GENERALISED CONSTANT WIDTH FOR MANIFOLDS

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1. INTRODUCTION

The notion of constant width may be formulated as follows. Let H be a compact C^∞ n-manifold without boundary, convexly imbedded in real (n + 1)-space R^{n+1}. A chord of H is *normal* if it is normal to H at one of its two end-points, and *binormal* if it is normal to H at both end-points (compare Morse [3, p. 183]). Therefore H has *constant width* if every normal chord is binormal; for then all normal chords have the same length. Such a manifold is of course diffeomorphic to S^n. Conversely, any compact connected closed hypersurface in R^{n+1} of constant width is convex and diffeomorphic to S^n.

In this paper we formulate more general conditions for manifolds imbedded with arbitrary codimension, and in some cases we obtain corresponding classification theorems.

Let V denote a smooth (that is C^∞) connected n-manifold without boundary, smoothly imbedded in R^m as a closed subset, for some $m > n$. We write $\nu(p)$ for the (m - n)-plane in R^m normal to V at $p \in V$, and we say that V is *transnormal* in R^m if, for each pair $p, q \in V$, the relation $q \in \nu(p)$ implies that $\nu(p) = \nu(q)$. Thus transnormality generalises constant width. It is easy to show that the map ν from a transnormal manifold V to the space of normal (m - n)-planes of V is a covering map. We say that V has *order* r or is r-*transnormal* if ν is r-fold. The main result is as follows.

THEOREM 1.1. *Any transnormal* n-*manifold of order* 2 *in* R^m *is diffeomorphic to the cartesian product* $V_1 \times V_2$ *of differential manifolds* V_1, V_2, *where* V_1 *is homeomorphic to* S^j *and* V_2 *is homeomorphic to* R^{n-j} $(0 < j \leq n)$.

We do not know whether V_1 can have an unusual differential structure. For instance, can any of the 27 unusual 7-spheres be transnormally imbedded in R^9?

We show in Section 4 that for any transnormal manifold V and any $p, q \in V$, the sets $\nu(p) \cap V$ and $\nu(q) \cap V$ are isometric as subsets of R^m. We call $\nu(p) \cap V$ a *generating frame* of V, and we prove that the generating frame always admits a transitive group of isometries. This fact, together with Theorem 1.1, yields the following.

THEOREM 1.2. *If* V *is a transnormal* n-*manifold in* R^{n+1} *of order* r, *then* $r = 2$ *or* $r = 1$.

Since it is easy to show that R^n is (up to homeomorphisms) the only transnormal manifold of order 1, Theorems 1.1 and 1.2 classify transnormal hypersurfaces of finite order. In particular, the sphere, cylinder and plane are the only surfaces that can be transnormally imbedded in R^3 with finite order. The standard imbedding of the torus in R^4 is 4-transnormal.

The last statement is a consequence of the easily proved fact that if M and N can be transnormally imbedded in R^a, R^b with orders λ and μ respectively, then $M \times N$

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can be transnormally imbedded in R^{a+b} with order $\lambda\mu$ (see the end of Section 8 for a slightly stronger statement). It now follows that any cartesian product

$$S^{i_1} \times S^{i_2} \times \cdots \times S^{i_k} \times R^{i_{k+1}}$$

can be r-transnormally imbedded in R^m, with $r = 2^k$ and $m = k + \Sigma\, i_s$. This suggests two questions: (1) If V is r-transnormal, is r a power of 2? (2) Do there exist r-transnormal manifolds not diffeomorphic (or homeomorphic) to a product of standard spheres with some euclidean space?

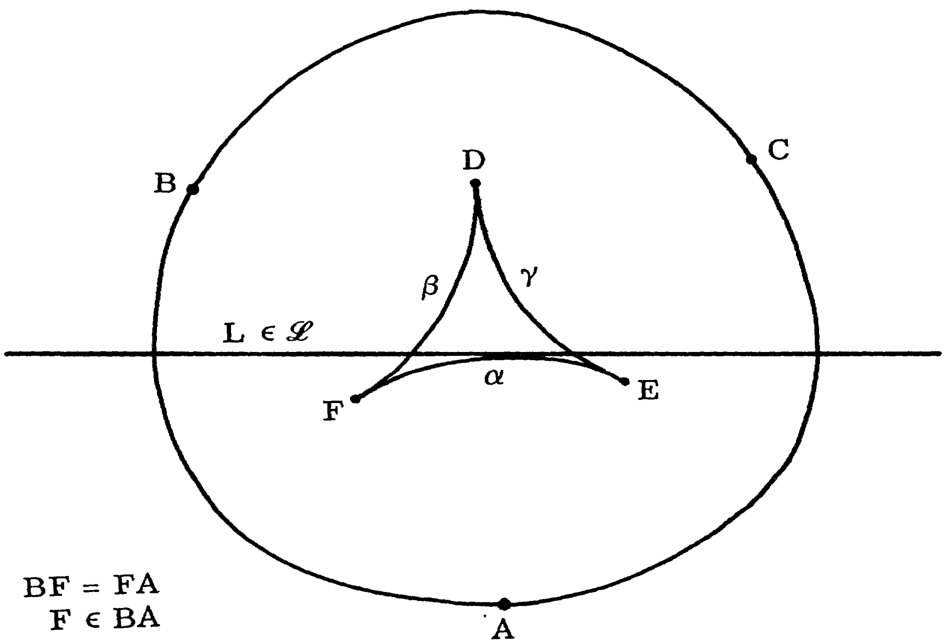
The remainder of the paper is directed towards a solution of these two problems. We prove for example the existence of r integrable distributions on V whose complementary (orthogonal) distributions are also integrable, provided the covering ν is regular.

These and the preceding results emerge from a study of the properties of the distance function $\Lambda_q : V \to R$, where $\Lambda_q(x) = \|x - q\|^2$ and $q \in V$. We use the elementary parts of Morse theory.

2. A TRANSNORMAL 2-SPHERE

For illustrative purposes, we modify the Reuleaux triangle construction (see [1] or [5], for example) to obtain a transnormal imbedding of S^2 in R^3 as a C^∞-surface of nonconstant curvature.

Let T be an equilateral triangle in $R^2 \subset R^3$ with vertices A, B, C. Let α, β, γ be concave arcs inside T, joining the mid-points D, E, F of the sides of T in pairs, such that β and γ have C^∞-contact at D and their common tangent at D bisects the angle EDF; the pairs γ, α and α, β form similar configurations at E and F respectively, and α, β, γ meet nowhere else (see the figure).



BF = FA
F ∈ BA

Let $\mathscr{L}$ denote the 1-parameter C^∞-family of straight lines tangent to α, β, or γ. Then the orthogonal trajectory of $\mathscr{L}$ that passes through A, B, and C is a C^∞ noncircular simple closed curve of constant width in R^2.

Rotate this curve in R^3 about the median of T through A. The closed surface Σ so generated is a C^∞ 2-sphere of constant width, but with nonconstant curvature.

3. FOCAL POINTS AND THE DISTANCE FUNCTION

In this section we use certain facts about focal points of a manifold imbedded in a euclidean space. Proofs of these standard results can be found, for example, in Milnor [2].

Let V be a transnormal n-manifold imbedded in R^m. For any $q \in V$, let $\Lambda_q: V \to R$ be the C^∞ *distance function* defined by $\Lambda_q(p) = \|p - q\|^2$. We set

$$V^* = \{(p,x): p \in V,\ x \in \nu(p)\}.$$

Thus V^* is the normal bundle space of V in R^m: it is a C^∞ m-dimensional submanifold of $V \times R^m$. The *projection* $\pi: V^* \to V$ and the *end-point* map $\eta: V^* \to R^m$ are defined by $\pi(p, x) = p$ and $\eta(p, x) = x$ respectively. Both are C^∞-maps.

If $(p, x) \in V^*$ is a singularity of η (that is if the rank ρ of the Jacobian of η at (p, x) is less than m), then x is a *focal* point of V with *base* p and *multiplicity* $\mu = m - \rho$. For each $p \in V$, F_p will denote the set of focal points of V with base p. We also set

$$F_V = \bigcup_{p \in V} F_p$$

and call this the *focal set* of V. Directly from the definition, we see that $F_p \subset \nu(p)$, for each $p \in V$.

Now put $V^+ = (V \times V) \cap V^*$. Then $\eta | V^+$ is a locally diffeomorphic covering map of V^+ onto V. Notice however that V^+ is not necessarily connected, since it contains the diagonal of $V \times V$. We observe that there are open neighbourhoods E of V in R^m and E^* of V^+ in V^* such that $\eta | E^*$ is a locally diffeomorphic map of E^* onto E.

LEMMA 3.1. *No transnormal manifold meets its focal set.*

Proof. If $q \in V$ is a focal point of V with base $p \in V$, then $(p, q) \in V^+$ is a singular point of η. But η immerses a neighbourhood of V^+ in R^m, so we have a contradiction.

LEMMA 3.2. *The function* Λ_q *is nondegenerate, for each* $q \in V$.

Proof. Recall that $p \in V$ is a critical point of Λ_q if and only if the line joining p to q is perpendicular to V at p. Hence the critical set of Λ_q is the set $\nu(q) \cap V$. Further, a critical point $p \in V$ of Λ_q is nondegenerate if and only if $q \notin F_p$. Hence Λ_q is nondegenerate if and only if $q \notin F_V$. The required result now follows by Lemma 3.1.

4. GENERATING FRAMES

As in the previous section, and throughout the remainder of this paper, V will denote a transnormal n-manifold in R^m. The map $\nu: V \to G_{n,m}$ that sends $p \in V$ to the (m - n)-plane $\nu(p)$ normal to V at p is a C^∞-map into the open Grassmannian of (m - n)-planes in R^m. Set $W = \nu(V)$. It is an immediate consequence of the definition of transnormality that W is an n-manifold and that ν is a C^∞-immersion of V in W. We prove a little more than this.

LEMMA 4.1. *The map* ν *is a covering map of* W *by* V.

Proof. The map $\pi: V^* \to V$ is a locally trivial fibre-map. Let $\xi \in W$ and let $p \in \xi \cap V$. Let M be an open cell-neighbourhood of p on V such that $\pi^{-1}(M)$ is bundle-equivalent to $\xi \times M$. Put $N = \nu(M)$. Then $\nu^{-1}(N) = \eta(\pi^{-1}(M) \cap V^+)$, and each of its components is mapped diffeomorphically by ν onto N. Hence ν is a covering map, and this proves the lemma.

Next we look at the metric properties of the sets $\nu^{-1}(\xi)$ $(\xi \in W)$.

Let $C: I \to W$ be a (piecewise differentiable) arc beginning at $\xi = C(0)$ and ending at $\zeta = C(1)$, say. Then for each $p \in \nu^{-1}(\xi)$, there is a unique piecewise differentiable arc $C_p: I \to V$ beginning at $p = C_p(0)$ and such that $\nu C_p = C$. Hence C induces a map $C^\#: \nu^{-1}(\xi) \to \nu^{-1}(\zeta)$.

From now on we write $p \sim q$ to mean that $p, q \in V$ and $\nu(p) = \nu(q)$. We also put $p_t = C_p(t)$, $q_t = C_q(t)$ for $t \in [0, 1] = I$.

LEMMA 4.2. $C^\#$ *is an isometry.*

Proof. Let $p \sim q$ and $\nu(p) = \xi$ as above. Then $p_t \sim q_t$ for each $t \in I$, and the tangents to C_p, C_q at p_t, q_t are normal to the join of these points. Hence $\|p_t - q_t\|$ is independent of the value of t. We conclude that the map $C^\#$, given by $C^\#(p) = p_1$, for each $p \in \nu^{-1}(\xi)$ is an isometry of $\nu^{-1}(\xi)$ onto $v^{-1}(\zeta)$.

Thus $\nu^{-1}(\xi)$ is independent of $\xi \in W$ up to isometry. We choose some base point $\xi_0 \in W$ and call $\nu^{-1}(\xi_0)$ the *generating frame* $\phi(V)$ of V. By choosing C to be a closed arc, we get the following.

COROLLARY 4.3. $\phi(V)$ *admits a transitive group of isometries.*

We denote this group by G(V).

5. DISTRIBUTIONS AND CRITICAL POINTS

For each $q \in V$, ∇_q will denote the gradient C^∞ vector field grad Λ_q of Λ_q on V. The zeros of ∇_q are the critical points of Λ_q and so lie in $\nu(q)$. Recall that the integral curves of ∇_q are orthogonal trajectories of the contours of Λ_q. Each such curve, parametrised by the values of Λ_q, begins at some critical point of Λ_q and either ends at some other, distinct critical point or else continues to infinite length. We write Γ_q for the set of critical points of Λ_q.

Let $p \sim q$ on V. Then $p \in \Gamma_q$ and has index j, say. The *stable manifold* S(p, q) (or S) of Λ_q at p is the set of points of V that lie on integral curves of ∇_q ending at p, together with p itself. Then S is homeomorphic to R^j. The *unstable manifold* U(p, q) (or U) at p is likewise homeomorphic to R^{n-j}, and it consists of p together with points on integral curves of ∇_q beginning at p. Thus the tangent spaces at p to S and U are orthogonal complementary subspaces of the tangent space $\tau(p)$ to V at p.

We suppose from now on that the map ν: V $\to$ W is a *regular* covering. In other words, the group G(V) of isometries of the generating frame ϕ(V) operates without fixed points; that is, for each closed curve K_p in V covering a closed curve K on W as in Section 4, and for each $h \sim p$, the curve K_p is also closed.

With this assumption, we can assign to each pair (p, q) a C^∞ distribution of j-planes, as follows.

Suppose that C_p is an arc on V from p to $x \in V$. Let $y = C^{\#}(q)$. Then, with the above hypothesis, the point y depends only on x and (p, q), and not on the choice of C_p. Now define $\lambda_{pq}: V \to V$ by setting $\lambda_{pq}(x) = y$. Then λ_{pq} is a diffeomorphism of V onto itself such that $\lambda_{pq}\lambda_{qh} = \lambda_{ph}$. Further, for each $x \in V$, $\|x - y\| = \|p - q\|$ and $x \sim y$, where $y = \lambda_{pq}(x)$.

The index of x as a critical point of y varies continuously with x over V, in view of Lemmas 3.1 and 3.2. Hence the index of x is j.

We define $\triangle_{pq}(x)$ to be the tangent space at x to S(x, y). Thus $\triangle_{pq}$ (or $\triangle$) is a well-defined C^∞ distribution of j-planes on V. A distribution $\triangle^*_{pq} = \triangle^*$ of (n - j)-planes is obtained on replacing S by U in the above definition; $\triangle$ and $\triangle^*$ are orthogonal and complementary.

6. LOCAL STRUCTURE OF V

Let A_p be an open cell-neighbourhood of p on V, so small that

(i) it is disjoint from $A_q = \lambda(A_p)$ and

(ii) the line xq joining x to q is not tangent to V at x, for any $x \in A_p$.

Then the orthogonal projection of xq into $\nu(x)$ is a straight line L_x through x, normal to V.

LEMMA 6.1. $\lambda(x) \in L_x$.

Proof. Let T_α denote the set of points $v \in R^m$ such that $\|u - v\| = \alpha$, for some $u \in V$ for which $(u, v) \in V^*$. We write T^*_α for the set of all such pairs (u, v). Then T^*_α is a C^∞ (m - 1)-manifold, a bundle of (m - n - 1)-spheres over V.

Now let $\alpha = \|p - q\|$. Then $V \subset T_\alpha$, and T^*_α admits the cross-section $x \to (x, \lambda_{pq}(x))$, the image being contained in V^+. Put $A = \nu(A_p)$.

Define f: $A_p \to T_\alpha$ by taking f(x) to be the point on L_x at distance α from x, chosen from the two possibilities in such a way that f is continuous and f(p) = q. Since (p, q) is not a singular point of the end-point map η, neither is (x, f(x)) for x sufficiently near p on V. We can suppose without loss of generality that each $x \in A_p$ has this property. Then f imbeds A_p in T_α. Put $B_p = f(A_p)$, and let B^*_p denote the set containing (p, q) in T^*_α which is mapped diffeomorphically onto B_p by η.

The set Q of nonsingular points of η on T^*_α is an open (m - 1)-manifold imbedded in T^*_α and containing B^*_p. Let Y be the C^∞-distribution of n-planes on Q associated with the connexion in the bundle T^*_α given by the metric of R^m. Thus the elements of Y are orthogonal (in the metric on T^*_α induced from R^m) to the (m - n - 1)-sphere fibres in T^*_α. Hence Y is an integrable distribution. The integral manifold of Y through (p, q) is mapped by η onto V.

Now for each $x \in A_p$, Y(x, f(x)) is by definition tangent to B^*_p at (x, f(x)). Thus B^*_p is contained in an integral manifold of Y. But $(p, q) \in B^*_p$, and so $B_p \subset V$. Thus f and λ_{pq} agree on A_p. This proves the lemma.

Next we examine the differential $d\lambda$ of $\lambda = \lambda_{pq}$.

Choose a system of cartesian coordinates $x_1, \cdots, x_m$ for R^m such that the n-plane $\tau(p)$ tangent to V at p has equations $x_k = 0$ $(k = n+1, \cdots, m)$, and the (m - n)-plane $\nu(p)$ has equations $x_i = 0$ $(i = 1, \cdots, n)$, with $p = (0, \cdots, 0)$. Then $q = (0, \cdots, 0, q_{n+1}, \cdots, q_m)$, for some $q_k \in R$ not all zero. Thus some open neighbourhood of p, say the neighbourhood A_p above, is given by equations $x_k = g_k(x_1, \cdots, x_n)$, for some C^∞ functions g_k $(k = n+1, \cdots, m)$ defined on an open neighbourhood Ω of O in R^n. From now on, the indices i, k will run through $1, \cdots, n$ and $n+1, \cdots, m$, respectively.

Any tangent vector T to A_p at $x = (x_1, \cdots, x_m)$ can be expressed in the form $T = (\tau_1, \cdots, \tau_m) \in R^m$ with

(6.2)
$$\tau_k = \sum_i \tau_i D_i g_k(x_*),$$

where $x_* = (x_1, \cdots, x_n) \in R^n$. (Here, as elsewhere, we identify the tangent space to R^m at any point of R^m with the space R^m itself.)

Likewise, any normal vector N to A_p at x is of the form $N = (\nu_1, \cdots, \nu_m)$, where

(6.3)
$$\nu_i = -\sum_k \nu_k D_i g_k(x_*).$$

Suppose now that T, N are respectively the orthogonal projections of the vector x - q into $\tau(x)$, $\nu(x)$. Then the components of T, N are related by the equations

(6.4)
$$\tau_i - \sum_i \nu_k D_i g_k(x_*) = x_i,$$

and

(6.5)
$$\sum_i \tau_i D_i g_k(x_*) + \nu_k = g_k(x_*) - q_k.$$

Let $C_p\colon I \to V$ be a C^∞-curve beginning at p as before, with nonzero tangent $C_p'(0) = p'$ at p. Then the tangent $C_q'(0) = q'$ to C_q at q is equal to $d\lambda_{pq}(p')$.

By using Lemma 6.1 and the above equations, we find that $q' = p' - \zeta$, where

$$\zeta_i = -\sum_k q_k \sum_{r=1}^{n} D_{ir} g_k(0) p_r'.$$

Differentiating (6.3) with respect to t, we then see that $T' + \zeta = p'$ and so $q' = T'$. Now $T = M\nabla_q(x)$, where M is a positive real number (which may vary with x).

Hence

$$dT/dt = \nabla_q(x)\, dM/dt + M\, d\nabla_q(x)/dt, \quad \text{and so } T' = M\nabla_q'(p).$$

Further,

$$\nabla_q'(0) = \lim_{t\to 0} \{\nabla_q(x) - \nabla_q(p)\}/t = \lim_{t\to 0}\{\nabla_q(x)/t\}.$$

We therefore conclude that the line Q spanned by q' is the limit as $t \to 0$ of the line spanned by $\nabla_q(p_t)$. Since $\tau(p)$ is parallel to $\tau(q)$, and $\tau(p)$ has equations $x_k = 0$, we can identify these spaces with R^n in the obvious way. Then $d\lambda_{pq}$ is an isomorphism of R^n with itself.

LEMMA 6.6. *The isomorphism* $d\lambda_{pq}$ *has exactly two eigenvalues, one positive and one negative, of which* $\Delta^*(p)$ *and* $\Delta(p)$ *are the respective eigenspaces.*

Proof. We see that $p' \in \Delta(p)$ if and only if p' is a negative multiple of $\nabla_q'(p)$. But q' is a positive multiple of $\nabla_q'(p)$, by the above. This yields the statement for $\Delta(p)$, and $\Delta^*(p)$ is dealt with in a similar way.

The following elementary fact should also be noted.

LEMMA 6.7. *If* $x \sim y$, *then* $\nabla_q(x)$ *and* $\nabla_q(y)$ *are parallel, for any* $x, y, q \in V$.

Proof. The tangent space to R^m at x is $\nu(x) \oplus \tau(x)$, and so $x - q = \alpha + \beta$, where $\alpha \in \nu(x)$, $\beta \in \tau(x)$. Now

$$y - q = (y - x) + (x - q) \text{ and } y - x \in \nu(x).$$

Hence $y - q = \alpha' + \beta$, where $\alpha' = (y - x) + \alpha \in \nu(x)$. But $\nu(x) = \nu(y)$, and $\tau(x)$ is parallel to $\tau(y)$. Thus $x - q$ and $y - q$ project onto identical vectors β in $\tau(x)$, $\tau(y)$ respectively. Since these projections are (positive) multiples of $\nabla_q(x)$, $\nabla_q(y)$, the lemma is proved.

7. INTEGRABILITY OF Δ AND Δ^*

THEOREM 7.1. Δ *and* Δ^* *are integrable.*

Proof. As before, let p, q be distinct, with $p \sim q$. Then p is a critical point of Λ_q, of index j, say. If $j = 0$ or $j = n$, then Δ and Δ^* are trivially integrable. We may therefore suppose that $0 < j < n$.

Let A_p be the open neighbourhood of p on V introduced in Section 6, and let

$$x \in A_p \cap S(p, q) = S_p.$$

Then, by Lemma 6.1, $y = \lambda(x) \in L_x$. Also, the line L_x is the orthogonal projection of the line xq on $\nu(x)$, and $\nabla_q(x)$ is contained in the orthogonal projection of xq on $\tau(x)$. Directly from the definition of S_p, we see that $\nabla_q | S_p = \nabla(\Lambda_q | S_p)$. Consider the projections of xq on the $(m - j)$-plane $\nu_p(x)$ normal to S_p at x and the j-plane $\tau_p(x)$ tangent to S_p at x. Clearly $\nu_p(x) \supset \nu(x)$ and $\tau_p(x) \subset \tau(x)$. But the projection of xq into $\tau_p(x)$ coincides with its projection into $\tau(x)$, these being spanned by the vectors $\nabla(\Lambda_q | S_p)(x)$ and $\nabla_q(x)$, respectively. Hence the projections of xq into $\nu_p(x)$ and $\nu(x)$ coincide also.

Now x is a critical point of $\Lambda_y | S_p$, and it is nondegenerate since y cannot be a focal point of S_p with base x (see the proofs of Lemmas 3.1 and 3.2). Since $\Lambda_q | S_p$ has a local maximum at p, we see that x is a local maximum of $\Lambda_y | S_p$. Now we need only apply Lemma 6.6 to the above observations to see that $\Delta(x)$ is tangent to S_p at x. This proves that Δ is integrable. Similar arguments apply to Δ^*.

8. PROOF OF THEOREM 1.1

We begin by treating 1-transnormal manifolds.

THEOREM 8.1. *Any* 1-*transnormal* n-*manifold in any* R^m *is homeomorphic to* R^n.

Proof. Let V be such a manifold, and let $q \in V$. Then q is a nondegenerate minimum of Λ_q, and Λ_q has no other critical point. Hence V is homeomorphic to an open n-cell, which proves the theorem.

Remark. Conversely, R^n can be r-transnormally imbedded in R^m if and only if $r = 1$. For then R^n is a covering space of finite order r.

Proof of Theorem 1.1. The only reason for considering S_p rather than S(p, q) itself in the proof of Theorem 7.1 was to ensure that xq is not tangent to V at x. Should it happen that for some $z \in S(p, q)$, the line zq is tangent to V at z, then L_z is not determined by projection of zq on $\nu(z)$. However, the line L_z joining z to $w = \lambda_{pq}(z)$ is well-defined. Also, by Lemma 6.7, it cannot happen that both zq and wq are tangent to V (at z and, w respectively). Suppose then that V is 2-transnormal. The hypothesis that ν is regular is automatically satisfied, and $(\lambda_{pq})^2$ is the identity. Thus if necessary we can interchange the rôles of x and y in Theorem 7.1 to obtain the result that for 2-transnormal manifolds, S(p, q) is contained in a single integral manifold of $\triangle$. Similar remarks apply to U(p, q) and $\triangle^*$.

Let V be 2-transnormal. One of the two critical points of Λ_q $(q \in V)$ is q itself, a nondegenerate minimum. The index j of the second critical point p of Λ_q cannot be 0, since V is connected. If $j = n$, then V is homeomorphic to S^n, by Reeb's theorem.

Suppose therefore that $0 < j < n$. The unstable manifold U of Λ_q at p is homeomorphic to R^{n-j}, and every integral curve of ∇_q which lies upon it is infinite in length; U is an integral manifold of $\triangle^*$. The closure Cl (S) of the stable manifold of Λ_q at p is a smooth manifold consisting of the j-cell S attached to the point q. Thus the integral manifolds of $\triangle$ are homeomorphic to S^j.

We have now shown that V can be decomposed into two mutually orthogonal families of differential manifolds such that each member of one family meets each member of the other in exactly one point. This implies at once that V is diffeomorphic to $Cl(S) \times U$, which is equivalent to the statement of Theorem 1.1 with $V_1 = Cl(S)$, $V_2 = U$.

Finally, we observe that if V is a compact hypersurface, then V is diffeomorphic to S^n. It is convex because p is the point at which Λ_q attains its maximum value on V, and p is the nearest point to q on $\tau(p)$.

Concluding remarks.

(1) Identify R^m, R^ℓ with the orthogonal complements $R^m \oplus O$, $O \oplus R^\ell$ in $R^m \oplus R^\ell = R^{m+\ell}$. Then imbeddings of V in R^m and V' in R^ℓ determine an imbedding of $V \times V'$ in $R^{m+\ell}$. It is a straightforward matter to verify that $V \times V'$ is transnormally imbedded if and only if V and V' are transnormally imbedded, and that the normal maps ν, ν' of V, V' are regular if and only if the normal map ν'' of $V \times V'$ is regular. The order of ν'' is the product of the orders of ν and ν'.

(2) The statement of Lemma 3.2 suggests the following classification problem. Find all manifolds V that can be imbedded in a euclidean space in such a way that for each $q \in V$ the distance function Λ_q is nondegenerate.

REFERENCES

1. H. G. Eggleston, *Convexity,* Cambridge Univ. Press, 1958.

2. J. W. Milnor, *Morse theory,* Annals of Mathematics Studies no. 51, Princeton Univ. Press, 1963.

3. M. Morse, *Calculus of variations in the large,* Amer. Math. Soc. Colloquium Publications 18, Amer. Math. Soc., New York, 1934.

4. F. Reuleaux, *Kinematics of machinery,* London, 1876.

5. I. M. Yaglom and V. G. Boltyanskiĭ, *Convex figures;* translation by P. J. Kelly and L. F. Walton; Holt, Rinehart and Winston, New York, 1961.

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