GROUPS WITH A FINITE NUMBER OF INDECOMPOSABLE INTEGRAL REPRESENTATIONS

Alfredo Jones

1. INTRODUCTION

Let G be a finite group, let K be an algebraic number field, and let R be the algebraic integers of K. We let RG be the group ring consisting of all linear combinations of elements of G with coefficients in R. By an RG-module we understand a unital, left RG-module that is finitely generated and torsion-free as R-module. Thus in particular, if R = Z, the rational integers, every ZG-module gives rise to a representation of G by matrices with entries in Z and conversely. We denote by n(RG) the number of non-isomorphic indecomposable RG-modules.

Recently, Heller and Reiner [4] proved that, if G is a p-group, then n(ZG) is finite if and only if G is cyclic of order p or p^2 (see [6] and [7]). The necessity of this condition was also proved by Dade [2]. This extends earlier results of Diederichsen [4] and Reiner [9] on the cyclic group of order p, and results of Roĭter [13] and Troy [14] on the cyclic group of order four. For arbitrary G, Heller and Reiner showed that, if n(ZG) is finite, then for every p all p-Sylow subgroups of G are cyclic of order p or p^2 (see [6] and [12]).

In this paper we complete these results, proving that for arbitrary G, if all Sylow subgroups of G are cyclic of order at most p^2, then n(ZG) is finite.

The general reference for the results used, as well as for the notation, will be [1].

2. REDUCTION TO THE LOCAL CASE

Given a prime ideal P in R, we denote by R_P the ring of the P-adic valuation of K, that is,

$$R_P = \{a/b;\ a,\ b \in R,\ b \notin P\}\ .$$

If R' is a ring extension of R and M is an RG-module, we denote the R' G-module $R' \otimes_R M$ by R'M.

LEMMA 1. *Given any group* G, *let* R_0 *be the ring of elements of* K *integral at all primes* P *which divide the order of* G. *An* RG-*module* M *is decomposable if and only if the* R_0 G-*module* R_0 M *is decomposable.*

Proof. Suppose that R_0 M is decomposable, and let L be an R_0 G-summand. Set $L \cap M = N$. Then N is an RG-module which is a pure R-submodule of M; therefore it is an R-direct summand of M, and $R_0 N = L$. Since $R_0 \subset R_P$ for all

Received August 24, 1962.

The results contained in this paper are included in the author's dissertation written under Professor Irving Reiner at the University of Illinois. The author also thanks Professor S. Takahashi for helpful suggestions. The research was supported in part by a contract with the Office of Naval Research.

prime ideals P of R which divide the order of G, it follows that $R_P N = R_P L$ is an $R_P G$-summand of $R_P M$. This implies that N is an RG-summand of M (see [10]). The implication in the other direction is obvious.

LEMMA 2. *Given any group* G *and* $R_0 G$-*modules* M *and* N, N *is isomorphic to an* $R_0 G$-*summand of* M *if and only if for every prime ideal* P *in* R *which divides the order of* G $R_P N$ *is isomorphic to an* $R_P G$-*summand of* $R_P M$.

Proof. For each P let $R_P M \cong R_P N \dotplus M_P$. Then $KM \cong KN \dotplus KM_P$. From the Krull-Schmidt Theorem for KG-modules, it follows that all the $R_P G$-modules M_P are K-isomorphic. Then, by a theorem of Maranda (see [8]), there exists an R_0 G-module L such that for all P which divide the order of G, $M_P \cong R_P L$. Therefore, for all such P, $R_P M \cong R_P (N \dotplus L)$; and, by a result of Maranda [8], this implies $M \cong N \dotplus L$.

For every fixed t let E_t be the set of all t-tuples $(n_1, \cdots, n_t)$, where n_i, $1 \leq i \leq t$, are non-negative integers not all 0. Consider E_t to be partially ordered by letting

$$(n_1, \cdots, n_t) \leq (n_1', \cdots, n_t') \quad \text{if } n_i \leq n_i' \text{ for all } i.$$

LEMMA 3. *Every non-empty subset* S *of* E_t *has a finite number of minimal elements.*

Proof. The result is obviously true for E_1; assume it has been proved for E_{t-1}. Let $(\bar{n}_1, \cdots, \bar{n}_t)$ be any fixed element of S. For each k $(1 \leq k \leq t)$ and each m $(0 \leq m \leq \bar{n}_k)$ consider

$$S_k^m = \{(n_1, \cdots, n_t) \in S;\ n_k = m\}.$$

By the induction hypothesis, the number of minimal elements of the ordered set S_k^m is finite. Now let $\bar{S}$ be the set formed by $(\bar{n}_1, \cdots, \bar{n}_t)$ and all the minimal elements of all ordered sets S_k^m $(1 \leq k \leq t,\ 0 \leq m \leq \bar{n}_k)$. Then $\bar{S}$ is finite and every minimal element of S is in $\bar{S}$.

LEMMA 4. *Let* R' *be a ring extension of* R *with* $R \subset R' \subset K$, *and let* G *be any group. Then given any* R'G-*module* M', *there exists an* RG-*module* M *such that* $M' \cong R'M$.

Proof. Let $\{m_i\}$ $(1 \leq i \leq t)$ be a set of generators of M' over R'. Take

$$M = \sum_{\substack{1 \leq i \leq t \\ g \in G}} Rgm_i .$$

Then M is an RG-module, and it is easily seen that $M' \cong R'M$.

PROPOSITION 5. *Given any group* G, n(RG) *is finite if and only if for every prime ideal* P *of* R *which divides the order of* G, $n(R_P G)$ *is finite.*

Proof. If n(RG) is finite, then $n(R_P G)$ must be finite because, since $R \subset R_P \subset K$, by Lemma 4 every $R_P G$-module is obtained from an RG-module by taking the tensor product with R_P.

Assume now $n(R_P G)$ finite for all P dividing the order of G. By Lemma 1, if an $R_0 G$-module M is indecomposable, then $R_0 M$ is indecomposable. Since there can only be a finite number of RG-modules R_0-isomorphic to a fixed R_0G-module (see [15]), to prove that n(RG) is finite it suffices to show that $n(R_0G)$ is finite.

For each P which divides the order of G assume the set of indecomposable R_PG-modules to be numbered from 1 to r(P). Then given any R_0G-module M, for each such P we can assign to M a sequence of non-negative integers $(n_1^P, \cdots, n_{r(P)}^P)$ by letting n_i^P be the number of times that the i-th indecomposable R_PG-module appears in some fixed decomposition of R_PM into indecomposable R_PG-modules. Assume also that the set of prime ideals in R which divide the order of G is numbered from 1 to s, and order the preceding sequences accordingly. Then to each R_0G-module M, there corresponds a well-defined sequence of $t = r(P_1) + \cdots + r(P_s)$ non-negative integers $(n_1(M), \cdots, n_t(M))$. Furthermore, that $n_i(M) = n_i(N)$ for all i $(1 \leq i \leq t)$ implies that $R_PM \cong R_PN$ for all P which divide the order of G. Therefore $M \cong N$.

We observe now that, if $M \not\cong N$ and $n_i(N) \leq n_i(M)$ for all i $(1 \leq i \leq t)$, then for all P which divide the order of G, R_PN is isomorphic to an R_PG-summand of R_PM. Thus by Lemma 2, N is isomorphic to an R_0G-summand of M. It follows that, if M is indecomposable, the sequence $(n_1(M), \cdots, n_t(M))$ must be minimal in the set E_t defined above. From Lemma 3 we conclude that $n(R_0G)$ is finite.

3. COMPLETE VALUATION RINGS

PROPOSITION 6. *Let* G *be any group, and let* H *be a* p-Sylow *subgroup of* G. *Consider the algebraic number field* K *with a* P-*adic valuation such that* $p \in P$, *and let* K*, *with valuation ring* R*, *be the completion of* K. *Then* n(R* G) *is finite if and only if* n(R* H) *is finite.*

Proof. Suppose n(R* G) is finite. Let L be an indecomposable R* H-module. Form the induced R* G-module

$$L^G = R^*G \otimes_{R^*H} L.$$

Given any R* G-module M, M_H indicates the R* H-module obtained by restriction of the operation of R* G on M to R* H.

Then $(L^G)_H \cong L \dotplus L'$ for some R* H-module L'. If we take $L^G \cong M_1 \dotplus \cdots \dotplus M_r$, where the M_i $(1 < i < t)$ are indecomposable R* G-modules, it follows that

$$(L^G)_H \cong M_{1H} \dotplus \cdots \dotplus M_{rH}.$$

Since the Krull-Schmidt Theorem holds for representations over complete valuation rings (see [11]), and since L is indecomposable, L must be isomorphic to a summand of M_{iH} for some i $(1 \leq i \leq r)$. This shows that the ranks of the indecomposable R* H-modules are bounded; therefore, n(R* H) is finite.

Now suppose n(R* H) is finite. Let M be an indecomposable R* G-module. Then $[(M_H)^G]_H \cong M_H \dotplus L'$, where L' is an R* H-module. Let π be the projection $[(M_H)^G]_H \to M_H$, and choose $g_1, \cdots, g_m \in G$, to be m representatives of the cosets of G over H. Since $m = [G : H]$ is prime to p, $m^{-1} \in R_P$; therefore,

$$\pi' = \sum_{i=1}^{m} g_i m^{-1} \pi g_i^{-1}$$

is well defined. π' is an R^*G-homomorphism which maps $(M_H)^G$ onto M, leaving fixed the elements of $(M_H)^G$ which are in M, so M is an R^*G-summand of $(M_H)^G$. If $M_H \cong L_1 \dotplus \cdots \dotplus L_t$, where the L_i $(1 \leq i \leq t)$ are indecomposable R^*H-modules, then

$$(M_H)^G \cong L_1^G \dotplus \cdots \dotplus L_t^G;$$

so, from the Krull-Schmidt Theorem for R^*G-modules, it follows that M is isomorphic to a summand of L_i^G for some i $(1 \leq i \leq t)$. This implies that $n(R^*G)$ is finite.

For the proof of the two implications in Proposition 6 we only need the fact that the Krull-Schmidt Theorem holds for R^*G and R^*H-modules. The result then also holds, for example, for $R_P G$ and $R_P H$-modules if R_P is a valuation ring of a splitting field for G and H (see [5]).

PROPOSITION 7. *Let* K *be an algebraic number field with valuation ring* R_P, *and let* K^* *(with valuation ring* R^**) be the completion of* K. *Then for any* G, *if* $n(R^*G)$ *is finite, so is* $n(R_P G)$.

Proof. Suppose $n(R^*G)$ is finite. To prove that $n(R_P G)$ is finite, it suffices to show that the ranks of the indecomposable $R_P G$-modules are bounded.

To each indecomposable $R_P G$-module M assign a sequence of $t = n(R^*G)$ integers: $(n_1(M), \cdots, n_t(M))$, where $n_i(M)$ is the multiplicity with which the i-th indecomposable R^*G-module appears in some fixed decomposition of R^*M into indecomposable R^*G-modules. Then if

$$(n_1(L), \cdots, n_t(L)) \leq (n_1(M), \cdots, n_t(M)),$$

it follows that R^*L is isomorphic to an R^*G-summand of R^*M. By Lemma 3, the set of all such sequences has a finite number of minimal elements; therefore for every $R_P G$-module M of rank larger than a fixed number, there exist an $R_P G$-module L and an R^*G-module N^* such that $R^*M \cong R^*L \dotplus N^*$. It follows that $K^*M \cong K^*L \dotplus K^*N^*$. The proof of the Noether-Deuring Theorem (see [3]) shows that KL is a KG-summand of KM if K^*L is a K^*G-summand of K^*M. So there exists a KG-module $\bar{N}$ such that $KM \cong KL \dotplus \bar{N}$. Thus we see that

$$K^*L \dotplus K^*N^* \cong K^*L \dotplus K^*\bar{N};$$

so by the Krull-Schmidt Theorem, $K^*N^* \cong K^*\bar{N}$.

Thus we arrive at the case of an R^*G-module N^*, such that K^*N^* comes from a KG-module $\bar{N}$. It is known (see [1] and [5]) that this situation implies the existence of an $R_P G$-module N such that $N^* \cong R^*N$. Therefore, $R^*M \cong R^*(L \dotplus N)$, and this implies $M \cong L \dotplus N$ (see [8]).

This proves that the ranks of the indecomposable R_P G-modules are bounded.

4. PROOF OF THE THEOREM

THEOREM 8. *Given any group* G, *if for every prime* p *which divides the order of* G *the* p-*Sylow subgroups of* G *are cyclic of order* p *or* p^2, *then* $n(ZG)$ *is finite.*

Proof. Suppose that all Sylow subgroups of G are cyclic of order at most p^2. To show that $n(ZG)$ is finite, by Proposition 5, it is enough to prove that $n(Z_p G)$ is finite for all primes p which divide the order of G.

Let p be any such prime, and let H be a p-Sylow subgroup. Since H is cyclic of order at most p^2, we know that n(ZH) is finite; therefore, $n(Z_p H)$ is finite. Let Z* be the completion of Z_p. In [6] it is shown that if H is a cyclic p-group every Z* H-module is the tensor product of a Z_p H-module with Z*, so n(Z* H) must be finite. From Proposition 6 it follows that n(Z* G) is finite, and Proposition 7 shows that $n(Z_p G)$ is finite.

REFERENCES

1. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Interscience Publishers, New York, 1962.

2. E. C. Dade, *Some indecomposable group representations*, to appear in Ann. of Math.

3. M. Deuring, *Galoissche Theorie und Darstellungstheorie*, Math. Ann. 107 (1932), 140-144.

4. F. E. Diederichsen, *Über die Ausreducktion ganzzahliger Gruppendarstellungen bei arithmetischer Äquivalenz*, Abh. math. Sem. Hansische Univ. 13 (1940), 375-412.

5. A. Heller, *On group representations over a valuation ring*, Proc. Nat. Acad. Sci. U.S.A. 47 (1961), 1194-1197.

6. A. Heller and I. Reiner, *Representations of cyclic groups in rings of integers* I, Ann. of Math. (2) 76 (1962), 73-92.

7. ———, *On groups with finitely many indecomposable integral representations*, Bull. Amer. Math. Soc. 68 (1962), 210-212.

8. J. M. Maranda, *On the equivalence of representations of finite groups by groups of automorphisms of modules over Dedekind rings*, Canad. J. Math. 7 (1955), 516-526.

9. I. Reiner, *Integral representations of cyclic groups of prime order*, Proc. Amer. Math. Soc. 8 (1957), 142-146.

10. ———, *On the class number of representations of an order*, Canad. J. Math. 11 (1959), 660-672.

11. ———, *The Krull-Schmidt Theorem for integral group representations*, Bull. Amer. Math. Soc. 67 (1961), 365-367.

12. ———, *Indecomposable representations of non-cyclic groups*, Michigan Math. J. 9 (1962), 187-191.

13. A. V. Roĭter, *On the representations of the cyclic group of fourth order by integral matrices*, Vestnik Leningrad. Univ. 15 (1960), no. 19, 65-74.

14. A. Troy, *Integral representations of cyclic groups of order* p^2, Doctoral thesis, Univ. of Illinois, 1961.

15. H. Zassenhaus, *Neuer Beweis der Endlichkeit der Klassenzahl bei unimodularer Äquivalenz endlicher ganzzahliger Substitutionsgruppen*, Abh. math. Sem. Hansische Univ. 12 (1938), 276-288.

University of Illinois