e - L(e)\}x \tag{3}$$
$$= 3ex\{e L(e) + L(e)e - L(e)\} - 3\{e L(e) + L(e)e - L(e)\}xe .$$

Proof. The verification that

$$[[[xe]e]e] = [xe] \tag{4}$$

for all $x \in R$ is straightforward. Repeated application of L to (4) results in

$$[[[L(x), e] + [x, L(e)], e] + [[[xe], L(e)], e] + [[[xe]e], L(e)] \tag{5}$$
$$= [L(x), e] + [x, L(e)] .$$

Expansion and simplification of (5) then yields the desired conclusion.

LEMMA 5. $L(e) = [es] + z$, *for some* $s \in R$ *and* $z \in Z$.

Proof. Setting $L(e) = \Sigma f_{ij}$, $f_{ij} \in R_{ij}$ and substituting in (3), we obtain the relation

$$x(2f_{11} - f_{22}) - (2f_{11} - f_{22})x = 3ex(f_{11} - f_{22}) - 3(f_{11} - f_{22})xe \tag{6}$$

for all $x \in R$. If $x \in R_{12}$, (6) reduces to $f_{11}x = xf_{22}$, whence we conclude

$$(f_{11} + f_{22})x = x(f_{11} + f_{22}) \qquad (x \in R_{12}) .$$

Similarly,

$$(f_{11} + f_{22})x = x(f_{11} + f_{22}) \qquad (x \in R_{21}) .$$

Now let $x \in R_{11}$ and $y \in R_{12}$. Then

$$\{(f_{11} + f_{22})x - x(f_{11} + f_{22})\}y = (f_{11} + f_{22})xy - xy(f_{11} + f_{22})$$
$$= (f_{11} + f_{22})xy - (f_{11} + f_{22})xy$$
$$= 0 ,$$

since $y, xy \in R_{12}$. It follows from Lemma 1 that

$$(f_{11} + f_{22})x - x(f_{11} + f_{22}) = 0 \qquad (x \in R_{11}) .$$

Similarly,

$$(f_{11} + f_{22})x = x(f_{11} + f_{22}) \qquad (x \in R_{22}) ,$$

and so $f_{11} + f_{22} = z \in Z$. Hence, $L(e) = (f_{12} + f_{21}) + z$; and, setting $s = f_{12} - f_{21}$, one easily verifies that $L(e) = (es - se) + z$.

4. DEFINITION OF D AND T

Throughout this section and the next we impose the additional assumption that $L(e)$ *is an element of* Z.

LEMMA 6. $L(R_{ij}) \subset R_{ij}$, $(i \neq j)$.

Proof. Let $x \in R_{12}$, and set $L(x) = \Sigma y_{ij}$, $y_{ij} \in R_{ij}$. Then

$$\sum y_{ij} = L(x) = L([ex]) = [L(e), x] + [e, L(x)] = [e, L(x)] = y_{12} - y_{21}.$$

It follows that $y_{11} = y_{21} = y_{22} = 0$, and thus $L(x) \in R_{12}$. A similar argument holds if $x \in R_{21}$.

LEMMA 7. $L(R_{ii}) \subset \overline{R}_{ii} + Z'$.

Proof. Let $x \in R_{11}$, and set $L(x) = \Sigma y_{ij}$, $y_{ij} \in R_{ij}$. Then

$$0 = L([ex]) = [L(e), x] + [e, L(x)] = [e, L(x)] = y_{12} - y_{21},$$

whence $y_{12} = y_{21} = 0$ and $L(x) \in R_{11} + R_{22}$. Similarly, if $x \in R_{22}$, then $L(x) \in R_{11} + R_{22}$. Now let $x \in R_{11}$ and $y \in R_{22}$ with $L(x) = a_{11} + a_{22}$ and $L(y) = b_{11} + b_{22}$ $(a_{ii}, b_{ii} \in R_{ii})$. Then

$$0 = L([xy]) = [L(x), y] + [x, L(y)] = [a_{22}y] + [xb_{11}] = 0,$$

and so in particular $[a_{22}y] = 0$. In view of Lemma 2, $[a_{22}y] = 0$ for all $y \in \overline{R}_{22}$; and thus, by Lemma 3, $a_{22} = (1 - e)z$ for some $z \in Z'$. Therefore

$$L(x) = a_{11} + (1 - e)z = [(a_{11} - ez) + z] \in \overline{R}_{11} + Z'.$$

In the same fashion one sees that that $L(R_{22}) \subset \overline{R}_{22} + Z'$.

We summarize the results we have obtained thus far:

(7) $$\text{if } x \in R_{ij},\ (i \neq j), \text{ then } L(x) = x^* \in R_{ij};$$

(8) $$\text{if } x \in R_{ii}, \text{ then } L(x) = x^* + z,\ x^* \in \overline{R}_{ii},\ z \in Z'.$$

Relations (7) and (8) enable us to define in a natural way a mapping D of R into $\overline{R}$ according to the rule

$$D(x) = x^* \text{ if } x \in R_{ij} \text{ for all } i, j.$$

A mapping T of R into Z' is then defined by the rule

$$T(x) = L(x) - D(x) \qquad (x \in R).$$

5. PROPERTIES OF D AND T

LEMMA 8. $T(x + y) = T(x) + T(y)$ *for all* $x, y \in R$.

Proof. It suffices to show that T is additive on R_{ii}. If $x, y \in R_{ii}$, then

$$T(x + y) - T(x) - T(y) = L(x + y) - D(x + y) - L(x) + D(x) - L(y) + D(y)$$

$$= [D(x) + D(y) - D(x + y)] \in \overline{R}_{ii} \cap Z' = 0.$$

COROLLARY. $D(x + y) = D(x) + D(y)$ *for all* $x, y \in R$.

LEMMA 9. $D(xyx) = D(x)yx + xD(y)x + xyD(x)$ *for all* $x \in R_{ij}$ $(i \neq j)$ *and all* $y \in R$.

Proof. Letting $x \in R_{ij}$ $(i \neq j)$, we may write $2xyx = [[xy]x]$. Then

$$2D(xyx) = L(2xyx) = L([[xy]x]) = [[L(x), y] + [x, L(y)], x] + [[xy], L(x)]$$

$$= [[D(x), y] + [x, D(y)], x] + [[xy], D(x)]$$

$$= 2\{D(x)yx + xD(y)x + xyD(x)\};$$

we make use of the fact that, for $i \neq j$, $\overline{R}_{ij}^2 = 0$ and $D(R_{ij}) \subset R_{ij}$. Since the characteristic of R is not 2, the desired conclusion follows.

LEMMA 10. *For* $x \in R_{ii}$ *and* $y \in R_{jk}$ $(j \neq k)$, $D(xy) = D(x)y + xD(y)$.

Proof. We may assume that $x \in R_{11}$ and $y \in R_{12}$. Then

$$D(xy) = L(xy) = L([xy]) = [L(x), y] + [x, L(y)]$$

$$= [D(x), y] + [x, D(y)] = D(x)y + xD(y).$$

LEMMA 11. *For* $x \in R_{ii}$ *and* $y \in R_{jj}$, $D(xy) = D(x)y + xD(y)$.

Proof. We may assume that $x, y \in R_{11}$. Choosing $r \in R_{12}$, we may write, using Lemma 10, the relations

$$D(xy)r = D(xyr) - xyD(r) = D(x)yr + xD(yr) - xyD(r)$$

$$= D(x)yr + x\{D(y)r + yD(r)\} - xyD(r)$$

$$= \{D(x)y + xD(r)\}r.$$

Hence $\{D(xy) - D(x)y - xD(y)\}r = 0$ for all $r \in R_{12}$. Therefore, by Lemma 1, $D(xy) - D(x)y - xD(y) = 0$.

THEOREM 1. D *is an ordinary derivation of* R *into* $\overline{R}$.

Proof. In order to prove that $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$, we may assume, in view of Lemmas 10 and 11, that $x \neq 0 \in R_{12}$ and $y \in R_{21}$. From the relations

$$T([xy]) = L([xy]) - D([xy]) = [L(x), y] + [x, L(y)] - D([xy])$$

$$= [D(x), y] + [x, D(y)] - D(xy) + D(yx)$$

we obtain

$$\{D(x)y + xD(y) - D(xy)\} + \{D(yx) - D(y)x - yD(x)\} = z \in Z'. \tag{9}$$

If $z = 0$, $[D(x)y + xD(y) - D(xy)] \in (R_{11} \cap R_{22})$ and hence is equal to 0. Thus suppose $z \neq 0$. Multiplication of (9) on the left by x yields the formula

$$xD(yx) - xD(y)x - xyD(x) = xz.$$

Applying Lemma 10, we find that

$$D(xyx) - D(x)yx - xD(y)x - xyD(x) = xz;$$

and, by Lemma 9, we see that $xz = 0$. It follows that $x = 0$, which is a contradiction.

COROLLARY. $T(xy - yx) = 0$ *for all* $x, y \in R$.

6. THE MAIN RESULT

We now drop the assumption (used in Sections 4 and 5) that $L(e) \in Z$. However, by Lemma 5, $L(e) = [es] + z$ where $s \in R$, $z \in Z$. Letting I be the inner derivation determined by s, that is, $I(x) = xs - sx$ for all $x \in R$, we see that $L' = L - I$ is a Lie derivation of R into itself such that $L'(e) = z \in Z$. According to Section 4, L' may be written in the form $L' = D + T$, that is, $L = (I + D) + T$. Our main result now follows from Theorem 1 and its corollary.

THEOREM 2. *Let* L *be a Lie derivation of a primitive ring* R *into itself, where* R *contains a nontrivial idempotent and the characteristic of* R *is not* 2. *Then* L *is of the form* $D + T$, *where* D *is an ordinary derivation of* R *into a primitive ring* $\overline{R}$ *containing* R *and* T *is an additive mapping of* R *into the center of* $\overline{R}$ *that maps commutators into zero.*

We conclude by remarking that if the ring R in Theorem 2 is simple, then D maps R into itself and T maps R into the center of R. To see this one need only note that $D(R_{ij}) \subset R_{ij} \subset R$ $(i \neq j)$ and that the subring generated by the R_{ij} $(i \neq j)$ is an ideal and hence is equal to R.

REFERENCES

1. I. N. Herstein, *Lie and Jordan structures in simple, associative rings*, Bull. Amer. Math. Soc. 67 (1961), 517-531.

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