AN ANALOGUE OF A PROBLEM OF LITTLEWOOD

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A well-known problem of Littlewood (see [1], [2], [3]) concerning Diophantine approximation can be stated as follows: for any real θ, ϕ and any $\varepsilon > 0$, does there exist a positive integer n such that

$$n\|n\theta\|\,\|n\phi\| < \varepsilon \ ? \tag{1}$$

Here $\|\alpha\|$ denotes the difference between α and the nearest integer, taken positively. In view of the difficulty of the problem (and of even conjecturing the answer), it may be of interest to consider the analogous problem for some other systems of elements, which have many properties in common with the integers and real numbers.

Let K be any field, and let t be an indeterminate. Let N be any polynomial in t with coefficients in K, say

$$N = n_0 + n_1 t + \cdots + n_h t^h \quad (n_h \neq 0), \tag{2}$$

and introduce the valuation

$$|N|_K = e^h. \tag{3}$$

Any rational function in t is expressible uniquely as a formal power series of the type

$$m_h t^h + \cdots + m_0 + m_{-1} t^{-1} + \cdots, \tag{4}$$

where the coefficients are in K and satisfy ultimately a linear recurrence relation. We denote by $\mathfrak{P}_K$ the field of *all* formal power series of the type (4), with arbitrary coefficients in K, and we extend the definition of valuation in (3) to apply to such series.

We regard $\mathfrak{P}_K$ as analogous to the field of real numbers, and the rational functions in t and polynomials in t as analogous to the rational numbers and integers respectively. We ask whether, given Θ and Φ in $\mathfrak{P}_K$ there exists N of the form (2) such that

$$|N|_K \, \|N\Theta\|_K \, \|N\Phi\|_K \tag{5}$$

is arbitrarily small, where the double modulus sign denotes that the valuation is applied to the fractional part of the series for $N\Theta$ or $N\Phi$, that is, to the part which comprises only the negative powers of t.

We shall prove that *if* K *is the field of real numbers (or, a fortiori, the field of complex numbers), the answer is in the negative.* Our proof applies for any infinite field of constants; however, we are unable to solve the problem if K is the field of residue-classes to a prime modulus.

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Since the integral parts of Θ and Φ are irrelevant in (5), we make a slight change of notation and write

$$\Theta = \theta_1 t^{-1} + \theta_2 t^{-2} + \cdots, \tag{6}$$

and similarly for Φ. Then the fractional part of $N\Theta$ is

$$L_1(\theta)t^{-1} + L_2(\theta)t^{-2} + \cdots, \tag{7}$$

where

$$L_r(\theta) = n_0\theta_r + n_1\theta_{r+1} + \cdots + n_h\theta_{r+h}, \tag{8}$$

and similarly for Φ.

We shall show that it is possible to choose real numbers $\theta_1, \theta_2, \cdots$ and $\phi_1, \phi_2, \cdots$ in such a way as to ensure that for every $h = 1, 2, \cdots$ and every $r \geq 0$ and $s \geq 0$ with $r + s = h + 1$, the equations

$$L_1(\theta) = \cdots = L_r(\theta) = 0, \quad L_1(\phi) = \cdots = L_s(\phi) = 0 \tag{9}$$

have no solution in $n_0, \cdots, n_h$ except the trivial solution $0, \cdots, 0$. It is to be understood that, if $r = 0$ or $s = 0$, the corresponding set of equations in (9) is empty.

If this is so, the case $r = h + 1$ of the assertion states that, for any N of the form (2) with $n_h \neq 0$, it is impossible that

$$L_1(\theta) = \cdots = L_{h+1}(\theta) = 0.$$

Hence there is some r $(0 \leq r \leq h)$ such that

$$L_1(\theta) = \cdots = L_r(\theta) = 0, \quad L_{r+1}(\theta) \neq 0.$$

It then follows from the assertion that there is some $q \leq h + 1 - r$ for which $L_q(\phi) \neq 0$. Thus

$$|N|_K = e^h, \quad \|N\Theta\|_K = e^{-r-1}, \quad \|N\Phi\|_K \geq e^{-(h+1-r)},$$

whence

$$|N|_K \|N\Theta\|_K \|N\Phi\|_K \geq e^{-2}.$$

Let $\Delta_{r,s}$ denote the determinant of the $r + s$ linear forms in (9) in the $r + s = h + 1$ variables $n_0, \cdots, n_h$. Then

$$\Delta_{r,s} = \begin{vmatrix} \theta_1, & \cdots, & \theta_{r+s} \\ & \cdots & \\ \theta_r, & \cdots, & \theta_{2r-1+s} \\ \phi_1, & \cdots, & \phi_{r+s} \\ & \cdots & \\ \phi_s, & \cdots, & \phi_{2s-1+r} \end{vmatrix},$$

where if $r = 0$ the θ's are absent and if $s = 0$ the ϕ's are absent. We have to choose $\theta_1, \theta_2, \cdots$ and $\phi_1, \phi_2, \cdots$ to satisfy all the conditions

$$\Delta_{r,s} \neq 0 \qquad (r \geq 0,\ s \geq 0,\ r + s \geq 1).$$

For then the equations (9) will not be soluble in $n_0, \cdots, n_h$, with some $n_i \neq 0$.

We note that

$$\Delta_{r,s} = \pm\theta_{2r-1+s}\Delta_{r-1,s} \pm\phi_{2s-1+r}\Delta_{r,s-1} + F(\theta_1, \cdots, \theta_{2r-2+s}; \phi_{2s-2+r}),$$

where F is a polynomial with integral coefficients, and where it is understood that $\Delta_{0,0} = 1$, and $\Delta_{-1,s} = \Delta_{r,-1} = 0$. We pick out all those determinants $\Delta_{r,s}$ which contain any of $\theta_{2m-1}, \theta_{2m}, \phi_{2m-1}, \phi_{2m}$ and contain no θ_j or ϕ_j with $j > 2m$. These are the determinants with

$$2r - 1 + s \leq 2m, \quad 2s - 1 + r \leq 2m$$

and either $2r - 1 + s \geq 2m - 1$ or $2s - 1 + r \geq 2m - 1$. They are finite in number, and each of them is of one of the six types

$$\Delta\theta_{2m-1} + F(\theta_1, \cdots, \theta_{2m-2}; \phi_1, \cdots, \phi_{2m-2}),$$

$$\Delta\phi_{2m-1} + F(\theta_1, \cdots, \theta_{2m-2}; \phi_1, \cdots, \phi_{2m-2}),$$

$$\Delta\theta_{2m-1} + \Delta\phi_{2m-1} + F(\theta_1, \cdots, \theta_{2m-2}; \phi_1, \cdots, \phi_{2m-2}),$$

$$\Delta\theta_{2m} + F(\theta_1, \cdots, \theta_{2m-1}; \phi_1, \cdots, \phi_{2m-1}),$$

$$\Delta\phi_{2m} + F(\theta_1, \cdots, \theta_{2m-1}; \phi_1, \cdots, \phi_{2m-1}),$$

$$\Delta\theta_{2m}\phi_{2m-1} + \Delta\theta_{2m-1}\phi_{2m} + \theta_{2m}G(\theta_1, \cdots, \theta_{2m-2}; \phi_1, \cdots, \phi_{2m-2})$$
$$+ \phi_{2m}H(\theta_1, \cdots, \theta_{2m-2}; \phi_1, \cdots, \phi_{2m-2})$$
$$+ F(\theta_1, \cdots, \theta_{2m-1}; \phi_1, \cdots, \phi_{2m-1}),$$

where the various Δ's (not all the same, of course) are instances of $\Delta_{p,q}$ which involve only $\theta_1, \cdots, \theta_{2m-2}$ and $\phi_1, \cdots, \phi_{2m-2}$, and where G, H and the various F's are polynomials in the variables stated. Not all six types need occur.

Suppose the unknowns $\theta_1, \cdots, \theta_{2m-2}, \phi_1, \cdots, \phi_{2m-2}$ have already been chosen in such a way that all determinants that involve only these unknowns are different from zero. We can then choose $\theta_{2m-1}, \phi_{2m-1}, \theta_{2m}, \phi_{2m}$ so as to ensure that all determinants which involve these variables and no variable of greater suffix shall be different from zero. For this means making a finite number of polynomials assume non-zero values, each of these polynomials being of one of the six types described above. None of these polynomials in $\theta_{2m-1}, \theta_{2m}, \phi_{2m-1}, \phi_{2m}$ vanishes identically, since each has at least one non-zero coefficient Δ. The set of points in the four-dimensional real space of $\theta_{2m-1}, \phi_{2m-1}, \theta_{2m}, \phi_{2m}$ at which any such polynomial vanishes is an algebraic set of dimension 3, and a finite number of such sets cannot exhaust the space. This suffices for the construction, and we can therefore proceed by induction on m. For $m = 1$, we have only to choose $\theta_1, \phi_1, \theta_2, \phi_2$ so that

$$\theta_1 \neq 0, \quad \phi_1 \neq 0, \quad \theta_1 \phi_2 - \theta_2 \phi_1 \neq 0.$$

We note finally that it is further possible to choose $\theta_1, \theta_2, \cdots$ and $\phi_1, \phi_2, \cdots$ so that the absolute values of θ_j and ϕ_j tend to zero rapidly as $j \to \infty$. Thus we can construct Θ and Φ in the more restricted system of power series in t^{-1} with real coefficients which converge for all $t \neq 0$.

REFERENCES

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