A BASIS THEOREM FOR CUSP FORMS ON GROUPS OF GENUS ZERO

John Roderick Smart

1. Let $\Gamma(1)$ denote the modular group, that is, the group of linear fractional substitutions $w = (az + b)/(cz + d)$ with a, b, c and d integers and $ad - bc = 1$. Let μ denote the dimension of the vector space of cusp forms of dimension $-r$ for $\Gamma(1)$. (See Section 2 for definitions.) It is known that if r is an even integer, then $\mu = [r/12]$ if $r \not\equiv 2 \mod 12$ and $\mu = [r/12] - 1$ if $r \equiv 2 \mod 12$. In 1939 Petersson [6] proved the following theorem: *The first* μ *Poincaré series* (see (2.3)) $g_{-r}(z, 1), g_{-r}(z, 2), \cdots, g_{-r}(z, \mu)$ *span the vector space of cusp forms of dimension* $-r$ *for the modular group, where* $r > 2$ *is an even integer.* The object of this paper is to show that essentially the same proof applies to all real zonal horocyclic groups of genus zero and finite signature. The modular group is such a group.

To our knowledge this theorem is not implied by any of the more recent basis theorems. The proof fails if the genus of the group is larger than zero. In the next section we make the necessary definitions.

2. A *horocyclic* group Γ is a group of linear fractional transformations which maps a disk or half-plane one-to-one onto itself, is discontinuous at each interior point of the disk and is not discontinuous at any of the boundary points. (Equivalent terms are *Fuchsian group of the first kind* and *Grenzkreisgruppen* [9].) A *real* horocyclic group maps the upper half-plane onto itself; furthermore, if it possesses a translation it is termed *zonal*. The signature of a discontinuous group is an n + 2-tuple describing certain characteristics of a fundamental region; namely, the genus g, the number n of inequivalent fixed points (with respect to Γ) and an integer (possibly infinite) at least 2 associated with each of these fixed points. In case n is finite we write $(g, n; k_1, k_2, \cdots, k_n)$ for the signature. At an elliptic fixed point we associate the order of the transformation in Γ fixing the point; at a parabolic fixed point we associate the number ∞. If Γ is a zonal horocyclic group which has a finite signature $(g, n; k_1, k_2, \cdots, k_n)$, then we may assume that

$$2 \leq k_1 \leq k_2 \leq \cdots \leq k_s < \infty, \; k_{s+1} = \cdots = k_n = \infty, \quad \text{and } s < n.$$

(For a discussion of the signature of discontinuous groups and related matters see [1; pp. 203-209], [4] and [5; Ch. VII].) With few exceptions the converse is true: given

$$(g, n; k_1, k_2, \cdots, k_n), \; 2 \leq k_1 \leq k_2 \leq \cdots \leq k_s < \infty,$$

$$k_{s+1} = \cdots = k_n = \infty, \quad \text{and } s < n,$$

there exists a zonal horocyclic group Γ with the given signature. Since we are concerned with $g = 0$, we need only require that $n > 3$ or if $n = 3$ then $1/k_1 + 1/k_2 + 1/k_3 < 1$ [5].

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A canonical polygon R for Γ is a fundamental region bounded by $4g + 2n$ hyperbolic lines arranged in a special order (which need not concern us) [5], [4; pp. 186 ff.]. This polygon has the special feature that each of the elliptic and parabolic fixed points are inequivalent and it contains exactly one accidental cycle (the sum of the angles meeting at the vertices of this cycle is 2π). The following theorem can be found in Fricke-Klein [4; pp. 310 ff.]: *A horocyclic group of finite signature possesses a canonical polygon.* (See also [5; Ch. VII, 4D].)

Thus if Γ is a zonal horocyclic group of finite signature $(0, n; k_1, \cdots, k_n)$ with $2 \leq k_1 \leq \cdots \leq k_s < \infty$, $k_{s+1} = \cdots = k_n = \infty$ and $s < n$, then Γ possesses a fundamental region R which is bounded by $2n$ sides, has s inequivalent elliptic vertices, $n - s$ parabolic vertices, and one accidental cycle consisting of n vertices. Moreover, the orders of the elliptic substitutions fixing these s vertices are $k_1, k_2, \cdots, k_s$, respectively.

Let $\overline{\Gamma}$ be the group of two-by-two matrices defined by $\Gamma = \overline{\Gamma}/\{\pm I\}$, where $I = (1\ 0 \mid 0\ 1)$ (we write our matrices in one line with a bar separating rows).

An everywhere regular automorphic form of dimension $-r$ for Γ, r integral and even, is a function f regular in $\mathscr{H}$, the upper half plane, which satisfies the equation

$$f(Vz) = (cz + d)^r f(z) \tag{2.1}$$

for each $V = (\cdot\ \cdot \mid c\ d) \in \overline{\Gamma}$ and each $z \in \mathscr{H}$. It is further assumed that f is regular at each of the parabolic cusps p_j, $j = 1, \cdots, n - s$. That is, f possesses the "Fourier expansions"

$$f(z) = (c_j z + d_j)^{-r} \sum_{n=0}^{\infty} a_n(A_j)\, e^{2\pi i n A_j z/\lambda_j} \tag{2.2}$$

where $A_j^{-1}\infty = p_j$, $A_j = (\cdot\ \cdot \mid c_j\ d_j)$ is a real two-by-two matrix of determinant 1 and

$$A_j^{-1} U^{\lambda_j} A_j \in \overline{\Gamma} \quad (\lambda_j > 0,\ U = (1\ 1 \mid 0\ 1))$$

generates the subgroup of transformations fixing p_j. We take $A_1 = I$. We say that $f(z)$ is a *cusp form* if $a_0(A_j) = 0$ for $j = 1, 2, \cdots, n - s$.

Let $r > 2$ be an even integer and consider the Poincaré series

$$g_{-r}(z, \nu) = \nu^{r-1} \sum_{V \in \mathscr{M}(\Gamma)} (cz + d)^{-r} e^{2\pi i \nu V z/\lambda}, \tag{2.3}$$

where $V = (\cdot\ \cdot \mid c\ d)$ ranges over all matrices in $\overline{\Gamma}$ with distinct lower rows and ν is an integral valued parameter. If $r > 2$, these series converge uniformly and absolutely on compact subsets of $\mathscr{H}$, and thus they represent automorphic forms of dimension $-r$ for Γ. The Poincaré series (2.3) with $\nu > 0$ are everywhere regular cusp forms.

The everywhere regular automorphic forms of dimension $-r$ for Γ form a finite dimensional complex vector space denoted by $\mathscr{C}^+(\Gamma, -r)$. The cusp forms form a subspace denoted by $\mathscr{C}^0(\Gamma, -r)$. Let the dimension of $\mathscr{C}^0$ be μ, and let $\phi_1, \phi_2, \cdots, \phi_\mu$ be a basis. Then each of these functions has an expansion like (2.2) at ∞. Let

(2.4) $$\phi_k(z) = \sum_{n=1}^{\infty} b_n(k)\, e^{2\pi i n z/\lambda} \quad (b_n(k) = b_n(I, k);\ k = 1, 2, \cdots, \mu)$$

be their expansions. The vectors

(2.5) $$B_n = B_n(I) = \{b_n(1), b_n(2), \cdots, b_n(\mu)\},$$

with μ components, are called the *Fourier vectors* of the basis $\phi_1, \phi_2, \cdots, \phi_\mu$.

Petersson [7] introduced the inner product

(2.6) $$(f, g) = \iint_R f(z)\overline{g(z)}\, y^{r-2}\, dx\, dy$$

between cusp forms $f, g \in \mathscr{C}^0(\Gamma, -r)$, $r > 0$. The integral can be shown to be independent of the choice of the fundamental region R used. In particular, if $g(z) = g_{-r}(z, \nu)$, we have the important formula [7; p. 505]

(2.7) $$(f, g_{-r}(z, \nu)) = e_r\, a_\nu(I) \qquad (e_r = (r-2)!\,(4\pi)^{1-r}\lambda^r),$$

where $a_\nu(I)$ is the ν-th coefficient in the expansion (2.2) with $A_j = I$. With the aid of (2.7) one can prove the Fundamental Theorem on Linear Relations between Poincaré Series [7; p. 517]: *The linear relation*

(2.8) $$\sum_{k=1}^{\mu} \xi_k\, g_{-r}(z, \nu_k) = 0$$

holds if and only if

(2.9) $$\sum_{k=1}^{\mu} \bar{\xi}_k\, B_{\nu_k} = 0,$$

where the ξ_k *are complex constants and the* B_{ν_k} *are the Fourier vectors of a basis.*

3. We are now in a position to state and prove our theorem.

THEOREM. *Let* Γ *be a real zonal horocyclic group of signature*

$$(0, n: k_1, k_2, \cdots, k_n) \quad (2 \le k_1 \le k_2 \le \cdots \le k_s < \infty;\ k_{s+1} = \cdots = k_n = \infty).$$

Let r *be an even integer greater than* 2. *Then the first* μ *Poincaré series* $g_{-r}(z, 1), \cdots, g_{-r}(z, \mu)$ *span the space* $\mathscr{C}^0(\Gamma, -r)$ *of cusp forms of dimension* $-r$ *on* Γ.

Proof. We suppose to the contrary that there exist constants ξ_k not all zero such that

$$\sum_{k=1}^{\mu} \xi_k\, g_{-r}(z, k) = 0.$$

Then by the theorem on linear relations,

$$\sum_{k=1}^{\mu} \overline{\xi}_k B_k = 0,$$

where B_k is the Fourier vector of a basis $\phi_1, \phi_2, \cdots, \phi_\mu$. Writing out the above relations for each component of the Fourier vectors we find

$$\sum_{k=1}^{\mu} \overline{\xi}_k b_k(j) = 0 \qquad (j = 1, 2, \cdots, \mu).$$

This homogeneous set of equations has a non-trivial solution by assumption; hence, the transposed set also has a non-trivial solution $\xi'_1, \xi'_2, \cdots, \xi'_\mu$:

$$\text{(3.1)} \qquad \sum_{k=1}^{\mu} \xi'_k b_j(k) = 0 \qquad (j = 1, 2, \cdots, \mu).$$

Now consider the cusp form

$$\text{(3.2)} \qquad \Phi(z) = \sum_{k=1}^{\mu} \xi'_k \phi_k(z),$$

which is not identically zero; therefore, it has a zero of finite order n_0 at ∞. Expanding Φ at ∞, we find that

$$\Phi(z) = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\mu} \xi'_k b_n(k) \right) e^{2\pi i n z/\lambda}.$$

Thus by (3.1),

$$\text{(3.3)} \qquad n_0 \geq \mu + 1.$$

We have shown that Φ has a zero of order at least $\mu + 1$ at $z = \infty$. Φ is a cusp form hence it has zeros at the remaining cusps; moreover, it may have zeros at the elliptic fixed points. It is known [2] that an everywhere regular automorphic form of dimension $-r$ for Γ has a certain fixed number $N = N(\Gamma, -r)$ of zeros in a fundamental region R. The proof now proceeds as follows. Upon adding up the orders of the zeros of Φ and computing $N(\Gamma, -r)$ and μ, we find that Φ has more than N zeros. Thus Φ is identically zero, contradicting our assumption that $\xi_1, \xi_2, \cdots, \xi_\mu$ are not all zero.

We now calculate μ. The dimension of $\mathscr{C}^+(\Gamma, -r)$ is given by [3; p. 26]

$$(r - 1)(p - 1) + \sigma_0 r/2 + \sum_{i=1}^{e_0} [(1 - 1/k_i)r/2],$$

where p is the genus of Γ, σ_0 is the number of inequivalent parabolic cusps, e_0 is the number of inequivalent elliptic vertices in a fundamental region, and k_i is the order of the i-th elliptic vertex. The bracket [u] denotes the greatest integer function. In our case we obtain the number

$$1 + (n - s - 2)r/2 + \sum_{i=1}^{s} [(1 - 1/k_i)r/2].$$

Furthermore, it is known [8] that the dimension of $\mathscr{N}(\Gamma, -r)$, where

$$\mathscr{C}^+(\Gamma, -r) = \mathscr{C}^0(\Gamma, -r) + \mathscr{N}(\Gamma, -r)$$

is $n - s$. Thus

$$\mu = 1 + (n - s - 2)r/2 + \sum_{i=1}^{s} [(1 - 1/k_i)r/2] - (n - s). \tag{3.4}$$

In counting the zeros $N = N(\Gamma, -r)$ of a form $f \in \mathscr{C}^+(\Gamma, -r)$ in a fundamental region R certain conventions are necessary for counting zeros which lie on the boundary. We have already mentioned that the zero at a parabolic cusp is measured in the appropriate local variable (see (2.2)). If f has a zero at an interior point of a side of R, then f also has a zero of the same order at the corresponding point of the paired side of R. We agree to count only one of these zeros. If f has a zero of order m at an elliptic vertex of order k, then we agree to count m/k in $N(\Gamma, -r)$. At the points of an accidental cycle f has the same order; we agree to count only one of these orders in N.

Ford [2; p. 114] gives the formula $N = (n - 1 - \Sigma 1/k)r/2$ for a group which possesses a fundamental region bounded by $2n$ sides. The sum $\Sigma 1/k$ extends over the orders of the cycles, $k = 1$ for an accidental cycle, and $k \geq 2$ for an elliptic cycle. In our case we find that

$$N = N(\Gamma, -r) = \left(n - 2 - \sum_{i=1}^{s} 1/k_i\right)r/2. \tag{3.5}$$

At an elliptic fixed point of order k in R an automorphic form of dimension $-r$ has a zero unless k divides $r/2$ and the order m of this zero satisfies the congruence $(r/2 + m)/k \equiv 0 \mod 1$ [2; p. 110]. Suppose

$$r/2 \equiv m_1 \mod k_1,\ r/2 \equiv m_2 \mod k_2, \cdots, r/2 \equiv m_s \mod k_s$$

$$(0 < m_j \leq k_j \text{ for } j = 1, 2, \cdots, s).$$

Then Φ has a zero of order at least $(k_j - m_j)/k_j$ at the elliptic fixed point of R of order k_j $(1 \leq j \leq s)$.

We have accounted for

$$\rho = n_0 + n - s - 1 + \sum_{i=1}^{s} (k_i - m_i)/k_i \tag{3.6}$$

zeros of Φ at the elliptic vertices and parabolic cusps of R. By assumption $\rho \leq N$. We now show that this is incompatible with (3.3). Consider the equality

$$N - \rho = \left(n - 2 - \sum_{i=1}^{s} 1/k_i \right) r/2 - n_0 - n + s + 1 - \sum_{i=1}^{s} (k_i - m_i)/k_i .$$

We write $n_0 = \mu + 1 + m_0$ $(m_0 \geq 0)$ and substitute from (3.4); then

$$N - \rho = rs/2 - (r/2) \sum_{i=1}^{s} (1/k_i) - 1 - m_0 - \sum_{i=1}^{s} [(1 - 1/k_i)r/2] - \sum_{i=1}^{s} (k_i - m_i)/k_i .$$

We see from the definition of m_i that

$$(r/2)(1 - 1/k_i) - [(1 - 1/k_i)r/2] - (k_i - m_i)/k_i = 0$$

for each $i = 1, 2, \cdots, s$. Thus

$$N - \rho = - 1 - m_0 \qquad (m_0 \geq 0),$$

which is a contradiction. This completes the proof.

Added in proof. It has come to the author's attention that the results of this paper are covered by a paper of Hans Petersson, *Über Weierstrasspunkte und die expliziten Darstellungen der automorphen Formen von reeller Dimension*, Math. Z. 52 (1949), 32-59. He proves that on a discontinuous group of genus g a basis for the cusp forms of dimension $-r$ can be chosen from the first $\mu + g$ Poincaré series. There is no restriction that r even be integral or that the group be zonal.

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University of Wisconsin