HOMOGENEITY OF CERTAIN MANIFOLDS

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1. INTRODUCTION

The connected n-dimensional topological manifold M^n is said to be *homogeneous* if for any two locally flat embeddings f_1 and f_2 of the closed n-cell D^n into M^n, there can be found a homeomorphism h of M^n onto itself such that $hf_1 = f_2$. The manifold M^n, if it is orientable, is *homogeneous up to orientation* if the above homeomorphism h exists provided that f_1 and f_2 induce the same orientation on M^n from a given orientation on D^n.

This paper is concerned with the following conjecture.

HOMOGENEITY CONJECTURE. *If* M^n *is orientable but does not admit an orientation reversing homeomorphism, then* M^n *is homogeneous up to orientation. Otherwise,* M^n *is homogeneous.*

The corresponding conjecture in piecewise linear topology was proved by Newman [11] and Gugenheim [8], and in differential topology by Palais [12]. The present conjecture has been proved for $n \leq 3$ by the triangulation theorems of Bing [1] and Moise [10]. In addition, S^n and R^n are homogeneous according to Brown [2, 3]; $S^{n-1} \times S^1$, and more generally the n-sphere with handles, is homogeneous according to Brown and Gluck [4, 5, 6, 7].

If M^n is permitted to have a boundary, then the homogeneity conjecture for the closed n-cell D^n coincides with the n-dimensional *annulus conjecture*, which claims that the closed region between any two disjoint locally flat (n - 1)-spheres in S^n is homeomorphic to $S^{n-1} \times [0, 1]$. It is known [5, 6] that a solution of the annulus conjecture for dimensions less than or equal to n would yield a solution of the homogeneity conjecture for manifolds of dimension less than or equal to n. In this sense, the real problem is a purely local one, but all known solutions in dimensions greater than three ignore this fact and depend instead on some convenient global properties of the manifold M^n.

In this paper we present a technique, embodied in the following theorem, for showing that certain manifolds are homogeneous.

THEOREM 1.1. *Let* P^k *be a connected finite polyhedron, piecewise linearly embedded in the* n*-sphere* S^n, $2k + 2 \leq n$. *Let* N^n *denote the interior of a regular neighborhood of* P^k *in* S^n, *and* M^{n-1} *its boundary. Then*

$$S^n - P^k, \quad N^n, \quad M^{n-1} \times R^1, \quad \textit{and} \quad M^{n-1} \times S^1$$

are homogeneous manifolds.

Some easy consequences are:

(1) *If* $k \neq n + 1$, *then* $S^n \times R^k$ *is homogeneous.*

(2) *If* $1 = p_1 \leq p_2 \leq \cdots \leq p_r$ *and* $p_r \geq p_{r-1} + p_{r-2} + \cdots + p_1$, *then* $S^{p_1} \times S^{p_2} \times \cdots \times S^{p_r}$ *is homogeneous.*

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(3) *If* M^2 *is a closed connected orientable two-manifold and* $k \geq 4$, *then* $M^2 \times R^k$ *is homogeneous.*

(4) *If* M^n *is a closed, connected differentiable* n-*manifold, then for each* $k \geq n + 2$ *there exists a differentiable* R^k-*bundle over* M^n *which is homogeneous.*

2. DEFINITIONS

The set of points $\{(x_1, \cdots, x_n): \Sigma x_i^2 \leq 1\}$ in Euclidean n-space R^n will be denoted by D^n, and its boundary by S^{n-1}. The set D^n and any space homeomorphic to D^n will be called a *closed* n-*cell.* S^{n-1} and any space homeomorphic to S^{n-1} will be called an (n - 1)-*sphere.*

A k-manifold M^k in an n-manifold M^n will be said to be *locally flat* if each point of M^k has a neighborhood U in M^n such that the pair $(U, U \cap M^k)$ is topologically equivalent to the pair (R^n, R^k). An embedding $f: M^k \to M^n$ is *locally flat* if $f(M^k)$ is locally flat in M^n. An embedding $f: D^n \to M^n$ is *locally flat* if f/S^{n-1} is locally flat. Note that $f: D^n \to M^n$ is locally flat if and only if the closure of $M^n - f(D^n)$ is a manifold with boundary. From this point of view the local flatness of f is a minimal "reasonable" requirement.

$Hom(D^n, M^n)$ will denote the set of all locally flat embeddings of D^n into M^n, and $H(M^n)$ will denote the group of all homeomorphisms of M^n onto itself. If $h \in H(M^n)$ and $f \in Hom(D^n, M^n)$, then $hf \in Hom(D^n, M^n)$. In this sense, $H(M^n)$ acts as a transformation group on $Hom(D^n, M^n)$. A manifold M^n is homogeneous if and only if this action is transitive.

3. THE MACHINERY

The next two sections summarize some material from [4, 5, 6], where the proofs of the theorems stated below can be found.

Let f_0 and f_1 be elements of $Hom(D^n, M^n)$ such that $f_0(D^n)$ lies in the interior of $f_1(D^n)$. If there exists an embedding $F: S^{n-1} \times [0, 1] \to M^n$ such that, for all $x \in S^{n-1}$, $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$, then F will be called a *strict annular equivalence* between f_0 and f_1, and we write both

$$f_0 \underset{A}{\sim} f_1 \quad \text{and} \quad f_1 \underset{A}{\sim} f_0 .$$

Strict annular equivalence is not an equivalence relation, but it induces one as follows. Two elements f and f' of $Hom(D^n, M^n)$ will be said to be *annularly equivalent,* written

$$f \underset{a}{\sim} f' ,$$

if there exists a finite sequence of elements $f = f_0, f_1, \cdots, f_k = f'$ of $Hom(D^n, M^n)$ such that $f_i \underset{A}{\sim} f_{i+1}$ for $i = 0, 1, \cdots, k - 1$. Annular equivalence is an equivalence relation. The following theorem states an elementary property of annular equivalence; it appears as Lemma 3.1 in [6], where the proof may be found.

THEOREM 3.1. *Let* f *be an element of* $Hom(D^n, M^n)$ *and* U *an open set in* M^n. *Then there exists an element* f' *of* $Hom(D^n, M^n)$ *such that* $f'(D^n) \subset U$ *and* $f \underset{a}{\sim} f'$.

The principal structure theorem about annular equivalence (proved as Theorem 3.4 in [6]):

THEOREM 3.2. *Let* f *and* f' *be annularly equivalent elements of* $\mathrm{Hom}(D^n, M^n)$ *with disjoint images. Then there is a* $g \in \mathrm{Hom}(D^n, M^n)$ *such that*

$$f \underset{A}{\sim} g \underset{A}{\sim} f'.$$

If M^n is S^n or R^n, or more generally if M^n is any *stable* manifold (see [6] for definitions), we have a stronger result (included in Theorem 14.1 of [6]):

THEOREM 3.3. *Let* M^n *be a stable manifold, and* f *and* f' *elements of* $\mathrm{Hom}(D^n, M^n)$ *such that* $f(D^n) \subset \mathrm{Int}\, f'(D^n)$. *If* $f \underset{a}{\sim} f'$, *then* $f \underset{A}{\sim} f'$.

4. MORE MACHINERY

Let h be a homeomorphism of M^n onto itself. If there exists a nonempty open set $U \subset M^n$ such that $h/U = 1$, we say that h is *somewhere the identity*. If there exists a closed n-cell E with locally flat boundary in M^n such that $h/M^n - E = 1$, we say that h is *almost everywhere the identity*.

$SH(M^n)$, the group of *stable homeomorphisms* of M^n, will consist of products of homeomorphisms, each of which is somewhere the identity. $SH_0(M^n)$ will consist of products of homeomorphisms, each of which is almost everywhere the identity.

Now let $f_1, f_2 \in \mathrm{Hom}(D^n, M^n)$. If there exists a stable homeomorphism $h \in SH(M^n)$ such that $hf_1 = f_2$, then we say that f_1 and f_2 are stably equivalent, and write

$$f_1 \underset{s}{\sim} f_2.$$

This is an equivalence relation, and the set of stable equivalence classes of elements of $\mathrm{Hom}(D^n, M^n)$ will be denoted by

$$\mathrm{Hom}_s(D^n, M^n).$$

Since $SH(M^n)$ is a normal subgroup of $H(M^n)$, $H(M^n)$ acts on $\mathrm{Hom}(D^n, M^n)$ by permuting the stable equivalence classes, and therefore it induces an action of $H(M^n)$ on $\mathrm{Hom}_s(D^n, M^n)$.

The description of this action that is given in the next theorem follows immediately.

THEOREM 4.1. *If an element of* $H(M^n)$ *leaves one stable equivalence class of* $\mathrm{Hom}(D^n, M^n)$ *fixed, it is an element of* $SH(M^n)$ *and therefore leaves all stable equivalence classes fixed. Hence* $H(M^n)/SH(M^n)$ *acts as a regular permutation group on* $\mathrm{Hom}_s(D^n, M^n)$, *and it is therefore in one-to-one correspondence with a subset of* $\mathrm{Hom}_s(D^n, M^n)$.

By the very definition of stable equivalence, $SH(M^n)$, and therefore surely $H(M^n)$, acts transitively on any single stable equivalence class of $\mathrm{Hom}(D^n, M^n)$. Hence the action of $H(M^n)$ on $\mathrm{Hom}(D^n, M^n)$ is transitive if and only if the action of $H(M^n)/SH(M^n)$ on $\mathrm{Hom}_s(D^n, M^n)$ is transitive. Therefore we have the following proposition.

THEOREM 4.2. M^n *is homogeneous if and only if* $H(M^n)/SH(M^n)$ *acts transitively on* $\mathrm{Hom}_s(D^n, M^n)$.

We close this section with the following information, which appears as Theorems 5.4 and 5.5 of [6], and which will be used repeatedly in this paper.

THEOREM 4.3. *Two elements of* $\mathrm{Hom}(D^n, M^n)$ *are stably equivalent if and only if they are annularly equivalent. Furthermore, if* f *and* f' *are stably equivalent, then there exists a homeomorphism* $h \in SH_0(M^n)$ *such that* $hf = f'$.

5. INHERITANCE OF HOMOGENEITY

If M^n is a connected topological manifold that is known to be homogeneous, and U is a connected open subset of M^n, it is not known in general whether U must be homogeneous. Later, we will try to show that certain open subsets of a homogeneous manifold are themselves homogeneous, and this section sets the stage for such an attempt.

Let

$$i: U \subset M^n$$

denote the inclusion map.

If f is an element of $\mathrm{Hom}(D^n, U)$, then f may also be considered an element of $\mathrm{Hom}(D^n, M^n)$. If f and f' are stably equivalent elements of $\mathrm{Hom}(D^n, U)$, then by Theorem 4.3, f and f' are annularly equivalent in U. This annular equivalence in U is *a fortiori* an annular equivalence in M^n, hence again by Theorem 4.3, f and f' are stably equivalent in M^n. We therefore get a natural map

$$i_*: \mathrm{Hom}_s(D^n, U) \to \mathrm{Hom}_s(D^n, M^n).$$

By Theorem 3.1, i_* is onto, but it may not be one-to-one, for example, if U is orientable while M^n is non-orientable.

Suppose now that $h \in H(U)$ and $f \in \mathrm{Hom}(D^n, U)$. Since M is homogeneous, there exists an element $H \in H(M^n)$ such that

$$Hf = hf.$$

That is, H and h agree on $f(D^n)$. The homeomorphism H is not uniquely determined by these conditions, but the coset of H in $H(M^n)/SH(M^n)$ clearly is. We therefore have a well-defined map

$$j: H(U) \times \mathrm{Hom}(D^n, U) \to H(M^n)/SH(M^n)$$

with the following properties:

(1) $j(h_2 h_1, f) = j(h_2, h_1 f) \cdot j(h_1, f)$,

(2) if $h \in SH(U)$, then $j(h, f) = SH(M^n)$, by Theorems 4.1 and 4.3,

(3) if $\mathrm{Int}\, f_1(D^n) \cap \mathrm{Int}\, f_2(D^n) \neq \emptyset$, then $j(h, f_1) = j(h, f_2)$.

Property (3) and the connectedness of U imply that $j(h, f)$ is independent of f, so define

$$j: H(U) \to H(M^n)/SH(M^n)$$

$$j(h) = j(h, f) \quad \text{for each } f \in \text{Hom}(D^n, U).$$

Properties (1) and (2) then indicate that the new j is a group homomorphism whose kernel includes SH(U), so we get an induced homomorphism

$$j_*: H(U)/SH(U) \to H(M^n)/SH(M^n).$$

By the definitions of i_* and j_*, the following diagram is commutative.

$$\begin{array}{ccccc} H(M^n)/SH(M^n) & \times & \text{Hom}_s(D^n, M^n) & \to & \text{Hom}_s(D^n, M^n) \\ j_* \uparrow & & i_* \uparrow & & i_* \uparrow \\ H(U)/SH(U) & \times & \text{Hom}_s(D^n, U) & \to & \text{Hom}_s(D^n, U). \end{array}$$

Concerning this diagram, we already have the following information:

(1) The upper action is regular.

(2) The lower action is regular.

(3) The upper action is transitive, that is, M^n is homogeneous.

(4) j_* is a group homomorphism.

(5) i_* is onto.

The following questions are suggested by the diagram:

(a) Is the lower action transitive, in other words, is U homogeneous?

(b) Is j_* onto?

(c) Is j_* one-to-one?

(d) Is i_* one-to-one?

These questions are related by the following lemmas.

LEMMA 5.1. *If* i_* *is one-to-one, then* j_* *is also one-to-one.*

Suppose $j_*([h]) = SH(M^n)$. Choose $f \in \text{Hom}(D^n, U)$. Then

$$j_*([h]) = j(h, f) = SH(M^n) = [H],$$

where $Hf = hf$. Since $H \in SH(M^n)$, f and hf are stably equivalent in $\text{Hom}(D^n, M^n)$. But i_* is one-to-one, hence they must already be stably equivalent in $\text{Hom}(D^n, U)$. By Theorem 4.1, $h \in SH(U)$, and therefore j_* is one-to-one.

LEMMA 5.2. *If the lower action is transitive (that is, if* U *is homogeneous), then* j_* *is onto.*

Let H be a given element of $H(M^n)$, and choose an $f \in \text{Hom}(D^n, U)$. Choose $H' \in H(M^n)$ so that $[H'] = [H]$ and so that $H'f \in \text{Hom}(D^n, U)$. Since U is homogeneous, there is an $h \in H(U)$ such that $hf = H'f$. But then

$$j_*([h]) = j(h, f) = [H'] = [H],$$

so j_* is onto.

LEMMA 5.3. *If* i_* *is one-to-one and* j_* *is onto, then the lower action is transitive.*

The hypotheses and Lemma 5.1 imply that both i_* and j_* are one-to-one and onto. The diagram then provides an isomorphism between the lower and upper actions. Since the upper action is transitive, the lower action must also be transitive.

LEMMA 5.4. *If* M^n *is a stable manifold, then* i_* *is one-to-one.*

This is part of Theorem 14.1 of [6]. Combining Lemmas 5.1 through 5.4, we obtain the following proposition.

THEOREM 5.5. *Let* M^n *be a homogeneous stable manifold, and* U *a connected open subset. Then* U *is homogeneous if and only if* j_* *is onto.*

If U is homogeneous, then the lower action is transitive, and hence j_* is onto by Lemma 5.2.

Suppose that j_* is onto. Since M^n is stable, i_* is one-to-one by Lemma 5.4. Then the lower action is transitive by Lemma 5.3, hence U is homogeneous.

6. RELATION BETWEEN THE HOMOGENEITY PROBLEMS FOR $M^{n-1} \times R^1$ AND $M^{n-1} \times S^1$

Let M^{n-1} be a compact, connected (n - 1)-manifold without boundary. R^1 will denote the real numbers, and S^1 the one-sphere parametrized by the reals modulo 1. If $t \in R^1$, then $[t] \in S^1$ will denote the equivalence class of reals congruent to t(mod 1).

Define

$$p: M^{n-1} \times R^1 \to M^{n-1} \times S^1$$

by

$$p(x, t) = (x, [t]) .$$

Then p is a covering map that exhibits $M^{n-1} \times R^1$ as a covering space over $M^{n-1} \times S^1$.

The map

$$\tau: M^{n-1} \times R^1 \to M^{n-1} \times R^1$$

defined by

$$\tau(x, t) = (x, t + 1)$$

generates the group of covering translations of $M^{n-1} \times R^1$.

The goal of the next six sections is to prove the following result.

THEOREM 6.1. *Let* M^{n-1} *be a compact, connected* (n - 1)*-manifold without boundary. If* $M^{n-1} \times R^1$ *is homogeneous, then so is* $M^{n-1} \times S^1$ *homogeneous.*

7. DEFINITION OF p^*

LEMMA 7.1. *The covering translation* τ *is a stable homeomorphism of* $M^{n-1} \times R^1$.

Define $\tau_1 : M^{n-1} \times R^1 \to M^{n-1} \times R^1$ by

$$\tau_1(x, t) = \begin{cases} (x, t) & \text{for } t \leq -1, \\ (x, 2t+1) & \text{for } -1 \leq t \leq 0, \\ (x, t+1) & \text{for } t \geq 0. \end{cases}$$

Define $\tau_2 : M^{n-1} \times R^1 \to M^{n-1} \times R^1$ by

$$\tau_2(x, t) = \begin{cases} (x, t+1) & \text{for } t \leq -1, \\ (x, (t+1)/2) & \text{for } -1 \leq t \leq 1, \\ (x, t) & \text{for } t \geq 1. \end{cases}$$

Direct computation verifies that

$$\tau = \tau_2 \tau_1 .$$

But τ_1 and τ_2 each restrict to the identity on open subsets of $M^{n-1} \times R^1$, and are therefore stable. Hence τ, and therefore every covering translation, is stable.

Suppose now that f is an element of $\mathrm{Hom}(D^n, M^{n-1} \times S^1)$. Then f lifts to an embedding $\tilde{f}: D^n \to M^{n-1} \times R^1$, that is, there exists an element

$$\tilde{f} \in \mathrm{Hom}(D^n, M^{n-1} \times R^1)$$

such that $p\tilde{f} = f$. The embedding $\tilde{f}$ is well-defined up to composition with a covering translation, which must be stable by Lemma 7.1, and therefore determines a unique element of $\mathrm{Hom}_s(D^n, M^{n-1} \times R^1)$. If f and f' are stably equivalent elements of $\mathrm{Hom}(D^n, M^{n-1} \times S^1)$, then by Theorem 4.3, f and f' are annularly equivalent in $M^{n-1} \times S^1$. This annular equivalence lifts directly to an annular equivalence between *some* coverings $\tilde{f}$ and $\tilde{f}'$ of f and f'. Then again by Theorem 4.3, $\tilde{f}$ and $\tilde{f}'$ are stably equivalent elements of $\mathrm{Hom}(D^n, M^{n-1} \times R^1)$. This lifting procedure therefore induces a map

$$p^*: \mathrm{Hom}_s(D^n, M^{n-1} \times S^1) \to \mathrm{Hom}_s(D^n, M^{n-1} \times R^1) .$$

8. DEFINITION OF q^*

The homeomorphism $\tilde{h} \in H(M^{n-1} \times R^1)$ is said to *cover* the homeomorphism $h \in H(M^{n-1} \times S^1)$, and h is said to *lift* to $\tilde{h}$, if

$$p\tilde{h} = hp .$$

If $\tilde{h}_1$ and $\tilde{h}_2$ both cover h, there exists an integer k such that

$$\tilde{h}_2 = \tau^k \tilde{h}_1 .$$

Since τ is stable, this means that the coset of $\tilde{h}$, $\tilde{h} \cdot SH(M^{n-1} \times R^1)$, is well-determined by h.

Let

$$HL(M^{n-1} \times S^1)$$

denote the subgroup of $H(M^{n-1} \times S^1)$ consisting of homeomorphisms that can be lifted to homeomorphisms of $M^{n-1} \times R^1$. Furthermore, let

$$SHL(M^{n-1} \times S^1) = HL(M^{n-1} \times S^1) \cap SH(M^{n-1} \times S^1).$$

So far, the lifting procedure furnishes a group homomorphism

$$HL(M^{n-1} \times S^1) \to H(M^{n-1} \times R^1) / SH(M^{n-1} \times R^1).$$

Suppose now that $h \in SHL(M^{n-1} \times S^1)$ is covered by $\tilde{h}$. Let

$$f \in Hom(D^n, M^{n-1} \times S^1)$$

be covered by $\tilde{f} \in Hom(D^n, M^{n-1} \times R^1)$. Then $\tilde{h}\tilde{f}$ covers hf. Then the fact that f and hf are stably equivalent in $Hom(D^n, M^{n-1} \times S^1)$ implies, according to Section 7, that $\tilde{f}$ and $\tilde{h}\tilde{f}$ are stably equivalent in $Hom(D^n, M^{n-1} \times R^1)$. But then $\tilde{h}$ is stable by Theorem 4.1. Therefore $SHL(M^{n-1} \times S^1)$ lies in the kernel of the above homomorphism, and we obtain an induced homomorphism

$$q^*: \frac{HL(M^{n-1} \times S^1)}{SHL(M^{n-1} \times S^1)} \to \frac{H(M^{n-1} \times R^1)}{SH(M^{n-1} \times R^1)}.$$

9. THE DIAGRAM

In the spirit of Section 5, we draw the following diagram which, by the very definitions of p^* and q^*, is commutative.

$$\begin{array}{ccccc} \dfrac{H(M^{n-1} \times R^1)}{SH(M^{n-1} \times R^1)} & \times & Hom_s(D^n, M^{n-1} \times R^1) & \to & Hom_s(D^n, M^{n-1} \times R^1) \\ q^* \uparrow & & p^* \uparrow & & p^* \uparrow \\ \dfrac{HL(M^{n-1} \times S^1)}{SHL(M^{n-1} \times S^1)} & \times & Hom_s(D^n, M^{n-1} \times S^1) & \to & Hom_s(D^n, M^{n-1} \times S^1) \end{array}$$

So far we have only the following information about this diagram:

(1) The upper action is regular.

(2) The lower action is regular.

(3) q^* is a group homomorphism.

The following questions are suggested:

(a) Is the upper action transitive, that is, is $M^{n-1} \times R^1$ homogeneous?

(b) Is the lower action transitive?

(c) Is p^* one-to-one?

(d) Is p^* onto?

(e) Is q^* one-to-one?

(f) Is q^* onto?

10. THE MAP p^* IS BIJECTIVE

THEOREM 10.1. *The map* p^* *is one-to-one.*

Let $f, f' \in \mathrm{Hom}(D^n, M^{n-1} \times S^1)$ have stably equivalent liftings $\tilde{f}, \tilde{f}'$. By Theorem 3.1, we may assume that $f(D^n)$ and $f'(D^n)$ lie in $M^{n-1} \times (0, 1) \subset M^{n-1} \times S^1$, in which case we can choose $\tilde{f}$ and $\tilde{f}'$ so that $\tilde{f}(D^n)$ and $\tilde{f}'(D^n)$ lie in

$$M^{n-1} \times (0, 1) \subset M^{n-1} \times R^1 .$$

By Theorem 4.3, $\tilde{f}$ and $\tilde{f}'$ are annularly equivalent in $M^{n-1} \times R^1$. The annular equivalence can clearly be compressed within $M^{n-1} \times (0, 1) \subset M^{n-1} \times R^1$, and then projected down to an annular equivalence between f and f'. Again by Theorem 4.3, f and f' are stably equivalent, so that p^* is one-to-one.

THEOREM 10.2. *The map* p^* *is onto.*

By Theorems 3.1 and 4.3, any element of $\mathrm{Hom}(D^n, M^{n-1} \times R^1)$ is stably equivalent to an element whose image lies in $M^{n-1} \times (0, 1)$, which can then be projected down to an element of $\mathrm{Hom}(D^n, M^{n-1} \times S^1)$. Hence p^* is onto.

11. THE HOMOMORPHISM q^* IS BIJECTIVE

THEOREM 11.1. *The homomorphism* q^* *is one-to-one.*

Suppose $h \in HL(M^{n-1} \times S^1)$ lifts to $\tilde{h} \in SH(M^{n-1} \times R^1)$. Choose any $f \in \mathrm{Hom}(D^n, M^{n-1} \times S^1)$ and lift f to $\tilde{f} \in \mathrm{Hom}(D^n, M^{n-1} \times R^1)$. Then $\widetilde{hf}$ covers hf. Since $\tilde{h}$ is stable, $\tilde{f}$ and $\widetilde{hf}$ are stably equivalent. By Theorem 10.1, f and hf are stably equivalent. Therefore h is stable by Theorem 4.1. Hence q^* is one-to-one.

LEMMA 11.2. *Let* H *be a homeomorphism of* $M^{n-1} \times R^1$ *onto itself such that* $H(M^{n-1} \times 0) \subset M^{n-1} \times (0, 1)$. *Then there exists a homeomorphism* H' *of* $M^{n-1} \times R^1$ *onto itself such that*

$$H'/M^{n-1} \times 0 = H/M^{n-1} \times 0$$

and

$$H'\tau = \tau H'.$$

It will be sufficient to construct an embedding

$$H' \colon M^{n-1} \times [0, 1] \to M^{n-1} \times R^1$$

such that

$$H'(x, 0) = H(x, 0)$$

and

$$H'(x, 1) = \tau H'(x, 0)$$

for all $x \in M^{n-1}$, for then the homeomorphism H' will be the obvious extension of the embedding H'.

Using the homeomorphism H, construct first an embedding

$$G_1: M^{n-1} \times [-1, 2] \to M^{n-1} \times R^1$$

such that

(1) $G_1(M^{n-1} \times 0)$ lies to the left of $M^{n-1} \times -1$,

(2) $G_1(x, 1) = H(x, 0)$ for all $x \in M^{n-1}$,

(3) $G_1(M^{n-1} \times 2)$ lies between $H(M^{n-1} \times 0)$ and $M^{n-1} \times 1$.

Next, using the embedding G_1, construct a homeomorphism

$$G_2: M^{n-1} \times R^1 \to M^{n-1} \times R^1$$

such that

(4) $G_2 G_1(x, 0) = G_1(x, 1) = H(x, 0)$ for all $x \in M^{n-1}$,

(5) G_2 restricts to the identity outside $G_1(M^{n-1} \times [-1, 2])$.

Finally, define

$$H' = G_2 \tau_1 G_1 / M^{n-1} \times [0, 1].$$

Then H' is an embedding such that

$$H'(x, 0) = G_2 \tau_1 G_1(x, 0) = G_2 G_1(x, 0) = H(x, 0)$$

and

$$H'(x, 1) = G_2 \tau_1 G_1(x, 1) = G_2 \tau_1 H(x, 0) = G_2 \tau H(x, 0)$$
$$= \tau H(x, 0) = \tau H'(x, 0).$$

This completes the proof of the lemma.

THEOREM 11.3. *The homomorphism* q^* *is onto.*

We must show that any homeomorphism H of $M^{n-1} \times R^1$ onto itself can be modified by a stable homeomorphism so that the result covers a homeomorphism of $M^{n-1} \times S^1$. Since the homeomorphism that sends (x, t) onto $(x, -t)$ already covers a homeomorphism of $M^{n-1} \times S^1$, we may assume that H does not interchange the ends of the space $M^{n-1} \times R^1$.

Now, since M^{n-1} is compact, first modify H by a stable "compression" of $M^{n-1} \times R^1$ so that $H(M^{n-1} \times 0) \subset M^{n-1} \times (0, 1)$. Now compare H with the homeomorphism H' whose existence is asserted by Lemma 11.2. $H^{-1} H'$ restricts to the identity on $M^{n-1} \times 0$ and does not interchange the ends of $M^{n-1} \times R^1$. It is a stable homeomorphism, for it can be written as the product of a homeomorphism that is the identity to the left of $M^{n-1} \times 0$ and agrees with $H^{-1} H'$ to the right, and a further homeomorphism that is the identity to the right of $M^{n-1} \times 0$ and agrees with $H^{-1} H'$

to the left. But the relation $H'\tau = \tau H'$ implies that H' covers a homeomorphism of $M^{n-1} \times S^1$. Therefore q^* is onto.

12. PROOF OF THEOREM 6.1

Theorems 10.1, 10.2, 11.1 and 11.3 show that the diagram of Section 9 provides an isomorphism between the upper and lower actions. If $M^{n-1} \times R^1$ is homogeneous, then the upper action is transitive by Theorem 4.2. Then the lower action must also be transitive. *A fortiori*, $H(M^{n-1} \times S^1) / SH(M^{n-1} \times S^1)$ acts transitively on $Hom_s(D^n, M^{n-1} \times S^1)$, so again by Theorem 4.2, $M^{n-1} \times S^1$ is homogeneous.

13. A THEOREM ABOUT $M^{n-1} \times R^1$

Let M^{n-1} be a compact, connected (n - 1)-manifold without boundary, and $f: M^{n-1} \to M^{n-1} \times R^1$ a locally flat embedding. The following three conditions are clearly equivalent.

(1) There exists a homeomorphism h of $M^{n-1} \times R^1$ onto itself such that $h(x, 0) = f(x)$ for all $x \in M^{n-1}$.

(2) There exists a homeomorphism h of $M^{n-1} \times R^1$ onto itself such that $h(M^{n-1} \times 0) = f(M^{n-1})$.

(3) $(M^{n-1} \times R^1) - f(M^{n-1})$ is a union of two open sets, each of whose closures is homeomorphic to $M^{n-1} \times [0, \infty)$.

The following theorem is useful.

THEOREM 13.1. *Let* M *be a locally flat copy of* M^{n-1} *in* $M^{n-1} \times R^1$ *with complementary domains* U *and* V. *If* $\overline{U}$ *is homeomorphic to* $M^{n-1} \times [0, \infty)$, *then so is* $\overline{V}$.

Call a copy of M^{n-1} in $M^{n-1} \times R^1$ *normal* if there exists a homeomorphism of $M^{n-1} \times R^1$ onto itself that takes this copy onto $M^{n-1} \times 0$.

According to [3], some closed neighborhood A of M in $\overline{V}$ is homeomorphic to $M^{n-1} \times [0, 1]$. Let M' denote the other boundary of A. Note that $U \cup A$ is homeomorphic to $\overline{U}$, and hence to $M^{n-1} \times [0, \infty)$. Note also that U must contain one of the ends of the space $M^{n-1} \times R^1$, and that therefore there are plenty of normal copies of M^{n-1} in U. In fact, if W is any neighborhood of M' in A, then normal copies of M^{n-1} can be found in W - M'. Simply take a normal copy of M^{n-1} in U, and, using the fact that $U \cup A$ is homeomorphic to $M^{n-1} \times [0, \infty)$, push it close to M' by a homeomorphism of $M^{n-1} \times R^1$ that restricts to the identity on M'.

To show that $\overline{V}$ is homeomorphic to $M^{n-1} \times [0, \infty)$, we will construct a homeomorphism $h: A - M' \to \overline{V}$. To do this, let $M_1, M_2, \cdots$ be an infinite sequence of normal copies of M^{n-1} in Int A, such that M_{i+1} lies to the «right» (assuming that U contains the left end and V the right end of $M^{n-1} \times R^1$) of M_i and such that the M_i converge setwise to M'. Since M_1 is a normal copy of M^{n-1}, some homeomorphism h_1 of $\overline{V}$ onto itself pushes M_1 far to the right but restricts to the identity of M. Inductively, let $h_{i+1} = h_i$ to the left of M_i and push M_{i+1} even further to the right, so that $h_i(M_i)$ lies to the right of $M^{n-1} \times i$. Then the h_i converge to a homeomorphism $h: A - M' \to \overline{V}$, and this proves the theorem.

COROLLARY 1. *Let* X *be a topological manifold with compact, connected boundary* M, *such that* X - M *is homeomorphic to* $M \times (0, \infty)$. *Then* X *is homeomorphic to* $M \times [0, \infty)$.

Add $M \times (-\infty, 0]$ to X by matching $M \times 0$ with M; this yields the space X'. Note that X' must be homeomorphic to X - M, that is, to $M \times R^1$. Then apply Theorem 13.1.

COROLLARY 2. *Let* C *be a closed* n-*cell with locally flat boundary in the* n-*sphere* S^n. *Then the closure of* S^n - C *is also a closed* n-*cell. Therefore* S^n *is homogeneous.*

Remove a point from the interior of C and another point from S^n - C, obtaining a space homeomorphic to $S^{n-1} \times R^1$, and then apply Theorem 13.1.

REMARK. The proof of Theorem 13.1 is an adaptation of a technique exploited by Brown in [2]. Corollary 2, therefore, comes as no surprise.

14. PROOF OF THEOREM 1.1

We are going to show that certain connected open subsets U of S^n are homogeneous. Since S^n is homogeneous [2, 3], Theorem 5.5 applies, and we need only show that

$$j_*: H(U)/SH(U) \to H(S^n)/SH(S^n)$$

is onto. To do this, we will start with an arbitrary homeomorphism H of S^n, modify it by stable homeomorphisms to obtain H', and then find a homeomorphism h of U onto itself that agrees with H' on some open set. In the simplest cases, H' will already take U onto itself, so we can let h = H'.

The tool for modifying H to H' will be the following theorem of Homma [9].

HOMMA'S THEOREM. *Let* M^n, $\tilde{M}^n$ *and* $\tilde{P}^k$ *be two finite combinatorial* n-*manifolds and a finite polyhedron such that* $\tilde{M}^n$ *is topologically embedded in* M^n, $\tilde{P}^k$ *is piecewise linearly embedded in* Int $\tilde{M}^n$ *and* $2k + 2 \leq n$. *Then for each* $\varepsilon > 0$ *there exists an* ε-*homeomorphism* F *of* M^n *onto* M^n *such that*

$$F/M^n - U_\varepsilon(\tilde{P}^k) = 1\,,$$

$F/\tilde{P}^k$ *is piecewise linear.*

Now let P^k be a connected finite polyhedron, piecewise linearly embedded in the n-sphere S^n, $2k + 2 \leq n$, as in the hypothesis of Theorem 1.1. Let H be any homeomorphism of S^n onto itself.

To apply Homma's theorem, let

$M^n = S^n$ with its given piecewise linear structure,

$\tilde{M}^n = S^n$ with the piecewise linear structure induced by H,

$\tilde{P}^k = H(P^k)$ with a triangulation carried over from P^k by H.

Since P^k was piecewise linearly embedded in $S^n = M^n$, $\tilde{P}^k$ is piecewise linearly embedded in $\tilde{M}^n$. Now by Homma's theorem, there exists an ε-homeomorphism H_1 of S^n onto itself such that $H_1/\tilde{P}^k$ is piecewise linear, that is, such that $H_1 H/P^k$ is piecewise linear. For small ε, H_1 is somewhere the identity and therefore stable.

Now the inclusion $P^k \subset S^n$ and $H_1 H/P^k$ are homotopic piecewise linear embeddings. Since $2k + 2 \leq n$, there exists an orientation-preserving piecewise linear homeomorphism H_2 of S^n onto itself such that $H_2 H_1 H/P^k$ is the inclusion, by [8]. And H_2 is stable by [5].

Therefore $H' = H_2 H_1 H$ is a stable modification of H such that $H'/P^k = 1$. In particular, $H'(S^n - P^k) = (S^n - P^k)$. Since H was arbitrary, this means that

$$j_*: H(S^n - P^k) / SH(S^n - P^k) \to H(S^n)/SH(S^n)$$

is onto, or equivalently that $S^n - P^k$ is homogeneous. This is the first part of Theorem 1.1.

Now let N^n be the interior of a regular neighborhood of P^k in S^n, with boundary M^{n-1}. Then $N^n - P^k$ is homeomorphic to $M^{n-1} \times R^1$, [13]. Let $\overline{U^n}$ be another regular neighborhood of P^k, concentric with $\overline{N^n}$, and chosen so small that both $\overline{U^n}$ and $H'(\overline{U^n})$ lie in N^n. Then $N^n - U^n$ is homeomorphic to $M^{n-1} \times [0, \infty)$.

In order to construct a homeomorphism h of N^n onto itself that agrees with H' on $\overline{U^n}$, it is necessary and sufficient that $N^n - H'(U^n)$ be homeomorphic to $M^{n-1} \times [0, \infty)$. But this is now implied by Theorem 13.1 and the fact that $H'(\overline{U^n}) - P^k$ is homeomorphic to $M^{n-1} \times [0, \infty)$. The existence of the homeomorphism h then implies that

$$j_*: H(N^n)/SH(N^n) \to H(S^n)/SH(S^n)$$

and

$$j_*: H(N^n - P^k)/SH(N^n - P^k) \to H(S^n)/SH(S^n)$$

are onto, or equivalently that N^n and $N^n - P^k$ are homogeneous. Recalling that $N^n - P^k$ is homeomorphic to $M^{n-1} \times R^1$, we obtain the next two parts of Theorem 1.1.

Finally, Theorem 6.1 and the homogeneity of $M^{n-1} \times R^1$ yield the homogeneity of $M^{n-1} \times S^1$. This completes the proof of Theorem 1.1.

15. SOME APPLICATIONS

THEOREM 15.1. *If* $k \neq n + 1$ *then* $S^n \times R^k$ *is homogeneous.*

First embed S^n in S^{n+k}, $k \geq n + 2$, and apply Theorem 1.1 to the interior of a regular neighborhood of S^n. This gives the present theorem for $k \geq n + 2$.

Next embed S^{k-1} in S^{n+k}, $k \leq n$. Then $2(k - 1) + 2 \leq n + k$, and Theorem 1.1 applied to $S^{n+k} - S^{k-1} \approx S^n \times R^k$ completes the argument.

THEOREM 15.2. *If* $1 = p_1 \leq p_2 \leq \cdots \leq p_r$ *and* $p_r \geq p_{r-1} + p_{r-2} + \cdots + p_1$, *then* $S^{p_1} \times S^{p_2} \times \cdots \times S^{p_r}$ *is homogeneous.*

Embed $S^{p_2} \times S^{p_3} \times \cdots \times S^{p_{r-1}}$ in S^{Σ}, where $\Sigma = p_1 + p_2 + p_3 + \cdots + p_r$. Then $\Sigma \geq 2(p_2 + p_3 + \cdots + p_{r-1}) + 2$, and the boundary of a regular neighborhood of $S^{p_2} \times S^{p_3} \times \cdots \times S^{p_{r-1}}$ is homeomorphic to $S^{p_2} \times S^{p_3} \times \cdots S^{p_r}$. The fourth case of Theorem 1.1 then completes the proof.

THEOREM 15.3. *If* M^2 *is a closed connected orientable two-manifold and* $k \geq 4$, *then* $M^2 \times R^k$ *is homogeneous.*

This follows from the fact that the interior of a regular neighborhood of M^2 in S^{k+2} is homeomorphic to $M^2 \times R^k$.

THEOREM 15.4. *If* M^n *is a closed connected differentiable* n-*manifold, then for each* $k \geq n + 2$ *there exists a differentiable* R^k-*bundle over* M^n *which is homogeneous.*

Triangulate S^{n+k} so that a differentiable embedding of M^n into S^{n+k} appears piecewise linear. The corresponding normal bundle is homeomorphic to the interior of a regular neighborhood of M^n, and the theorem then follows from the second case in Theorem 1.1.

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