COHOMOLOGY FIBRE SPACES, THE SMITH-GYSIN SEQUENCE, AND ORIENTATION IN GENERALIZED MANIFOLDS

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As indicated by the title, this paper concerns three related, but largely independent, topics. In Section I we study the concept of what we call a cohomology fibre space. Very roughly speaking, this is to Čech Cohomology as a fibre space is to singular cohomology. Non-trivial examples arise in the theory of transformation groups, as mentioned in Section I and elsewhere in the paper, and it is mainly these examples that motivate the generality of the definition.

In Section II we derive an exact sequence for a very general notion of a sphere fibration with singularities. The sequence, which we call the Smith-Gysin sequence, is an analogue of the Smith sequences and a generalization of the Gysin sequence. Although Section I is largely intended to provide a proper foundation for applications of the Smith-Gysin sequence, Definition 1.1 is all that is needed to read Section II.

In Section III we show that every cohomology n-manifold M over a Dedekind domain $L(n\text{-cm}_L)$ has an orientable double covering (the emphasis is on "double") and that every self-homeomorphism of an orientable $n\text{-cm}_L$ either preserves or reverses orientation. These facts are almost obvious if $L = Z$, since Z has only two automorphisms (as an abelian group), but they are by no means clear for general L ($L = Z_p$, for example) if M is not an $n\text{-cm}_Z$. These facts are obtained from the general Poincaré duality of [3] by showing that the tensor product of the orientation sheaf of any $n\text{-cm}_L$ with itself is a constant sheaf. Section III can be read independently of the rest of the paper except for the last two results which need Definition 1.1.

In Section IV we consider the analogue of the local Smith theorem for sphere fibrations with singularities.

Except for Section III, we shall always take either the integers Z or a field for the coefficient domain L. This is done only for simplicity and it would certainly suffice for L to be a principal ideal domain. For notation, definitions and simple properties of cohomology manifolds and cohomology dimension we refer the reader to [2; Chapter I]; also see [4]. R denotes the field of rational numbers. A map is said to be proper if the inverse image of a compact set is compact.

I. COHOMOLOGY FIBRE SPACES

DEFINITION 1.1. *A cohomology fibre space over* L (*abbreviated* cfs_L) *is a triple* (X, B, π) *where* X *and* B *are locally compact spaces and* π *is an open, proper and continuous map of* X *onto* B *such that the Leray sheaf* $\mathscr{L}^*$ *of* π *is locally constant with* $\mathscr{L}^0$ *the constant sheaf* L. *We say that it is* k-regular (*and call it a* k-cfs_L) *if, moreover, the stalks of* $\mathscr{L}^*$ *are finitely generated,* $\mathscr{L}^i = 0$ *for* $i > k$, *and the stalks of* $\mathscr{L}^k$ *are isomorphic to* L.

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Examples:

(1) If G is a compact connected Lie group acting on a space X with all isotropy groups finite, then $(X, X/G, \pi)$ is a k - cfs_R where π is the orbit map.

(2) Let Σ be a solenoid (inverse limit of circle groups) acting freely on a space X with orbit space B and orbit map π. Then $(X, B\ \pi)$ is a cfs_Z but is not regular since, although $\mathscr{L}^*$ is constant, $\mathscr{L}^1 \approx R$ and is not finitely generated. However, it is a 1-cfs_R. (See [6].) The latter is true even if Σ does not act freely but has no stationary points.

If (X, B, π) is a cfs and $b \in B$, we denote by F_b the set $\pi^{-1}(b)$. Note that the stalks of $\mathscr{L}^*$ are $H^*(F_b; L)$.

We shall let $k = \max\{i \mid \mathscr{L}^i \neq 0\}$ which will be finite for all the cases we consider.

THEOREM 1.2. *Let* (X, B, π) *be a* cfs_L *with* $\dim_L X = n < \infty$. *If* L *is a field, then* $\dim_L B \leq n - k$. *If* $L = Z$, *then* $\dim_L B \leq n + 1$.

Proof. Let L be a field. We shall first show that if $b \in B$ then there exists a neighborhood A_b of b in B such that if $C \subset A_b$ is any compact set, then $H^i(C; L) = 0$ for $i > n - k$. We may assume that $\mathscr{L}^*$ is constant.

Since $\dim_L X = n < \infty$, $\dim_L F_b \leq n$. Let α be a non-zero element of $H^k(F_b; L)$, and (using the constancy of $\mathscr{L}^k$) let α_c be the element corresponding to α in $H^k(F_c; L)$ for all $c \in B$. Since $\mathscr{L}^*$ is constant, there is a compact neighborhood A_b of b such that

$$\alpha_c \in \operatorname{Im}(H^k(\pi^{-1}(A_b)) \to H^k(F_c))$$

for every $c \in A_b$. Thus $\alpha_c \in \operatorname{Im}(H^k(\pi^{-1}(C)) \to H^k(F_c))$ for every compact set $C \subset A_b$ with $c \in C$.

Consider the spectral sequence of $\pi: \pi^{-1}(C) \to C$. The above remarks show that $1 \otimes \alpha_c \in H^0(C) \otimes H^k(F_c) \approx E_2^{0,k}$ is a permanent cocycle of this spectral sequence. Thus for any $\beta \in H^i(C; L)$, $\beta \otimes \alpha_c$ is a permanent cocycle. $\beta \otimes \alpha_c$ cannot be a coboundary, since $H^j(F_c) = 0$ for $j > k$. It follows that $i + k \leq n$ and hence that $H^j(C; L) = 0$ for $j > n - k$ for all compact sets $C \subset A_b$. (For a justification of these arguments see [2; XVI, Section 1].)

Now let $V \subset A_b$ be an open subset of B. Then $H^j(\overline{V}) = 0$ for $j > n - k$ and $H^j(\overline{V} - V) = 0$ for $j > n - k$. From the exact cohomology sequence

$$\cdots \to H_c^j(V) \to H^j(\overline{V}) \to H^j(\overline{V} - V) \to H_c^{j+1}(V) \to \cdots$$

it follows that $H_c^j(V) = 0$ for $j > n - k + 1$. However, if $H_c^{n-k+1}(V) \neq 0$ then we would conclude, from the spectral sequence of $\pi^{-1}(V) \to V$, that $H_c^{n+1}(\pi^{-1}(V)) \neq 0$ (since $\mathscr{L}^*$ is constant over V). Thus $H_c^j(V) = 0$ for $j > n - k$ and thus $\dim_L B \leq n - k$ [2, Chapter I]. This completes the case in which L is a field.

If $L = Z$, then an easy universal coefficient argument shows that (X, B, π) is a cfs_K for any field K. Moreover, $\dim_K X \leq n$, and hence, by the Theorem for fields, $\dim_K B \leq n$ for any field K. A standard universal coefficient argument then shows that $\dim_Z(B) \leq n + 1$.

Remark. If (X, B, π) is a k-cfs_Z then an easy argument on the spectral sequence of $\pi^{-1}(V) \to V$, for V small, yields the inequality that $\dim_Z B \leq n - k$.

THEOREM 1.3. *Let* (X, B, π) *be a* cfs_L *with the stalks of* $\mathscr{L}^*$ *finitely generated and assume that* X *is an orientable* $n\text{-}cm_L$. *Then* (X, B, π) *is a* $k\text{-}cfs_L$ *for some* $k \leq n$ *and* B *is an* $(n-k)\text{-}cm_L$. *Moreover, if* $\mathscr{L}^*$ *is constant then* B *is orientable* (also see 3.8).

Proof. We may as well assume that $\mathscr{L}^*$ is constant. Let $k = \max\{i \mid \mathscr{L}^i \neq 0\}$. We may assume that $k > 0$. We shall first show that B is clc_L. For this it suffices to show that for U and V open subsets of B with $\overline{U}$ compact and $\overline{U} \subset V$ that $\operatorname{Im}(H_c^i(U) \to H_c^i(V))$ is finitely generated for all i (see [2; I, 2.2]). Let s be the smallest i for which this is not true for some such U and V. Let W be such that $\overline{U} \subset W$, $\overline{W} \subset V$, and $\overline{W}$ is compact. Consider the spectral sequence (with compact supports) $E_r^{p,q}(U)$ of the map $\pi^{-1}(U) \to U$ and similarly for W and V. In the diagram

$$\begin{array}{ccccc} & & E_2^{s,0}(U) & \to & E_3^{s,0}(U) \\ & & \downarrow f & & \downarrow u \\ E_2^{s-2,1}(W) & \to & E_2^{s,0}(W) & \to & E_3^{s,0}(W) \\ \downarrow v & & \downarrow g & & \\ E_2^{s-2,1}(V) & \to & E_2^{s,0}(V) & & \end{array}$$

the image of v is finitely generated while that of $g \circ f$ is not. It follows that the image of u is not finitely generated (see [2; V, 1.1]). Continuing in this way, we can find a Y with $\overline{U} \subset Y$ and such that $\operatorname{Im}(E_r^{s,0}(U) \to E_r^{s,0}(Y))$ is not finitely generated for any r (it suffices to do this for $r = s + 1$) and in particular for $r = \infty$. It then follows that $\operatorname{Im}(H_c^s(\pi^{-1}(U)) \to H_c^s(\pi^{-1}(Y))$ is not finitely generated, contrary to the fact that X is clc (an n-cm is clc by [2; I, 2.2]).

We shall now complete the proof for the case in which L is a field. Let $U \subset B$ be an open connected set and let $j = \max\{i \mid H_c^i(U) \neq 0\}$. Then in the spectral sequence of $\pi^{-1}(U) \to U$ we see that $E_2^{j,k} \neq 0$ and cannot be killed. Since

$$H_c^n(\pi^{-1}(U)) \approx L,$$

it follows that $j = n - k$, that $H_c^{n-k}(U) \approx L$, and that $\mathscr{L}^k \approx L$. Moreover, if $U \subset V$ are open and connected, then the map $H_c^{n-k}(U) \to H_c^{n-k}(V)$ must be an isomorphism, since this is true of $H_c^n(\pi^{-1}(U)) \to H_c^n(\pi^{-1}(V))$. We must show that for any given V and $b \in V$ there is an open set U with $b \in U \subset V$ and with $H_c^i(U) \to H_c^i(V)$ trivial for $i < n - k$. Since X is clc and since

$$H^i(F_b) = \operatorname{dir\,lim}(H^i(\pi^{-1}(U)) \quad (b \in U),$$

we can find a neighborhood U of b such that $H^i(\pi^{-1}(V)) \to H^i(\pi^{-1}(U))$ is trivial for all $i > k$. By Poincaré duality [2; II, 2.3] we then see that $H_c^j(\pi^{-1}(U)) \to H_c^j(\pi^{-1}(V))$ is trivial for $j < n - k$. Thus, in the spectral sequence, we conclude that $E_\infty^{j,0}(U) \to E_\infty^{j,0}(V)$ is trivial for $j < n - k$. Now from an inductive argument, similar to the one above which proved that B is clc, we see that by choosing U sufficiently small we can assure that $H_c^j(U) \to H_c^j(V)$ is trivial for $j < n - k$. This completes the proof if L is a field.

We turn now to the case $L = Z$. Let $j_p = n - k_p$ be the dimension for which B is a j_p - cm over Z_p (j_0 over the rationals). Clearly $k_p \geq k_0$, and hence $j_p \leq j_0$ for all primes p. Thus, by a standard universal coefficient argument, $\dim_Z B \leq j_0 + 1$

and $H_c^{j_0+1}(B; Z)$ is all torsion. From the spectral sequence of $X \to B$ it follows that $H_c^{j_0}(B; \mathscr{L}^{k_0})$ modulo its torsion subgroup is isomorphic to Z, since $E_2^{j_0,k_0}$ consists of permanent cocycles, $E_2^{p,q}$ is all torsion for $p > j_0$ or $q > k_0$, and $H_c^n(X; Z) \approx Z$. Since $\mathscr{L}^{k_0}$ is constant with finitely generated stalks, it follows that $H_c^{j_0}(B; Z)$ modulo its torsion group is isomorphic to Z. Thus $H_c^{j_0}(B; Z_p) \neq 0$ for any p, and hence $j_p = j_0$ for all p (we now drop these subscripts). It is now easy to see that $\mathscr{L}^k \approx Z$, since $H_c^j(B; Z_p) \approx Z_p$ for all p. Moreover, the spectral sequence now shows that

$$H_c^j(B; Z) \approx E_2^{j,k} \approx E_\infty^{j,k} \approx H_c^n(X; Z) \approx Z$$

($E_\infty^{j+1,k-1} = 0$ since it is a torsion subgroup of $H_c^n(X; Z) \approx Z$). We conclude the same facts for open connected subsets $U \subset B$ and, since the isomorphism $H_c^j(U; Z) \approx Z$ is natural with respect to inclusions, the result follows from [2; I, 4.11].

We leave it as an exercise for the reader to prove the partial converse: if (X, B, π) is a cfs_L, X is an orientable n-cm_L, and B is an $(n - k)$-cm_L, then (X, B, π) is k-regular (and therefore, in particular, the stalks of $\mathscr{L}^*$ are finitely generated).

Remark. Theorem 1.3 would not be true if X were non-orientable over a field L of characteristic different from two, even if (X, B, π) were k-regular, as the following example shows. Let Z_2 act on S^1 and on E^{2m+1} by the antipodal map and on $E^{2m+1} \times S^1$ by the diagonal action. Let $X = (E^{2m+1} \times S^1)/Z_2$ and $B = E^{2m+1}/Z_2$ with the map $\pi: X \to B$ induced by the projection $E^{2m+1} \times S^1 \to E^{2m+1}$. Then X is a $(2m + 2)$-manifold and (X, B, π) is a 1-cfs_L for any field L of characteristic different from two, but B is not a cm_L. (Note that for $m = 0$, X is an open Moebius band and B is a ray.) Note also that every fibre but one has an orientable neighborhood in X.

However, we are able to prove the following result for $L = Z$.

THEOREM 1.4. *Let* (X, B, π) *be a* cfs_Z *with the stalks of* $\mathscr{L}^*$ *finitely generated and where* X *is an* n - cm_Z. *The following statements are then equivalent and imply that* B *is an* $(n - k)$-cm_Z:

(1) *Some fibre has an orientable neighborhood.*

(2) *The fibration is* k*-regular.*

(3) *Every fibre has an orientable neighborhood.*

Proof. If F_b has an orientable neighborhood, say $\pi^{-1}(U)$, then Theorem 1.3 implies that $\pi: X \to B$ is k-regular for some k and that U is an $(n - k)$-cm_Z. Thus it suffices to prove that (2) implies (3). Note that X is orientable over Z_2, and thus B is an $(n - k)$-cm_{Z_2}. Suppose that F_b has no orientable neighborhood in X. Let U be an open connected neighborhood of b in B, and consider the spectral sequence of $\pi^{-1}(U) \to U$. We know that

$$H_c^{n-k}(U; Z) \approx E_2^{n-k,k} \approx E_\infty^{n-k,k} \approx H_c^n(\pi^{-1}(U); Z) \approx Z_2.$$

Moreover, for any open connected V with $b \in V \subset U$ the inclusion map induces an isomorphism

$$Z_2 \approx H_c^{n-k}(V; Z) \to H_c^{n-k}(U; Z) \approx Z_2.$$

By the universal coefficient theorem we see that the map

$$H_c^{n-k-1}(V; Z_2) \to H_c^{n-k-1}(U; Z_2)$$

is not trivial. This contradicts the fact that B is an (n - k)-cm_{Z_2} and finishes the proof.

Remark. It is to be expected in the situation of the preceding two theorems that the fibres have some resemblance to manifolds (see, for example, [10]). One may show without great difficulty, for example, that the fibres have (globally) a Poincaré duality. (To see this, one may make use of the Poincaré duality in open subsets of X and B and of the spectral sequence.) Note, however, that as far as is known it may even be possible for F to be non-locally connected; for if the solenoid Σ could act freely on a cm M, then $(M, M/\Sigma, \pi)$ is an 1-cfs over the rationals. We conjecture, in view of a theorem of Spanier and Whitehead [11], that if L is a field and X is contractible (or, perhaps, even when a fibre is homologically trivial in X) then the cohomology ring of a fibre is an exterior algebra on odd dimensional generators. In the case of a k-cfs (X, B, π) with singularities (not defined here in general, but see Definition 2.1 below) in which X is a cm, we conjecture that the fibres have the cohomology of a k-sphere. This is true in less general situations, and this is the reason that we restrict our attention to the k-sphere case below.

II. SPHERE FIBRATIONS WITH SINGULARITIES

DEFINITION 2.1. *A cohomology* k-*sphere fibre space over* L *with singularities* (k-$cfss_L$) *is a quintuple* (X, F, X^*, F^*, π) *where* X *and* X* *are locally compact Hausdorff spaces with closed subspaces* F *and* F* *respectively and* π *is an open, proper and continuous map of* X *onto* X* *such that*

(1) $F = \pi^{-1}(F^*)$

(2) $\pi|F$ *is a homeomorphism onto* F*

(3) $(X - F, X^* - F^*, \pi|(X - F))$ *is a* k-cfs_L *with* $\mathscr{L}^i = 0$ *for* $i \neq 0, k$.

We shall assume throughout that $k > 0$. We denote by $\mathscr{L}$ the locally constant sheaf $\mathscr{L}^k$ on $X^* - F^*$ (which has stalks isomorphic to L).

Our main result is the following one.

THEOREM 2.2. *There exists an exact "Smith-Gysin" sequence*

$$\cdots \to H_c^i(X^* - F^*) \xrightarrow{\alpha} H_c^i(X) \xrightarrow{(\beta,\gamma)} H_c^{i-k}(X^* - F^*; \mathscr{L}) \oplus H_c^i(F) \xrightarrow{\omega+\delta} H_c^{i+1}(X^* - F^*) \to \cdots,$$

where $\alpha = j^*\pi^*$ (j^* *is the natural map* $H_c^*(X - F) \to H_c^*(X)$), βj^* *and* ω *are the corresponding maps in the Gysin sequence of* $X - F \to X^* - F^*$, γ *is the natural restriction, and* δ *is the connecting homomorphism for the pair* (X^*, F^*) *preceded by the natural isomorphicm* π^*: $H_c^*(F) \approx H_c^*(F^*)$.

Proof. Let $X_0^* = X^*$, $X_1^* = X_2^* = \cdots = X_k^* = F^*$ and $X_{k+1}^* = \emptyset$, and put $X_i = \pi^{-1}(X_i^*)$. Consider the Fary spectral sequence $E_r^{p,q}$ associated with this filtration of X* and with π (see [2; XI, 3.3]). Then E_∞ is the graded module associated with some filtration of $H_c^*(X; L)$ and

$$E_2^{p,q} = \sum_{t\geq 0} H_c^{p+t}(X_t^* - X_{t+1}^*; \mathscr{L}^{q-t}) = H_c^p(X^* - F^*; \mathscr{L}^q) \oplus H_c^{p+k}(F^*; \mathscr{L}^{q-k}).$$

Thus we see that $E_2^{p,q} = 0$ if $q \neq 0$, k, and

$$E_2^{p,0} = H_c^p(X^* - F^*; L)$$

$$E_2^{p,k} = H_c^p(X^* - F^*; \mathscr{L}) \oplus H_c^{p+k}(F; L)$$

The exact sequence now follows from the exact sequence associated with any spectral sequence which has non-trivial terms in only two "fibre" dimensions.

We shall leave it to the reader to check that the homomorphisms in the sequence are as indicated. Various naturality properties of the sequence are evident from the construction and will not be stated explicitly.

Remark. The sheaf $\mathscr{L}$ is constant in each of the following cases (the last three of which are examples of a k-cfss):

(1) $L \approx Z_2$.

(2) π is the orbit map of an action of SO(2) or Sp(1) which is free outside the set F of stationary points.

(3) L is a field of characteristic zero and π is the orbit map of any action of SO(2) or of any action of Sp(1) without two-dimensional orbits.

(4) L is a field of characteristic zero and π is the orbit map of the action of a solenoid (see [6]).

Also see Lemma 4.2 and Theorems 4.3 and 4.4.

Using this sequence, it is evident that, if $\mathscr{L}$ is constant, several "Smith type" theorems may be derived in complete analogy to the exposition to be found, for example, in [4]. This procedure is somewhat easier than that of Conner and Dyer [7] who use the ordinary Gysin sequence.

Remark. If ψ is a general paracompactifying family of supports on X and $\phi = \{K \subset X^* \mid K = \pi(K^*)$ for some $K^* \in \psi\}$ then ϕ is also paracompactifying and ψ and ϕ are "very well adapted" since π is open and proper (and hence closed). (See [2; Chapter XI] for these notions.) Let $\mathscr{S}$ be any sheaf on X*, and let $\pi^*\mathscr{S}$ be its inverse image on X. Then, in the same way, there exists an exact Smith-Gysin sequence:

$$\cdots \to H^i_{\phi|X^*-F^*}(X^* - F^*; \mathscr{S}) \to H^i_{\psi}(X; \pi^*\mathscr{S})$$

$$\to H^{i-k}_{\phi|X^*-F^*}(X^* - F^*; \mathscr{L} \otimes \mathscr{S}) \oplus H^i_{\psi|F}(F; \pi^*\mathscr{S}) \to \cdots .$$

(That $\mathscr{L}^* \otimes \mathscr{S}$ is the Leray sheaf with coefficients in $\pi^*\mathscr{S}$ of $X - F \to X^* - F^*$ follows from the fact that the stalks of $\mathscr{L}^*$ are torsion free and requires some argument. Also it need not be assumed that $\mathscr{L}^*$ is locally constant on $X^* - F^*$ but only that it has stalks as indicated in (1.1) and (2.1).)

III. ORIENTATION

This section is a digression into the topic of orientation on an n-cm_L. (It will suffice for L to be a Dedekind domain in this section.) We shall assume familiarity with the results of [3]. M will denote a connected n-cm_L, and $\mathscr{L}$ will denote a locally constant sheaf on M with stalks isomorphic to L. $\mathscr{O}$ will denote the orientation sheaf on M (see [3; 7.5]).

LEMMA 3.1. *For* U *open in* M, $H_c^n(U; \mathscr{L}) \to H_c^n(M; \mathscr{L})$ *is onto.*

Proof. If $U \subset V$ are connected orientable open sets with $\mathscr{L}$ constant on V, then $H_c^n(U; \mathscr{L}) \to H_c^n(V; \mathscr{L})$ is an isomorphism [2, Chapter I], [4]. Let

$$G_U = \operatorname{Im}(H_c^n(U; \mathscr{L}) \to H_c^n(M; \mathscr{L}))$$

for any open $U \subset M$. Thus $G_U = G_V$ if U and V are as above. Since M is connected, G_U is constant, say $G_U = G$, for all U which are connected and orientable with $\mathscr{L}$ constant over U (thus for U sufficiently small). Let S be the collection of open subsets U such that $G_U = G$. The Mayer-Vietoris diagram

$$\begin{array}{ccc} H_c^n(U; \mathscr{L}) \oplus H_c^n(V; \mathscr{L}) & \to H_c^n(U \cup V; \mathscr{L}) & \to 0 \\ \downarrow & \downarrow & \\ H_c^n(M; \mathscr{L}) \oplus H_c^n(M; \mathscr{L}) & \to H_c^n(M; \mathscr{L}) & \to 0 \end{array}$$

shows that, if U; $V \in S$, then $U \cup V \in S$. Hence S contains every relatively compact open set. But $H_c^n(M; \mathscr{L}) = \lim H_c^n(U; \mathscr{L})$ over the relatively compact open sets U, and thus $M \in S$, which implies that $G_U = G_M = H_c^n(M; \mathscr{L})$ for all open U.

LEMMA 3.2. $\mathscr{L} \otimes \mathscr{O}$ *is the sheaf generated by the presheaf*

$$U \to \operatorname{Hom}(H_c^n(U; \mathscr{L}); L).$$

Proof. Let U be a connected open set over which $\mathscr{L}$ is constant. Then there exist the following isomorphisms each of which is natural with respect to inclusions of such open sets:

$$\operatorname{Hom}(H_c^n(U; \mathscr{L}); L) \approx H_n(U; \mathscr{L}) \approx H^0(U; \mathscr{L} \otimes \mathscr{O}) \approx (\mathscr{L} \otimes \mathscr{O})(U).$$

(The first isomorphism is by [3; 3.3] and the fact that $\mathscr{L}|_U$ is constant, the second by [3; 7.6]; the third is standard [9].) The result follows immediately. (Note that in using [3; 7.6] we do not need U to be paracompact since $\mathscr{L}$ is the constant sheaf L over U.)

THEOREM 3.3 $H_c^n(M; \mathscr{L}) \approx L$ *if and only if* $\mathscr{L} \otimes \mathscr{O}$ *is constant.*

Proof. Assume $H_c^n(M; \mathscr{L}) \approx L$. Then by Lemma 3.1, $H_c^n(U; \mathscr{L}) \to H_c^n(M; \mathscr{L})$ is an isomorphism for all connected open $U \subset M$, since any epimorphism $L \to L$ (as L modules) is necessarily an isomorphism. Thus for every connected open set U over which $\mathscr{L}$ is constant, there exists an isomorphism

$$L \approx \operatorname{Hom}(H_c^n(M; \mathscr{L}); L) \xrightarrow{\approx} \operatorname{Hom}(H_c^n(U; \mathscr{L}), L) \approx (\mathscr{L} \otimes \mathscr{O})(U)$$

natural with respect to inclusion (the last isomorphism is by the proof of Lemma 3.2). This provides a trivialization of $\mathscr{L} \otimes \mathscr{O}$ showing that $\mathscr{L} \otimes \mathscr{O}$ is constant. The converse will be proved following the next corollary.

COROLLARY 3.4. $\mathscr{O} \otimes \mathscr{O}$ *is constant.*

Proof. $H_c^n(M; \mathscr{O}) \approx H_0^c(M; \mathscr{O} \otimes \mathscr{O}^{-1}) = H_0^c(M; L) \approx L$, where $\mathscr{O}^{-1}$ is the inverse sheaf to $\mathscr{O}$. (That is, $\mathscr{O} \otimes \mathscr{O}^{-1}$ is the constant sheaf L. $\mathscr{O}^{-1}$ exists because $\mathscr{O}$ is locally constant [3; 7.7].) The first isomorphism is by [3; 7.9] and the last by [3; 6.6, 6.10(2)] (since an n-cm_L is always clc_L). Thus $\mathscr{O} \otimes \mathscr{O}$ is constant by the part of the theorem we have proved.

We will now complete the proof of the Theorem. We may take $\mathscr{O}^{-1} = \mathscr{O}$, and hence, if $\mathscr{L} \otimes \mathscr{O}$ is constant, then

$$H_c^n(M; \mathscr{L}) \approx H_0^c(M; \mathscr{L} \otimes \mathscr{O}^{-1}) = H_0^c(M; L) \approx L.$$

COROLLARY 3.5. *An* n-cm_L M *has an orientable double covering.*

Proof. Since $\mathscr{O}$ is the sheaf generated by $U \to \mathrm{Hom}(H_c^n(U; L), L)$, it follows that the automorphism g of L (as an L module) induced by the path (in the sense of [4]) $U, U_1, \cdots, U_m, U$ (by means of the induced maps on $H_c^n(U_i; L)$) is just the same as the automorphism of L induced (in the obvious way) by the induced maps on the $\mathscr{O}(U_i) \approx L$. Since $\mathscr{O} \otimes \mathscr{O}$ is the constant sheaf, it follows that this automorphism is of order two. Thus if $a = g(1)$, then $a^2 = 1$, and hence $a = \pm 1$ since L is an integral domain. Hence g is either the identity or -(identity), and the corollary follows from the results of [4] (see the remark below [4; 2.2]). (In fact, it is clear that any non-zero component of $\mathscr{O}$, as a bundle, is an orientable covering of M.)

COROLLARY 3.6. *Let* U *and* V *be connected open subsets of a connected orientable* n-cm_L M, *and let* $f: U \to V$ *be a homeomorphism. Choose an isomorphism* $H_c^n(M; L) \approx L$. *Then the composition*

$$g: L \approx H_c^n(M; L) \overset{\approx}{\leftarrow} H_c^n(V; L) \overset{f^*}{\to} H_c^n(U; L) \overset{\approx}{\to} H_c^n(M; L) \approx L$$

of isomorphisms is either the identity or minus the identity.

Proof. Let M' be the $(n+1)$-cm_L obtained from

$$U \times (-1, 1) \cup M \times (0, 1) \cup V \times (0, 2)$$

(open intervals) by identifying $\langle x, t\rangle$ with $\langle f(x), 2 + t\rangle$ for all $x \in U$ and $t \in (-1, 0)$.

The path (in the sense of [4])

$$M \times (0, 1);\ M \times (0, 1) \cup V \times (0, 2);\ V \times (1, 2) = U \times (-1, 0);$$

$$M \times (0, 1) \cup U \times (-1, 1);\ M \times (0, 1)$$

clearly induces the automorphism g of L. The result then follows from the proof of Corollary 3.5.

Remark. The last two results, of course, are not of interest if M is an n-cm_Z. For general L (particularly $L = Z_p$ or the rationals), however, they give a very satisfactory completion of the results of [4; Sections 2-5]. In particular, in Theorem 5.2 of [4], M* can be replaced by the orientable double covering of M (this is assumed to be two copies of M having opposite orientation if M is orientable). We would also like to point out at this time that Theorem 6.1 of [4] can be strengthened to include all topological transformation groups G, that is, any homeomorphism sufficiently close to the identity must preserve the local orientation. This can be seen by an argument on the Čech cohomology of the one point compactification of M (consider coverings containing the complement of the closure of some small open set $U \subset M$).

Now let (X, B, π) be a k-cfs$_L$ (see 1.1) with X an n-cm$_L$ and B an (n - k)-cm$_L$. Let $\mathcal{O}_X$ and $\mathcal{O}_B$ be the orientation sheaves on X and B, respectively, let $\mathcal{S}$ be any locally constant sheaf on B with stalks isomorphic to L, and let $\mathcal{L}$ be the sheaf $\mathcal{L}^k$ of (1.1). Consider the Leray spectral sequence with compact supports of π with coefficients in the inverse image $\pi^*\mathcal{S}$ of $\mathcal{S}$ (see [9; II, 2.11 and 4.17]). The Leray sheaf is defined by the presheaf

$$U \to H^*(\pi^{-1}(\overline{U}); \pi^*\mathcal{S}) \approx H^*(\pi^{-1}(\overline{U}); L) \otimes \mathcal{S}(U)$$

for U connected and such that $\mathcal{S}$ is constant over $\overline{U}$. Thus it is the sheaf $\mathcal{L}^* \otimes \mathcal{S}$. The spectral sequence has $E_2^{p,q} \approx H_c^p(B; \mathcal{L}^q \otimes \mathcal{S})$ and converges to $H_c^*(X; \pi^*\mathcal{S})$. Now $E_2^{p,q} = 0$ for $p > n - k$ or for $q > k$, and thus it follows that

$$H_c^n(X; \pi^*\mathcal{S}) \approx H_c^{n-k}(B; \mathcal{L} \otimes \mathcal{S}).$$

Since $\mathcal{L}$ is locally constant with stalks L, there is an inverse sheaf $\mathcal{L}^{-1}$ such that $\mathcal{L}^{-1} \otimes \mathcal{L}$ is the constant sheaf L. Now put $\mathcal{S} = \mathcal{L}^{-1} \otimes \mathcal{O}_B$, and we obtain from Theorem 3.3 the result that $\pi^*(\mathcal{L}^{-1} \otimes \mathcal{O}_B) \otimes \mathcal{O}_X$ is constant, and hence, by 3.4 and uniqueness [3; 7.7],

$$\mathcal{O}_X \approx \pi^*(\mathcal{L}^{-1} \otimes \mathcal{O}_B) \approx \pi^*(\mathcal{L}^{-1}) \otimes \pi^*(\mathcal{O}_B).$$

Tensoring this isomorphism with itself and using 3.4, we obtain the conclusion that $\pi^*(\mathcal{L}^{-1} \otimes \mathcal{L}^{-1})$ is constant; and, since the fibres of π are connected, it follows that $\mathcal{L} \otimes \mathcal{L}$ is constant and $\mathcal{L} \approx \mathcal{L}^{-1}$. We have proved the following result.

THEOREM 3.7. *$\mathcal{L} \otimes \mathcal{L}$ is constant, and* $\mathcal{O}_X \approx \pi^*(\mathcal{L}) \otimes \pi^*(\mathcal{O}_B)$.

The first fact is, of course, trivial if L = Z. The second yields a corollary.

COROLLARY 3.8. *If any two of the following conditions hold then the third also holds:*

(1) X *is orientable.*

(2) B *is orientable.*

(3) $\mathcal{L}$ *is constant.*

IV. THE LOCAL SMITH THEOREM

As we mentioned in Section II, several "Smith-type" theorems can be proved using our exact Smith-Gysin sequence. As examples of results which have analogues for a k-cfss$_L$ with $\dim_L X < \infty$ and $\mathcal{L}$ constant, we refer the reader to [4], Theorems 7.8, 7.13, 10.1, 10.5, 10.6, 11.2 and 11.4 and to [2], Chapter III, Theorems 4.3 and 4.4. Also see [7].

We shall restrict our attention in this section to the statement of the analogue of the local Smith theorem and to some results concerning the constancy of $\mathcal{L}$ and the possible values of k. We assume that (X, F, X^*, F^*, π) is a k-cfss$_L$ with X an n-cm$_L$ in this section.

THEOREM 4.1. *If $\mathcal{L}$ is constant, then each component of* F *is an* r-cm$_L$ *with* $r \equiv n$ (*modulo* k + 1). *If* U^* *is an open neighborhood in* X^* *of a point in* F^* *with* $F^* \cap U^*$ *connected and* $U = \pi^{-1}(U^*)$ *orientable, then the composition*

$$H_c^r(U \cap F) \xrightarrow{\delta} H_c^{r+1}(U^* - F^*) \xrightarrow{\omega} H_c^{(r+1)+(k+1)}(U^* - F^*) \xrightarrow{\omega} \cdots$$

$$\xrightarrow{\omega} H_c^{n-k}(U^* - F^*) \underset{\approx}{\xleftarrow{\beta}} H_c^n(U)$$

of homomorphisms in the Smith-Gysin sequence is an isomorphism. Moreover, F *is orientable if* X *is orientable.*

This can be proved by the method of [4; 7.4, 7.5] in case L is a field and for L = Z with some extra work. However, it is preferable to proceed roughly as follows. If U^* and V^* are open, relatively compact subsets of X^* with the closure of U^* contained in V^*, then it follows by an inductive argument (regressive) on the Smith-Gysin sequence that

$$\operatorname{Im}(H_c^i(U^* - F^*) \to H_c^i(V^* - F^*)) \text{ and } \operatorname{Im}(H_c^i(U^* \cap F^*) \to H_c^i(V^* \cap F^*))$$

are both finitely generated for all i (note that all spaces considered are finite dimensional and that this argument uses the fact that X is clc over L). Thus F is clc_L. By the same type of argument, it follows that the inverse families $\{H_c^i(U^* - F^*)\}$ and $\{H_c^i(U^* \cap F^*)\}$ are locally constant on $F^* \subset X^*$ in the sense of [8]. The fact that each component of F is an $r\text{-cm}_L$ then follows from [8; 7.5]. It also follows that the sequence of "local groups" induced by the Smith-Gysin sequence is exact [8]. That the composition of homomorphisms in our Theorem is an isomorphism follows from the corresponding fact in the local Smith-Gysin sequence (which is easy to establish) and from the natural map of the "local composition" into the global one. (See [4; 7.5].) The orientability of F follows from this isomorphism.

We now take up the question of the constancy of $\mathscr{L}$. Recall that if X is orientable, then $X^* - F^*$ is an $(n - k)\text{-cm}_L$ by Theorem 1.3, and that, if L = Z and $F \neq \emptyset$, then $X^* - F^*$ is an $(n - k)\text{-cm}_Z$ even if X is non-orientable by Theorem 1.4 (since any fibre near some point in F is in an orientable neighborhood).

LEMMA 4.2. *If* X *and* $X^* - F^*$ *are both orientable, then* $\mathscr{L}$ *is constant.*

Proof. From the Smith-Gysin sequence we obtain the isomorphisms

$$H_c^{n-k}(X^* - F^*; \mathscr{L}) \approx H_c^n(X; L) \approx L,$$

and hence $\mathscr{L} \approx \mathscr{L} \otimes \mathscr{O}$ is constant by Theorem 3.3 (with $M = X^* - F^*$). Of course, this also follows from the more general result (3.8).

THEOREM 4.3. *If* L = Z *and* $H_c^{n-1}(X; Z_2) = 0$, *then* X *and* $X^* - F^*$ *are orientable; hence* $\mathscr{L}$ *is constant.*

Proof. If X were non-orientable, then $H_c^n(X; Z) \approx Z_2$, and hence $H_c^{n-1}(X; Z_2) \neq 0$, which is a contradiction.

Now Theorem 4.1 is valid for coefficients in Z_2, and consequently

$$\dim_{Z_2}(X^* - F^*) = n - k \text{ and } \dim_{Z_2}(F) \leq n - k - 1.$$

Since $k \geq 1$, the Smith-Gysin sequence with coefficients in Z_2 shows that $H_c^{n-k-1}(X^* - F^*; Z_2) = 0$, and hence $X^* - F^*$ is orientable (over Z) by the same argument as that used for X.

THEOREM 4.4. *If* $L = Z$ *and if* $x^* \in F^*$, *then there exists a neighborhood* U^* *of* x^* *in* X^* *such that* $U^* - F^*$ *is orientable. Thus* $\mathscr{L}$ *is constant near* x^*.

Proof. This result is just a localization of the preceding Theorem 4.3. Let $U^* \subset V^*$ be open, connected neighborhoods of x^* in X^*, and let U and V be their inverse images under π. It may be assumed that

$$\mathrm{Im}\,(H_c^{n-1}(U; Z_2) \to H_c^{n-1}(V; Z_2)) = 0.$$

Then, the Smith-Gysin sequence shows that

$$\mathrm{Im}\,(H_c^{n-k-1}(U^* - F^*; Z_2) \to H_c^{n-k-1}(V^* - F^*; Z_2)) = 0.$$

However, if $U^* - F^*$ were not orientable, then a universal coefficient argument shows that this image could not be trivial.

COROLLARY 4.5. *If* $L = Z$, *then each component of* F *is an* $r\text{-cm}_Z$ *for some* $r \equiv n$ (*modulo* $k + 1$).

Remark. Theorem 4.4 would not be true if L were a field of characteristic different from two as the following example shows. Let X be the open cone over the unit tangent bundle $T_1(P^2)$ of the projective plane P^2, and let F be the vertex of the cone. Let X^* be the open cone over P^2 and $\pi: X \to X^*$ be the natural map. If L is a field of characteristic different from two, then X is a 4-cm_L (since $T_1(P^2)$ is a cohomology 3-sphere over L). Moreover, X is a cohomology E^4 over L. However, $X^* - F^*$ is not orientable, even near F^*, and $\mathscr{L}$ is not constant near F^*.

We know of no examples, however, for which F is not a cm_L.

Since we will be interested in the remainder of this section only in the neighborhood of a point $x^* \in F^*$, we may assume X to be orientable. We shall also assume that $\mathscr{L}$ is constant and, consequently, we may assume that F is a connected $r\text{-cm}_L$ with $n - r = h(k + 1)$ for some $h \geq 1$.

We shall also assume in the remainder of this section that x has a basic system of open neighborhoods $U = \pi^{-1}\pi(U)$ such that $U - F$ is paracompact, where $\pi(x) = x^*$. Then $\pi(U - F)$ is also paracompact since π is open and proper.

LEMMA 4.6. *If* $\mathscr{L}$ *is constant then the ring* Ind lim $H^*(U^* - F^*; L)$, U^* *ranging over neighborhoods of* x^*, *is a truncated polynomial ring* $L[\alpha]/\alpha^h$ *on one generator* α *of degree* $(k + 1)$ *and height* h.

Proof. The lemma makes sense since cup products may be introduced into the inductive limit. Since X and F are orientable cohomology manifolds of dimension n and r, respectively, it follows that for any connected open neighborhood U of x (with $\pi(x) = x^*$) there are arbitrarily small connected open neighborhoods V of x, $V \subset U$, such that

$$\mathrm{Im}\,(H_c^i(V - F; L) \to H_c^i(U - F; L))$$

is trivial for $i \neq n, r + 1$ and is isomorphic to L for $i = n, r + 1$. Moreover, this image is stable in V; that is, this inverse family forms a locally constant family at x in the sense of [8]. It follows easily from Poincaré-duality [2; II, 2.3] that

$$\text{Ind lim } H^i(U - F; L) \approx \begin{cases} L & \text{for } i = 0, n - r - 1 \\ 0 & \text{for } i \neq 0, n - r - 1. \end{cases}$$

Since inductive limits and exact sequences commute, there exists a limit Gysin sequence of the (ordinary) Gysin sequences with closed supports of the maps $U - F \to U^* - F^*$ (if $U = \pi^{-1}(U^*)$). The lemma follows immediately by a familiar argument on this sequence.

THEOREM 4.7. *If* $\mathscr{L}$ *is constant, with* $L = Z_p$ *and* $h > p$, *then* $(k + 1)$ *divides* $(p - 1)p^{\ell}$ *for some* ℓ. *If* $L = Z$ *and* $h > 3$, *then* k *is* 1 *or* 3. *If* $L = Z_2$ *and* $h > 2$, *then* k *is* 1, 3 *or* 7.

Proof. Since the Steenrod squares and reduced powers can be introduced into the inductive limit, the first statement follows from the arguments in [12; I, 4.5 and VI, 2.8]. The second statement follows by comparing the cases $L = Z_2$ and $L = Z_3$. Adams' operations $\phi_{i,j}$ [1] can also be introduced into the inductive limit and the last statement follows from his results.

COROLLARY 4.8. *If* $L = Z$, *then one of the following situations must occur:*

(1) $r = n - k - 1$,

(2) $k = 7$ *and* $r = n - 16$, *or* $r = n - 24$,

(3) $k = 1$ *or* 3.

This is merely a restatement of part of Theorem 4.7. Examples of all these cases come from cones over known sphere fiberings of spheres. Note that if the fibering is the orbit map of the action of a compact Lie group on X, then case (2) cannot occur (see [5]). It is also reasonable to conjecture that the case $k = 7$, $r = n - 24$ cannot occur.

REFERENCES

1. J. F. Adams, *On the non-existence of elements of Hopf invariant one,* Ann. of Math. (2) 72 (1960), 20-104.

2. A. Borel, et al, *Seminar on transformation groups,* Ann. of Math. Study no. 46, Princeton University Press, 1960.

3. A. Borel and J. C. Moore, *Homology theory for locally compact spaces,* Michigan Math. J. 7 (1960), 137-159.

4. G. Bredon, *Orientation in generalized manifolds and applications to the theory of transformation groups,* Michigan Math. J. 7 (1960), 35-64.

5. ———, *Transformation groups with orbits of uniform dimension,* Michigan Math. J. 8 (1961), 139-147.

6. G. Bredon, Frank Raymond and R. F. Williams, p-*adic groups of transformations,* Trans. Amer. Math. Soc. 99 (1961), 488-498.

7. P. E. Conner and Eldon Dyer, *On singular fiberings by spheres,* Michigan Math. J. 6 (1959), 303-311.

8. P. E. Conner and E. E. Floyd, *A characterization of generalized manifolds,* Michigan Math. J. 6 (1959), 33-43.

9. R. Godement, *Topologie algébrique et théorie des faisceaux,* Actual. Sci. Ind., no. 1252, Hermann et Cie., Paris, 1958.

10. F. Raymond, *Local triviality for Hurewicz fiberings of manifolds,* to appear.

11. E. H. Spanier and J. H. C. Whitehead, *On fibre spaces in which the fibre is contractible*, Comment. Math. Helv. 29 (1955), 1-8.

12. N. E. Steenrod and D. B. A. Epstein, *Cohomology operations*, Ann. of Math. Study no. 50, Princeton University Press, 1962.

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